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The standard $(\mathfrak{g}, K)$-modules of $Sp(2,\mathbf{R})$ I

The case of principal series

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Introduction

We investigate explicitly structure of the $(\mathfrak{g}, K)$-modules of the standard representations of $Sp(2,\mathbf{R})$ obtained by parabolic induction.

The group $Sp(2,\mathbf{R})$ has 3 non-trivial standard parabolic subgroups: the minimal parabolic subgroup P_{min}, the maximal parabolic subgroup P_J associated with the long root, and the maximal parabolic subgroup P_S associated with the short root. In this paper we discuss the case of the parabolic induction with respect to the minimal parabolic subgroup P_{min}.

0 The standard $(\mathfrak{g}, K)$-modules $SL(2,\mathbf{R})$

We start with a short review of the most classical case, i.e., the case of the group $SL(2,\mathbf{R})$.

0.1 The principal series

We write

$$G_0 = SL(2,\mathbf{R}),\ K_0 = SO(2),$$
$$A_0 = \{a_0 = \mathrm{diag}(r, r^{-1}) | r \in \mathbf{R}_{>0}\}, M_0 = \{\mathrm{diag}(\varepsilon,\varepsilon) | \varepsilon \in \{\pm 1\}\}$$

For a character σ in $\hat{M}_0 = \{\sigma_0(= \mathrm{id}), \sigma_1\}$ of M_0 and a linear form $\nu_0 \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_0, \mathbf{C})$ ($\mathfrak{a}_0 = Lie(A_0)$), the Hilbert space of the principal series represenation is defined as

$$H_{(\nu_0,\sigma)} = \{f : G \to \mathbf{C} | f(n_0 m_0 a_0 x) = \sigma(m_0) e^{(\nu_0+\rho_0)(\log(a_0))} f(x),$$
$$n_0 \in N_0, m_0 \in M_0, a_0 \in A_0, x \in G_0, \text{ and } f|K \in L^2(K_0)\}.$$

We have the irreducible decomposition of the K_0-module $L^2(K_0)$:

$$L^2(K_0) = \hat{\oplus}_{m\in\mathbf{Z}} \mathbf{C}\chi_m$$

where

$$\chi_m : k_0 = r_\theta \in K_0 \mapsto e^{im\theta} \in \mathbf{C}.$$

Then we have the natural identification:

$$H_{(\nu_0,\sigma)} \cong \begin{cases} \hat{\oplus}_{m\in 2\mathbf{Z}} \mathbf{C}\chi_m & \text{if } \sigma = \sigma_0; \\ \hat{\oplus}_{m\in 2\mathbf{Z}+1} \mathbf{C}\chi_m & \text{if } \sigma = \sigma_1. \end{cases}$$

Recall that

$$w\chi_m = \sqrt{-1}m\chi_m \text{ for } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the commutation relations

$$[w, x_\pm] = \pm 2\sqrt{-1}x_\pm \text{ for } x_\pm = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}.$$

And also use the Iwasawa decomposition:

$$x_\pm = \pm 2\sqrt{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + H_{12} \mp \sqrt{-1}w.$$

Then we have the following.

Proposition 0.1 (0) $x_+\chi_m \in \mathbf{C}\chi_{m+2}$ and $x_-\chi_m \in \mathbf{C}\chi_{m-2}$.
(i) $x_+\chi_m = (\nu_0 + \rho_0 + m)\chi_{m+2}$;
(ii) $x_-\chi_m = (\nu_0 + \rho_0 - m)\chi_{m-2}$.

0.1.1 Embedding of the discrete series and quotients as discrete series

Proposition 0.2 (i) If $\nu_0 + 1 = k$, there is an injective homomorphism $D_k^\pm \subset \pi_{(k-1)}$ of $(\mathfrak{g}_0, K_0)$ modues. Moreover the quotient $(\mathfrak{g}_0, K_0)$-modules $\pi_{(k-1)}/(D_k^+ \oplus D_k^-)$ is of dimension $k-1$. Note that we have $sgn(\sigma) = (-1)^k$.
(ii) If $\nu_0 = -(k-1)$, then

$$x_+\chi_{k-2} = 0 \text{ and } x_-\chi_{-(k-2)} = 0.$$

Moreover the $k-1$ dimensional space F_{k-2} generated by

$$\{\chi_m | m = -(k-2), m = -(k-2)+2, \cdots, m = k-2\}$$

is the space of the symmetric tensor represenation of degree $k-2$ of G_0. Moreover the quotient $\pi_{-(k-1)}/F_{k-2}$ is isomorphic to $D_k^+ \oplus D_k^-$. We have $sgn(\sigma) = (-1)^k$.

The proof of Propositions (0.1) and (0.2) are found in any introductory book on the theory of representations of $SL(2, \mathbf{R})$, or general theory of representations of real reductive groups (see for example, Wallach [2], §5.6).

In this paper, we are going to show the analogue of Proposition (0.1) for $Sp(2, \mathbf{R})$ (*cf.* Theorems 5.2 A and 5.2 B).

