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Asymptotic Analysis of Confluent Hypergeometric Partial Differential Equations in Many Variables

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1 Introduction

The confluent differential equation in one variable, known as the Kummer differential equation
\[ x \frac{d^2}{dx^2} w + (\gamma-x) \frac{d}{dx} w - \beta w = 0, \]
is studied by several authors in various ways. Among those, the so-called Borel-Laplace-Ecalle method is a powerful one, which is explained for example in [1]. This method is applicable to an analysis of the Humbert confluent hypergeometric differential equations $\Phi_2$ in 2 variables
\[ x \frac{\partial^2}{\partial x^2} w + y \frac{\partial}{\partial y} \frac{\partial}{\partial x} w + (\gamma-x) \frac{\partial}{\partial x} w - \beta w = 0, \]
and
\[ y \frac{\partial^2}{\partial y^2} w + x \frac{\partial}{\partial x} \frac{\partial}{\partial y} w + (\gamma-y) \frac{\partial}{\partial y} w - \beta' w = 0, \]
and we can obtain formal solutions, asymptotic solutions and so-called Stokes multipliers (see [2], [3]).

It is also applicable to an asymptotic analysis of the Humbert confluent hypergeometric partial differential equations in $m(>2)$ variables. Here, the author gives an overview of it.

2 Humbert confluent hypergeometric partial differential equations $\Phi_D$

The system of Humbert confluent hypergeometric partial differential equations $\Phi_D$ is as follows:

\[ x_k \frac{\partial^2 u}{\partial x^2_k} + \sum_{l \neq k} \frac{x_l}{x_k} \frac{\partial^2 u}{\partial x_k \partial x_l} + (\gamma-x_k) \frac{\partial u}{\partial x_k} - \beta_k u = 0, \]

where $\beta_k (k=1, \ldots, m)$ and $\gamma$ are not non-negative integers.

We consider this system in $M = (P^1(C))^m$. The system has irregular singularities on $H = \bigcup_{k=1}^m H_k$, where $H_k \approx P^1(C) \times \cdots \times \{\infty\} \times \cdots \times P^1(C)$.

For simplicity, let $p$ be a point in $H \setminus \bigcup_{k \neq i} (H_k \cap H_i)$, we consider the formal solutions and asymptotic solutions to $\Phi_D$ near the point.

Proposition 1. We have $(m+1)$ linearly independent formal solutions. Among them, $(m-1)$ formal solutions are convergent and 2 formal solutions are divergent.

Near a point $(\infty, x_2, \ldots, x_m)$ with bounded $x_2, \ldots, x_m$, we have divergent solutions of the following forms

\[ e^{x_1^{\beta_1}} x_1^{\beta_1-\gamma} \hat{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}), \]

and

\[ x_1^{-\beta_1} \hat{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}). \]

Here, we put

\[ \hat{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}) = \sum_{n=0}^{\infty} P_n(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) x_1^{-n}, \]
with the polynomials

\[
P_n(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m) = \sum_{\ell=0}^{n} \frac{(\gamma - \beta_1 + \ell)_{n-\ell}(1 - \beta_1)_{n-\ell}}{(n-\ell)!\ell!} \sum_{j_2+\ldots+j_m=\ell} \frac{(\beta_2 j_2 \ldots (\beta_m) j_m \ell!}{j_2!\ldots j_m!} x_2^{j_2} \ldots x_m^{j_m}
\]

and

\[
\hat{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}) = \sum_{n=0}^{\infty} \frac{\beta_1 n (\beta_1 + 1)_n}{n!} \Phi_{D}^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1 - n; x_2, \ldots, x_m)(-x_1)^{-n},
\]

where \(\Phi_{D}^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1 - n; x_2, \ldots, x_m)\) is the Humbert confluent hypergeometric function in \((m-1)\) variables with the parameter \((\beta_2, \ldots, \beta_m; \gamma - \beta_1 - n)\),

\[
\Phi_{D}^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1 - n; x_2, \ldots, x_m) = \sum_{j_2=0}^{\infty} \ldots \sum_{j_m=0}^{\infty} \frac{(\beta_2 \ldots (\beta_m) j_2 \ldots j_m x_2^{j_2} \ldots x_m^{j_m})}{(\gamma - \beta_1 - n)_{j_2+\ldots+j_m} j_2!\ldots j_m!}.
\]

Proposition 2. The divergent formal series

\[
\hat{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1})
\]

and

\[
\hat{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1})
\]

are of Gevrey order 1 as \(x_1 \to \infty\) uniformly on a bounded domain \(D\) in the \((x_2, \ldots, x_m)\)-space.

Definition. For a formal expression \(e^{\rho x_1} x_1^{-\lambda} \hat{p}(x)\) with a complex number \(\rho\), a non-negative integer \(A\) and a formal series \(\hat{p}(x) = \sum_{n=0}^{\infty} c_n(x_2, \ldots, x_m)x_1^{-n}\), we define the Borel transform as follows,

\[
\hat{B}_1(e^{\rho x_1} \sum_{n=0}^{\infty} c_n x_1^{-\lambda-n})(\xi_1) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+\lambda)} (\rho + \xi_1)^{n+\lambda-1}.
\]

Proposition 3. The Borel transforms of divergent solutions are holomorphic functions in a domain in \(\mathbb{C}^m\), which are analytically prolongeable.