1 The structure of $Sp(2, \mathbf{R})$

1.1 Basic objects

We use the case $n = 2$, but start from general n. Our whole group is

$$G = Sp(n; \mathbf{R}) := \{g \in M_{2n}(\mathbf{R}) | {}^t g J_n g = J_n, \det(g) = 1\},$$

where $J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. This is a split simple group of rank n of type C_n. We fix a maximal compact group K by

$$K := G \cap O(2n) = \{g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} | A + \sqrt{-1}B \in U(n)\}$$

once for all, which is also defined by using the Cartan involution $\theta : g \in G \to J_n {}^t g^{-1} J_n$ as

$$K = G^\theta = \{g \in G | g^\theta = g\}.$$

. Any maximal compact subgroup of G is conjugate to this standard one. The associated Lie algebras are given by

$$\mathfrak{g} = \{X \in M_{2n}(\mathbf{R}) | {}^t X J_n + J_n X = 0, \mathrm{tr}(X) = 0\}$$

and

$$\mathfrak{k} = \{X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} | A + \sqrt{-1}B \in \mathfrak{u}(n)\}.$$

Writing $\mathfrak{p} = \mathfrak{g}^{-\theta} = \{X \in \mathfrak{g} | {}^t X^\theta = -X\}$ we have a Cartan decompotion

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

1.2 Iwasawa decompostion

Notation We denote by e_{ij} the matrix unit in $M_n(\mathbf{C})$ with entry 1 at (i, j)-th component and 0 at other entries. Also by E_{ij} the matrix unit in $M_{2n}(\mathbf{C})$. For $x = {}^t x \in M_n(\mathbf{C})$ we set

$$p_\pm(x) = \begin{pmatrix} x & \pm\sqrt{-1}x \\ \pm\sqrt{-1}x & -x \end{pmatrix}.$$

The homomorphism of groups $\kappa : U(n) \to K$ is the inverse of

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K \mapsto A + \sqrt{-1}B \in U(n).$$

The induced homomorphisms of their Lie algebras and of their complexifications are denoted by the same symbol κ.

Lemma 1.1 *Put* $E_{2e_i} = E_{i+n,i}$, $E_{e_i+e_j} = E_{i+n,j} + E_{j+n,i}$ *and* $E_{e_i-e_j} = E_{i,j} - E_{j,i}$. *Then we have*

$$p_\pm(e_{ii}) = 2\sqrt{-1}E_{2e_i} + H_{i,n+i} \pm \kappa(e_{ii})$$

$$p_\pm(\frac{e_{ij} + e_{ji}}{2}) = (E_{e_i-e_j} \pm \sqrt{-1}E_{e_i+e_j}) \begin{cases} +\kappa(e_{ji}) & \text{if } (+) \\ -\kappa(e_{ij}) & \text{if } (-) \end{cases}.$$

Proof We can show this by direct computation.

1.3 New Notation for $n = 2$

When $n = 2$ we use the following notation.

New Notation We write

$$X_{\pm,ii} := e_{\pm}(e_{ii}) \quad (i = 1, 2),$$

and

$$X_{\pm,12} := e_{\pm}(\frac{e_{12} + e_{21}}{2}).$$

Then Iwasawa decomposition tells that

$$X_{\pm,ii} = \pm 2\sqrt{-1}E_{2e_i} + H_{i,i+2} \pm \kappa(e_{ii}),$$

and

$$X_{+,12} = E_{e_1-e_2} + \sqrt{-1}E_{e_1+e_2} + \kappa(e_{21}), \quad X_{-,12} = E_{e_1-e_2} - \sqrt{-1}E_{e_1+e_2} - \kappa(e_{12}).$$

Here

$$E_{e_1-e_2} = E_{12} - E_{43}, \text{ and } E_{e_1+e_2} = E_{14} + E_{23}.$$

1.4 The action of K on $\mathfrak{p}_{\pm}$

We denote the isomorphism between $U(2)$ and K and the associated isomorphism between their Lie algebras or the complexified Lie algebras by

$$\kappa : A + \sqrt{-1}B \in U(2), \text{ or } \in \mathfrak{u}(2)_{\mathbf{C}} \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K, \text{ or } \in \mathfrak{k}_{\mathbf{C}}.$$

Here $A, B \in M_2(\mathbf{C})$. Via κ, $U(2)$ or $\mathfrak{u}(2)_{\mathbf{C}} = \mathfrak{gl}(2, \mathbf{C})$ acts on $\mathfrak{p}_{\pm}$ through the adjoint action of K on $\mathfrak{p}_{\pm}$. For $p_{\pm}(x)$ $(x = {}^t x \in M_2(\mathbf{C}))$ this reads that

$$\begin{aligned} \kappa(A + \sqrt{-1}B) \cdot p_{\pm}(x) &= \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} x & \pm\sqrt{-1}x \\ \pm\sqrt{-1}x & -x \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \\ &= p_{\pm}((A \pm \sqrt{-1}B)x^t(A \pm \sqrt{-1}B)). \end{aligned}$$

Passing to the Lie algebra we have the following.

Lemma 1.2 For $\mathfrak{p}_+$ we have

$$\begin{array}{lll} \kappa(e_{11})X_{+,11} = 2X_{+,11}; & \kappa(e_{11})X_{+,12} = X_{+,12}; & \kappa(e_{11})X_{+,22} = 0. \\ \kappa(e_{12})X_{+,11} = 0; & \kappa(e_{12})X_{+,12} = X_{+,11}; & \kappa(e_{12})X_{+,22} = 2X_{+,12}. \\ \kappa(e_{21})X_{+,11} = 2X_{+,12}; & \kappa(e_{21})X_{+,12} = X_{+,22}; & \kappa(e_{21})X_{+,22} = 0. \\ \kappa(e_{22})X_{+,11} = 0; & \kappa(e_{22})X_{+,12} = X_{+,12}; & \kappa(e_{22})X_{+,22} = 2X_{+,22}. \end{array}$$

And for $\mathfrak{p}_-$ we have

$$\begin{array}{lll} \kappa(e_{11})X_{-,11} = -2X_{-,11}; & \kappa(e_{11})X_{-,12} = X_{-,12}; & \kappa(e_{11})X_{-,22} = 0. \\ \kappa(e_{12})X_{-,11} = -2X_{-,12}; & \kappa(e_{12})X_{-,12} = -X_{-,22}; & \kappa(e_{12})X_{-,22} = 0. \\ \kappa(e_{21})X_{-,11} = 0; & \kappa(e_{21})X_{-,12} = -X_{-,11}; & \kappa(e_{21})X_{-,22} = -2X_{-,12}. \\ \kappa(e_{22})X_{-,11} = 0; & \kappa(e_{22})X_{-,12} = X_{-,12}; & \kappa(e_{22})X_{-,22} = -2X_{-,22}. \end{array}$$

Proof By direct computation.

2 K-modules

2.1 The canonical basis for simple K-modules

Since K is a compact group, any irreducible continuous represenation (τ, W_τ) of K is of finite dimesion, and unitary. We refer to such (τ, W_τ) as a *simple K-module.* Since K is a connected Lie group, the category of continuous finite dimensional representations of K is equivalent to the category of finite dimensional representations of $\mathfrak{k} = \mathrm{Lie}(K)$. Since the complexification $\mathfrak{k}_{\mathbf{C}}$ of $\mathfrak{k}$ is isomorphic to $\mathfrak{gl}(2, \mathbf{C})$, the set of isomorphism classes of simple $\mathfrak{k}$-modules is parametrized by the set $L_K^+ = \{(l_1, l_2) \in \mathbf{Z}^2,\ l_1 \geq l_2\}$ of dominant integral weights of $\mathfrak{k}_{\mathbf{C}} = \mathfrak{gl}(2, \mathbf{C})$.

Each irreducible represenation, or simple module $\tau_{(m_{12}, m_{22})}$ of $\mathfrak{k}_{\mathbf{C}} = \mathfrak{gl}(2, \mathbf{C})$ associated with the dominant weight (m_{12}, m_{22}) has a basis parametrized by the Gelfand-Tsetlin patterns $M = \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix}$ $(m_{12} \leq m_{11} \leq m_{22})$.

Proposition 2.1 There exists a basis $\{f(M)\}_{M \in GZ(m_{12}, m22)}$ in $\tau_{(m_{12}, m_{22})}$ such that

$$\begin{aligned} e_{11} f(M) =& (m_{11} + m_{22}) f(M); \\ e_{22} f(M) =& (m_{12} - m_{11}) f(M); \\ e_{12} f(M) =& (m_{12} - m_{11}) f(M_{+1}); \\ e_{21} f(M) =& (m_{11} - m_{22}) f(M_{-1}). \end{aligned}$$

with respect to the simple roots $e_{i,i+1}, e_{i+1,i}$ $(i = 1, 2)$ in $\mathfrak{gl}(2, \mathbf{C})$. Here $M_{+1} = \begin{pmatrix} * & & * \\ & m_{11}+1 & \end{pmatrix}$, $M_{-1} = \begin{pmatrix} * & & * \\ & m_{11}-1 & \end{pmatrix}$ for $M = \begin{pmatrix} * & & * \\ & m_{11} & \end{pmatrix}$.
Proof This is well-known and classical fact.

Definition 2.1 A simple K-module τ equipped with a canonical basis is called a *marked simple module* or a simlple K-module with marking.

Note that the choice of a canonical basis in a simple K-module is unique up to scalar multiple by Schur's Lemma. The same lemma implies that if there is an isomorphism between K-simple modules with marking then it is unique strictly (not up to scalar). In particular the only automorphism of a simple K-module with marking is the identity map.

2.2 The K-modules $\mathrm{Ad}_{\mathfrak{p}_\pm}$.

Lemma 2.2 Up to scalar multiple there are identifications between natural basis:
(i) For the isomorphism $\mathfrak{p}_+ \cong \tau_{(2,0)}$ of K-modules,

$$X_{+,11} = f\begin{pmatrix} 2 & & 0 \\ & 2 & \end{pmatrix}, \quad X_{+,12} = f\begin{pmatrix} 2 & & 0 \\ & 1 & \end{pmatrix}, \quad X_{+,22} = f\begin{pmatrix} 2 & & 0 \\ & 0 & \end{pmatrix}.$$

(ii) For the isomorphism $\mathfrak{p}_- \cong \tau_{(0,-2)}$ of K-modules,

$$X_{-,22} = f\begin{pmatrix} 0 & & -2 \\ & 0 & \end{pmatrix}, \quad X_{-,12} = -f\begin{pmatrix} 0 & & -2 \\ & -1 & \end{pmatrix}, \quad X_{-,11} = f\begin{pmatrix} 0 & & -2 \\ & -2 & \end{pmatrix}.$$

Proof By direct computation.

Remark The above lemma tells that

$$(\mathfrak{p}_+, \{X_{+,11}, X_{+,12}, X_{+,22}\}) \text{ and } (\mathfrak{p}_-, \{X_{-,11}, -X_{-,12}, X_{-,22}\})$$

are simple K-modules with marking. From now on we always take these marking for $\mathfrak{p}_\pm$.

2.3 The symmetric tensor representations of K

Given a positive integer d, we define a square matrix $\mathrm{Sym}^d(S(k))$ of degree $d+1$ associated with $S(k)$ as follows.

For two independent variables U, V we define two linear forms by

$$U' = s_{11}U + s_{21}V \text{ and } V' = s_{12}U + s_{22}V,$$

or equivalently by

$$(U', V') = (U, V) \cdot S(k).$$

Then by using homogeneous forms $\{(U')^{d-i}(V')^i\}_{0 \le i \le d}$ of degree d, we define a $(d+1) \times (d+1)$ matrix $\mathrm{Sym}^d(S(k))$ by

$$((U')^d, \cdots, (U')^{d-i}(V')^i, \cdots, (V')^d) = (U^d, \cdots, U^{d-i}V^i, \cdots, V^d) \cdot \mathrm{Sym}^d(S(k)).$$

Here is a description of the (i,j)-th entry $(0 \le i, j \le d)$ of $\mathrm{Sym}^d(S(k))$.