In fact,

\[
\hat{B}_1(e^{\rho x_1} x_1^{\beta_1 - \gamma} \tilde{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}))(\xi_1) = \frac{1}{\Gamma(\gamma - \beta_1)} (\xi_1)^{\beta_1 - 1}(1 + \xi_1)^{\gamma - \beta_1 - 1} \times \Phi_{D}^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \ldots, (1 + \xi_1)x_m),
\]

\[
\hat{B}_1(x_1^{-\beta_1} \tilde{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}))(\xi_1) = \frac{1}{\Gamma(\beta_1)} (\xi_1)^{\beta_1 - 1}(1 + \xi_1)^{\gamma - \beta_1 - 1} \times \Phi_{D}^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1 + \xi_1)x_2, \ldots, (1 + \xi_1)x_m).
\]

Between

\[
\hat{B}_1(\hat{v})(\xi_1) = \hat{B}_1(e^{\rho x_1} x_1^{\beta_1 - \gamma} \tilde{V}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}))(\xi_1)
\]

and

\[
\hat{B}_1(\hat{u})(\xi_1) = \hat{B}_1(x_1^{-\beta_1} \tilde{U}(\beta_1, \beta_2, \ldots, \beta_m, \gamma, x_2, \ldots, x_m, x_1^{-1}))(\xi_1),
\]
we have a relation
\[ \Gamma(\beta_1)B_1(\xi)(\xi_1) = \Gamma(\gamma - \beta_1)(-1)^{-\beta_1+1}B_1(\xi)(\xi_1). \]

**Definition.** Consider a function \( f(\xi_1, x_2, \ldots, x_m) \) which is holomorphic and exponentially small in a tubular neighborhood in the first variable and a bounded domain in the other variables. We define the generalized Laplace transforms of \( f(\xi_1, x_2, \ldots, x_m) \), as follows
\[
\int_{C(q, \theta)} \exp(-x_1\xi_1)f(\xi_1, x_2, \ldots, x_m)d\xi_1,
\]
where \( C(q, \theta) \) is a following path of integral. For a point \( q \) in the tubular neighborhood, \( C(q, \theta) \) is a path on which \( \arg(\xi_1 - q) \) is taken to be initially \( \theta \) and finally \( \theta + 2\pi \).

**Proposition 4.** The Laplace transforms of Borel transforms of divergent solutions are holomorphic functions in a suitable angular domain with the summit \( p \) in \( P^1(C) \times C^{m-1} \), where they are actual solutions to the system \( \Phi_D \) with asymptotic expansions of Gevrey order 1. Here, the asymptotic expansions coincide with the divergent solutions, respectively.

In fact, for
\[-2\pi < \theta < 0,\]
the Laplace integral
\[
\frac{1}{\Gamma(\gamma - \beta_1)} \int_{C(-1, \theta)} \exp(-x_1\xi_1)(-\xi_1)^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1} \times \Phi_D^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1+\xi_1)x_2, \cdots, (1+\xi_1)x_m)d\xi_1
\]
is defined and represents a holomorphic function in the first variable \( x_1 \) in the angular domain (mod. 2\pi)
\[
\frac{\pi}{2} < \arg(-\xi_1x_1) < \frac{3\pi}{2},
\]
namely,
\[-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,
\]
because \( \exp(-x_1\xi_1) \) tends to 0 as \( \xi_1 \) tends to the infinity. By considering the analytic prolongation, we obtain an actual solution \( v \) in the angular domain
\[-\frac{5\pi}{2} < \arg x_1 < \frac{\pi}{2}.
\]

For
\[-\pi < \theta < \pi,
\]
the Laplace integral
\[
\frac{1}{\Gamma(\beta_1)} \int_{C(0, \theta)} \exp(-x_1\xi_1)\xi_1^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1} \times \Phi_D^{m-1}(\beta_2, \ldots, \beta_m; \gamma - \beta_1; (1+\xi_1)x_2, \cdots, (1+\xi_1)x_m)d\xi_1
\]
is defined and represents a holomorphic function in the first variable \( x_1 \) in the angular domain (mod. 2\pi)
\[
\frac{\pi}{2} < \arg(-\xi_1x_1) < \frac{3\pi}{2},
\]
namely,
\[-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,
\]
because \( \exp(-x_1\xi_1) \) tends to 0 as \( \xi_1 \) tends to the infinity. By considering the analytic prolongation, we obtain an actual solution \( u \) in the angular domain
\[-\frac{3\pi}{2} < \arg x_1 < \frac{3\pi}{2}.\]
**Proposition 5** We have fundamental systems of solutions
\[
(e_1 u, e_2 v, w_2, \ldots, w_m)
\]
in the angular domain
\[
-\frac{3\pi}{2} < \arg x_1 < \frac{\pi}{2}
\]
and
\[
(e_1 u, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)}, w_2, \ldots, w_m)
\]
in the angular domain
\[
-\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}
\]
where
\[
w_2 = x_1^{-\beta_1}x_{m}^{\beta_1-\gamma+1}h_2, \ldots, w_m = x_1^{-\beta_1}x_{m}^{\beta_1-\gamma+1}h_m
\]
with holomorphic functions \(h_2, \ldots, h_m\) at the point \(p\).

Then, we have the relations
\[
(e_1 u, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)})
\]
in the angular domain
\[
-\frac{\pi}{2} < \arg x_1 < \frac{\pi}{2}
\]
and
\[
(e_1 u(x_1 e^{-2i\pi})e^{2i\pi(-\beta_1)}, e_2 v(x_1 e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)})
\]
in the angular domain
\[
\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}
\]
In the above, we use the following constants
\[
e_1 = (e^{2i\pi\beta_1} - 1)^{-1},
\]
\[
e_2 = (e^{2i\pi(\gamma-\beta_1)} - 1)^{-1},
\]
\[
c_{12} = \frac{-2i\pi}{\Gamma(1-\beta_1)\Gamma(\gamma-\beta_1)},
\]
\[
c_{21} = \frac{-2i\pi e^{i\pi(\gamma-2\beta_1)}}{\Gamma(\beta_1)\Gamma(1-\gamma+\beta_1)}.
\]

**References**