Lemma 2.3 By the symbols $(a_1, \cdots, a_d), (b_1, \cdots, b_d)$ we denote the sequences of the elements in the set $\{1, 2\}$ with length d. For given j, we fix a sequence $(1, \cdots, 1, 2, \cdots, 2)$ with 1 in the first $d-j$ entries and 2 at the remaining j entries. For given i, we denote by $Sh(d-i, i)$ the set of all $(d-i, i)$-shuffles of two sets $\{1, \cdots, \}$ of cardinality $d-i$ and $\{2, \cdots, \}$ of cardinality i. Obviously the cardinality of $Sh(d-i, i)$ is $\binom{d}{i}$. Then the (i,j)-th entry of $\mathrm{Sym}^d(S(k))$ is given by

$$\sum_{(a_1, a_2, \cdots, a_d) \in Sh(d-i, i), (b_1, \cdots, b_d) = (1 \times (d-j), 2 \times j)} s_{a_1, b_1} \cdots s_{a_d, b_d}.$$

Proof The proof is a high-school mathematics.

The $d+1$ entries of each row vector of $\mathrm{Sym}^d(S(k))$ make a canonical basis of a simple subspace in $L^2(K)$ with highest weight $(d, 0)$. In fact the intertwining property

$$\mathrm{Sym}^d(x \cdot k) = \mathrm{Sym}^d(x)\mathrm{Sym}^d(k) \quad (x, k \in K)$$

implies that the entries of the each row generates a simple submodule of type $(d, 0)$ and the fact that this is proportional to the caninical basis is checked directly.

Definition We define a $d+1$ colum vectors $\{\mathbf{s}_i^{(d)}\}_{0 \le i \le d}$ of $d+1$ elementary functions by

$${}^t\mathrm{Sym}^d(S(k)) = (\mathbf{s}_0^{(d)}, \mathbf{s}_1^{(d)}, \cdots, \mathbf{s}_d^{(d)}).$$

Notation (matrices of elementary functions) For even d, we set

$$\mathbf{S}^{(d)}_{[0,2,\cdots,d]} = (\mathbf{s}^{(d)}_0, \mathbf{s}^{(d)}_2, \cdots, \mathbf{s}^{(d)}_d)$$

and

$$\mathbf{S}^{(d)}_{[1,3,\cdots,d-1]} = (\mathbf{s}^{(d)}_1, \mathbf{s}^{(d)}_3, \cdots, \mathbf{s}^{(d)}_{d-1}).$$

For odd d, we set

$$\mathbf{S}^{(d)}_{[0,2,\cdots,d-1]} = (\mathbf{s}^{(d)}_0, \mathbf{s}^{(d)}_2, \cdots, \mathbf{s}^{(d)}_{d-1})$$

and

$$\mathbf{S}^{(d)}_{[1,3,\cdots,d]} = (\mathbf{s}^{(d)}_1, \mathbf{s}^{(d)}_3, \cdots, \mathbf{s}^{(d)}_d).$$

For a colum vector ${}^t(a_0, a_1, \cdots, a_d)$ of size $d+1$, we define $*$-operator by

$$^*\begin{pmatrix} a_0 \\ \vdots \\ \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} a_d \\ -a_{d-1} \\ a_{d-2} \\ \vdots \\ (-1)^d a_0 \end{pmatrix}.$$

Also for the matrix $\bar{\mathbf{S}}^{(d)}_{[d,\cdots,0]}$ we set

$$^*\bar{\mathbf{S}}^{(d)}_{[d,\cdots,0]} = ({}^*\bar{\mathbf{s}}^{(d)}_d, \cdots, \bar{\mathbf{s}}^{(d)}_0).$$

Then we have a relation ${}^*\bar{\mathbf{S}}^{(d)}_{[d,\cdots,0]} = d(k)^{-d} \cdot \mathbf{S}^{(d)}_{[0,\cdots,d]}$.

Lemma 2.4 (Maching with the canonical basis) Let $< \mathbf{s}^{(d)}_i >$ be the simple K-module generated by the functions in the entries of the vector $\mathbf{s}^{(d)}_i$ for each i $(0 \leq i \leq d$. Then there is a (strictly) unique isomorphism of K-modules from this to $\tau_{(d,0)}$ which maps the $(a+1)$-th entry of $\mathbf{s}^{(d)}_i$ to the canonical basis $f\binom{d,\ 0}{a}$ in $\tau_{(d,0)}$.
Proof The proof is done by direct computation, utlizing Lemma (3.1) and the Leibniz rule.

2.4 Irreducible decompostion of $\tau_{(2,0)} \otimes \tau_{(d,0)}$

In later sections, we need irreducible decompostion of the tensor product $\mathfrak{p}_\pm \otimes \tau_{(l_1,l_2)}$ as K-modules. Since $\mathfrak{p}_+ \cong \tau_{(2,0)}, \mathfrak{p}_- \cong \tau_{(0,-2)} \cong \tau_{(2,0)}[-2]$ and $\tau_{(l_1,l_2)} \cong \tau_{(l_1-l_2,0)}[l_2]$, it suffices to consider only the irreducible decomposition of $\tau_{(2,0)} \otimes \tau_{(d,0)}$.

As we know, Clebsh-Gordan theorem tells that

$$\tau_{(2,0)} \otimes \tau_{(d,0)} \cong \tau_{(d,2)} \oplus \tau_{(d+1,1)} \oplus \tau_{(d+2,0)}.$$

Here the factor $\tau_{(d,2)}$ or $\tau_{(d+1,1)}$ is dropped if $d < 2$ or $d+1 < 1$ respectively. What we want to have is an explicit description of the injective K-homomorphism, which is unique up to scalar multiple,

$$\tau_{(d,2)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)}, \tau_{(d+1,1)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)} \text{ and } \tau_{(d+2,2)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)}$$

in terms of the canonical basis

Lemma 2.5

(i) The image $\{f'\binom{d,\ 2}{a}\}_{2\leq a\leq d}$ of the canonical basis $\{f\binom{d,\ 2}{a}\}_{2\leq a\leq d}$ with respect to $\tau_{(d,2)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)}$ is given by

$$f'\binom{d,2}{a} = f\binom{2,\ 0}{2} \otimes f\binom{d,\ 2}{a-2} - 2f\binom{2,\ 0}{1} \otimes f\binom{d,\ 0}{a-1} + f\binom{2,\ 0}{0} \otimes f\binom{d,\ 0}{a}.$$

(ii) The image $\{f'\binom{d+1,\ 1}{a}\}_{1\leq a\leq d+1}$ of the canonical basis $\{f\binom{d+1,\ 1}{a}\}_{1\leq a\leq d+1}$ with respect to $\tau_{(d+1,1)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)}$ is given by

$$\begin{aligned} f'\binom{d+1,\ 1}{a} &= \tfrac{a-1}{d} & f\binom{2,\ 0}{2} \otimes f\binom{d,\ 0}{a-2} \\ &+ \tfrac{d+2-2a}{d} & f\binom{2,\ 0}{1} \otimes f\binom{d,\ 0}{a-1} \\ &+ \tfrac{a-d-1}{d} & f\binom{2,\ 0}{0} \otimes f\binom{d,\ 0}{a}. \end{aligned}$$

(iii) The image $\{f'\binom{d+2,\ 0}{a}\}_{1\leq a\leq d+1}$ of the canonical basis $\{f\binom{d+2,\ 0}{a}\}_{1\leq a\leq d+1}$ with respect to $\tau_{(d+2,0)} \subset \tau_{(2,0)} \otimes \tau_{(d,0)}$ is given by

$$\begin{aligned} f'\binom{d+2,\ 0}{a} &= \tfrac{a(a-1)}{(d+2)(d+1)} & f\binom{2,\ 0}{2} \otimes f\binom{d,\ 0}{a-2} \\ &+ \tfrac{2a(d+2-a)}{(d+2)(d+1)} & f\binom{2,\ 0}{1} \otimes f\binom{d,\ 0}{a-1} \\ &+ \tfrac{(d+2-a)(d+1-a)}{(d+2)(d+1)} & f\binom{2,\ 0}{0} \otimes f\binom{d,\ 0}{a} \end{aligned}$$

for $0 \leq a \leq d+2$.

Proof One can confirm this by direct computation using Proposition (2.1).

3 Constituents in $L^2(K)$

In later sections, the representation spaces of standard representations of G is naturally identified with a subsapce of L^2. Therefore we have to analyse $L^2(K)$, which is a $K \times K$ bimodule by

$$f(x) \mapsto f(k_1^{-1}xk_2) \quad (f \in L^2)K), \ (k_1,k_2) \in K \times K).$$

Let $\hat{K}$ be the unitary dual of K, i.e., the set of unitary equivalence classes of finite dimensional irreducible continuous representations of K. Then the Peter-Weyl theorem tells that there is a decomposition of $K \times K$-bimodules

$$L^2(K) = \widehat{\oplus}_{\tau\in\hat{K}}\tau^* \boxtimes \tau.$$

Here $\tau^* \boxtimes \tau$ is the outer tensor product of τ and its contragradient representation τ^*. We construct each factor $\tau^* \boxtimes \tau$ explicitly in this subsection. .

Let (l_1, l_2) be the dominant weight which is the highest weight of each τ. Then we may rewrite

$$L^2(K) = \widehat{\oplus}_{(l_1.l_2)\in L^+}\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)}.$$

Thus we have to know each factor $\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)}$.

Note here that the representation $\tau_{(1,0)}$ is the tautological represenation $K \to U(2) \subset GL(2, \mathbf{C})$, $\tau_{(1,1)}$ is its determinant represenation. Moreover each $\tau_{(l_1,l_2)} \cong \tau_{(l_2,l_2)} \otimes \tau_{(l_1-l_2,0)}$ is the tensor product of $\tau_{(1,1)}^{\otimes l_2}$ and the symmteric tensor representation $Sym^{(l_1-l_2)}$ of the standard representation.

Let us start with small constituents:

$$\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)} \quad ((l_1, l_2) = (1,0), (0,-1), (2,0), (0,-2)$$

. Let

$$x = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K \mapsto S(x) = \begin{pmatrix} s_{11}(x) & s_{12}(x) \\ s_{21}(x) & s_{22}(x) \end{pmatrix} = A + \sqrt{-1}B \in U(2)$$

be the tautological representation. Then 4 entries $\{s_{ij}(x)\}$ constitute a basis of the space $\tau_{(0,-1)} \boxtimes \tau_{(1,0)}$.

Lemma 3.1 Let $\kappa :\to$ be the differential of $\kappa : U(2) \cong K$. Then the right regular action of $M_2(\mathbf{C}) = \mathfrak{u}(2)_{\mathbf{C}} = \mathfrak{gl}(2, \mathbf{C})$ on $\{s_{ij}\}_{1\le i,j\le 2}$ is given as follows:

$$\begin{array}{lll}
\kappa(e_{11})s_{i1} = s_{i1}, & \kappa(e_{11})s_{i2} = 0 & (i = 1, 2); \\
\kappa(e_{22})s_{i1} = 0, & \kappa(e_{22})s_{i2} = s_{i2} & (i = 1, 2); \\
\kappa(e_{12})s_{i1} = 0, & \kappa(e_{12})s_{12} = s_{i1} & (i = 1, 2); \\
\kappa(e_{21})s_{i1} = s_{i2}, & \kappa(e_{21})s_{i2} = 0 & (i = 1, 2).
\end{array}$$

The contragradient representation $\tau_{(0,-1))}$ of the tautological representation $\tau_{(1,0)}$ is the complex conjugation of $\tau_{(1,0)}$.

Lemma 3.2 (The dual of the tautological representation) For $\{\bar{s}_{ij}\}$ we have the following:

$$\begin{array}{lll}
\kappa(e_{11})\bar{s}_{i1} = -\bar{s}_{i1}, & \kappa(e_{11})\bar{s}_{i2} = 0 & (i = 1, 2); \\
\kappa(e_{22})\bar{s}_{i1} = 0, & \kappa(e_{22})\bar{s}_{i2} = -\bar{s}_{i2} & (i = 1, 2); \\
\kappa(e_{12})\bar{s}_{i1} = -\bar{s}_{i2}, & \kappa(e_{12})\bar{s}_{12} = 0 & (i = 1, 2); \\
\kappa(e_{21})\bar{s}_{i1} = 0, & \kappa(e_{21})\bar{s}_{i2} = -\bar{s}_{i1} & (i = 1, 2).
\end{array}$$

Proofs The above two Lemmata are proved by direct comptation

Now let us discuss the case of general (l_1, l_2):

$$\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)} \text{ in } L^2(K).$$

Notation Let $< \mathbf{s}_i^{(d)}\Delta^m >$ be the subspace of functions in $L^2(K)$ generated by the $(d+1)$ entries of the vector $\mathbf{s}_i^{(d)}\Delta^m = \Delta^m\mathbf{s}_i^{(d)}$ of elementary functions on K.

Proposition 3.3 (i) For each i $(0 \le i \le d)$ the space $< \mathbf{s}_i^{(l_1-l_2)}\Delta^{l_2} >$ is a simple K-module with dominant weight $(l_1, l_2) \in L_K^+$. Moreover the vector $\mathbf{s}_i^{(l_1-l_2)}\Delta^{l_2}$

is a vector of canonical basis in this space.
(ii) The sum

$$\sum_{i=0}^{l_1-l_2} < \mathbf{s}_i^{(d)} \Delta^{l_2} >$$

is a direct sum generating the $\tau_{(l_1,l_2)}$-isotypic component $\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)}$ in the right K-modules $L^2(K)$.
(iii) The value at the identity $e \in K$ of the vector $\mathbf{s}_i^{(l_1-l_2)}\Delta^{l_2}$ is the $(i+1)$-th unit vector ${}^t(0,\cdots,0,1,0,\cdots,0)$.
Proof The statements (i) and (ii) are classical facts. The claim (iii) follows from the fact that $\mathrm{Sym}(e)$ is the identity matrix of size $d+1$ and $\Delta(e) = 1$.

Definition The marking on $< \mathbf{s}_i^{(l_1-l_2)}\Delta^{l_2} >$ in $L^2(K)$ specified by (i) of the above lemma is called the *marking by elementary functions.*

4 The principal series represenations and their K-types

4.1 Definition of the principal series representations

In the begining we have to recall the standard minimal parabolic subgroup P_{min} in $G = Sp(2, \mathbf{R})$. Since G is a split group, this is also a Borel subgroup with split Cartan subgroup $T(A)$ with identity component

$$A = A_{min} = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1} | a_i \in \mathbf{R}_>\}.$$

The unipotent radical of P_{min} is given by $N_{min} = \exp(\mathfrak{n}_{min})$ with

$$\mathfrak{n}_{min} = \oplus_{\alpha \in \{2e_1, 2e_2, e_1-e_2, e_1+e_2\}} \mathfrak{g}_\alpha.$$

Here for the simple roots $\{e_1 - e_2, 2E_2\}$ in the positive root system $\{2e_1, 2e_2, e_1 - e_2, e_1 + e_2\}$, we put

$$\mathfrak{g}_{e_1-e_2} = \mathbf{R}(E_{12} - E_{34}) \text{ and } \mathfrak{g}_{2e_2} = \mathbf{R}E_{24}.$$

To specifty a quasi-character $e^\nu : A \to \mathbf{C}^*$ of A, we have to choose its logarithm $\nu \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C}) = \mathfrak{a} *_{\mathbf{C}}$. Here $\mathfrak{a} = \mathrm{Lie}(A)$. For

$$\log a = \mathrm{diag}(\log a_1, \log a_2, -\log a_1, -\log a_2) = \mathrm{diag}(t_1, t_2, -t_1, -t_2)$$

with $t_i = \log a_i \in \mathbf{R}$ $(i = 1, 2)$, we define the coordinates $(\nu_1, \nu_2) \in \mathbf{C}^2$ of ν by $\nu(\log a) = \nu_1 t_1 + \nu_2 t_2$. Then the half sum ρ of the postive roots

$$\rho = \frac{1}{2}\{2e_1 + 2e_2 + (e_1 - e_2) + (e_1 + e_2)\} = 2e_1 + e_2$$

has the coordinates $(2, 1)$.
We also have to prepare another data, i.e., a character σ of

$$M = Z_A(K) = T(A) \cap K = \{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) | \in \varepsilon_i \mu_2 = \{\pm 1\} \ (i = 1, 2)\} \cong (\mathbf{Z}/(2))^{\oplus 2}.$$

The outer tensor product of the quasi-character $e^{\nu+\rho}$ of A and $\sigma \in \hat{M}$ defines a 1-dimensional representation of the product AM and which is in turn extended to P_{min} via the natural surjection $P_{min} \to AM = P_{min}/N_{min}$.

With these data (σ, ν) given above, the parabolic induction

$$\pi_{(P_{min};\sigma,\nu)} := \mathrm{Ind}_{P_{min}}^{G}(e^{\nu+\rho})$$

is a Hilbert represenation of G by the right quasiregular action on the Hilbert space

$$\begin{aligned} H_\pi := \{ \quad & f : G \to \mathbf{C}, \text{locally integrable} \\ | \quad & f(nmax) = \sigma(m)e^{\nu}(a)f(x), \qquad x \in G, n \in N_{min}, m \in M, \\ & \textstyle\int_K |f(k)|^2 dk < \infty\}. \end{aligned}$$

with inner product

$$(f_1, f_2) = \int_K f_1(k)\bar{f}_2(k)dk.$$

Here dk is the Haar measure of K.

4.2 Canonical basis in the suspace $H_{\pi,K}$ of K-finite vectors

Retricting each function f in H_π to the subgroup K, we have an element in $L^2(K)$. Thus H_π is identified with a subspace of $L^2(K)$.

Proposition 4.1 (i) By the restriction map to K, the Hilbert space $H_{P_{min},\sigma,\nu}$ is identified with a closed subspace of $L^2(K)$:

$$\begin{aligned} L^2_{(M,\sigma)}(K) := \{ \quad & f : K \to \mathbf{C} \text{ in } L^2(K) \quad | \\ & f(mx) = \sigma(m)f(x) \text{ for a.e. } m \in M, x \in K\}. \end{aligned}$$

(ii) Moreover as a unitary representation of K, it has an irreducible decomposition :

$$L^2_{(M,\sigma)}(K) \cong \hat{\oplus}_{\tau\in\hat{K}}\{(\tau^*|M)[\sigma] \boxtimes \tau\}.$$

Here $(\tau^*|M)[\sigma]$ is the $[\sigma]$-isotypic component in $(\tau^*|M)$, which is considered as a trivial K-module here.

Proof The first claim is well-known fact. The second follows from the irreducible decomposition of $L^2(K)$ and the definition of $L^2_{(M,\sigma)}(K)$.

5 The shifts of K-types and contiguous relations

This section is the main result of this paper. We explain our problem conceptullay in the first subsection. After that in the follwoing sections, we compute the necessary data explicitly.

5.1 General setting

The K-finite part $H_{\pi,K}$ of the representation space H_π of the principal series π is also a $\mathfrak{k}$-module. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, in order to describe the action of $\mathfrak{g} = \mathrm{Lie}(G)$ or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Here $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are the holomorphic part and antiholomorpic part of $\mathfrak{p}_{\mathbf{C}}$, respectively.

Given a non-zero K-homomorphism $i : \tau = \tau_{(l_1,l_2)} \subset H_{\tau,K}$ from a simple K-module τ to $H_{\pi,K}$. Then the subspace $\mathfrak{p}_+\mathrm{Im}(i)$ in $H_{\pi,K}$ is the image of the canonical surjection

$$\mathfrak{p}_+ \otimes_{\mathbf{C}} \tau \to \mathfrak{p}_+\mathrm{Im}(i),$$

which is a K-homomorphism with $\mathfrak{p}_+$ endowed with the adjoint action Ad of K. Since $(Ad, \mathfrak{p}_+) \cong \tau_{(2,0)}$, the Clebsh-Gordan theorem implies that there are three injective K-homomorphisms

$$i_a : \tau_a \subset \mathfrak{p}_+ \otimes_{\mathbf{C}} \tau \quad (a = 1, 2, 3)$$

$$\tau_1 \cong \tau_{(l_1+2,l_2)}, \tau_2 \cong \tau_{(l_1+1,l_2+1)}, \tau_3 \cong \tau_{(l_1,l_2+2)}$$

for general (l_1, l_2). Then the composition:

$$\tau_a \subset \mathfrak{p}_+\tau \to\!\!\to \mathfrak{p}_+\mathrm{Im}(i) \to H_{\pi,K}$$

gives an element $j_a \in \mathrm{Hom}_K(\tau_a, H_\pi)$ determined by $i \in \mathrm{Hom}_K(\tau, H_\pi)$. Hence we have 3 $\mathbf{C}$-linear maps

$$\Gamma_a : \mathrm{Hom}_K(\tau, H_\pi) \to \mathrm{Hom}_K(\tau_a, H_\pi).$$

We we replace τ's by simple K-modules τ's endwowed with markings of canonical basis $\{f(\binom{l_1\ l_2}{k})\}$ in the above setting (we may say this is a kind of *rigidification*), then $\mathrm{Hom}_K(\tau, H_\pi)$ etc have induced canonical basis derived from the distiguished set of canonical basis by the entries of the vectors $\mathbf{s}_i^{(l_1-l_2)}\Delta^{l_2}$ $(0 \leq i \leq l_1 - l_2)$ in the τ-isotopic component $H_\pi([\tau])$.

Thus we have settle two problems:

Problem 5.A Describe i_a's in terms of canonical basis.

Problem 5.B Determine the matrix representations of the linear homomorphisms

$$\Gamma_a : \mathrm{Hom}_K(\tau, H_\pi) \to \mathrm{Hom}_K(\tau_a, H_\pi).$$

with respect to the induced basis.

The first problem is setteld in the next subsection of Dirac-Schmid operator, and the second problem is setted after that. As a result, we have infinite number of 'contiguous relation', a kind inifite system of differential-difference relations among vectors in $H_\pi([\tau])$'s and $H_\pi([\tau_a])$.

5.2 The canonical blocks of elementary functions

We define certain a matrix of elementary functions correpondinng to each $\tau_{(l_1,l_2)}$-isotypic component in our P_0 principal series.

Definition 5.1 The follwoing matrices are called *the canonical block of elementary functions* for $\tau_{(l_1.l_2)}$-isotypic component:
When $\pi_{P_0;\nu;\varepsilon}$ is even, we consider the matrices

$$\begin{array}{ll} \mathbf{s}^{(d)}_{[0,\cdots,d]}\Delta^{l_2} & \text{if } ((-1)^{l_1},(-1)^{l_2}) = (\varepsilon_1,\varepsilon_2); \\ \mathbf{s}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2} & \text{if } ((-1)^{l_1},(-1)^{l_2}) = (-\varepsilon_1,-\varepsilon_2). \end{array}$$

When $\pi_{P_0;\nu;\varepsilon}$ is odd, we consider the matrices

$$\begin{array}{ll} \mathbf{s}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2} & \text{if } ((-1)^{l_1},(-1)^{l_2}) = (\varepsilon_1,\varepsilon_2); \\ \mathbf{s}^{(d)}_{[1,\cdots,d]}\Delta^{l_2} & \text{if } ((-1)^{l_1},(-1)^{l_2}) = (-\varepsilon_1,-\varepsilon_2). \end{array}$$

The above definition amounts to fix basis in the image of the evaluation map of the $\tau_{(l_1,l_2)}$-isotypic compnent:

$$\tau_{(l_1,l_2)} \otimes_{\mathbf{C}} \mathrm{Hom}_K(\tau_{(l_1,l_2)}, H_\pi) \to H_\pi,$$

compatible with the tensor product decomposition.

5.3 The chirality operators or Dirac-Schmidt operators

We settle Problem 5.A in this subsection.

5.3.1 Construction of the operators

Firstly we have to introduce a notation to denote various diagonal matrices in the blocks of some matrices.

Notation 5.C Let the letteres a, b, a_1, a_2 be integral variables. Given two integers c_0, c_1 such that $c_0 \leq c_1$, and let $f(a)$ be a (polynomial or rational) function in a in the variable a. Then by

$$\mathbf{diag}_{c_0\leq a\leq c_1}(f(a)),$$

we denote the diagonal matrix of size $c_1 - c_0 + 1$ with the number $f(a)$ at the $((a-c_0)+1, (a-c_0)+1)$-th entry. This notation is used not only to denote a single (square) matrix, but also to denote some blocks of a (non-square) matrices.

Definition 5.2 (i)+ : We define a matrix $\mathcal{C}_{+;(-2)}$ of size $(d-1)\times(d+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\mathcal{C}_{+;(-2)} = L_0 \otimes X_{+,22} - 2L_1 \otimes X_{+,12} + L_2 \otimes X_{+,11}$$

with three constant matrices of size $(d-1)\times(d+1)$

$$\begin{aligned} L_0 &:= (E_{d-1}, \mathbf{0}_{(d-1)\times 1}, \mathbf{0}_{(d-1)\times 1}), \\ L_1 &:= (\mathbf{0}_{(d-1)\times 1}, E_{d-1}, \mathbf{0}_{(d-1)\times 1}), \\ L_2 &:= (\mathbf{0}_{(d-1)\times 1}, \mathbf{0}_{(d-1)\times 1}, E_{d-1}). \end{aligned}$$

(ii)+: Secondly we define a matrix $\mathcal{C}_{+;(0)}$ of size $(d+1)\times(d+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\mathcal{C}_{+;(0)} = -\frac{1}{d}M_0\otimes X_{+,22} - \frac{1}{d}M_1\otimes X_{+,12} + \frac{1}{d}M_2\otimes X_{+,11}$$

with 3 matrices M_0, M_1, M_2 of size $(d+1)\times(d+1)$:

$$\begin{aligned} M_0 &:= \begin{bmatrix} 0_{1,d} & 0 \\ \mathbf{daig}_{1\le a\le d}(a) & 0_{d,1} \end{bmatrix}, \\ M_1 &:= \mathbf{diag}_{0\le a\le d}(d-2a), \\ M_2 &:= \begin{bmatrix} 0_{d,1} & \mathbf{diag}_{1\le \le d}(d+1-a) \\ 0 & 0_{1,d} \end{bmatrix}. \end{aligned}$$

Remark We have a relation

$$(M_0+M_1+M_2)\begin{pmatrix}1\\ \cdot\\ \cdot\\ 1\end{pmatrix} = 0.$$

(iii)+: Thirdly we define 3 $(d+1)\times(d+1)$ diagonal matrices

$$\begin{aligned} N_0^{red} &= \mathbf{diag}_{0\le a\le d}((a+1)(a+2)), \\ N_1^{red} &= \mathbf{diag}_{0\le a\le d}((d+1-a)(a+1)) \end{aligned}$$

and

$$N_2^{red} = \mathbf{diag}_{0\le a\le d}((d+1-a)(d+2-a)).$$

Then we put

$$\mathcal{C}_{+;(+2)} = \frac{1}{(d+1)(d+2)}\{N_0\otimes X_{+,22} + 2N_1\otimes X_{+,12} + N_2\otimes X_{+,11}\}.$$

with

$$N_0 = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ N_0^{red}\end{pmatrix}, \quad N_1 = \begin{pmatrix} \mathbf{0} \\ N_1^{red} \\ \mathbf{0}\end{pmatrix}, \quad \text{and } N_2 = \begin{pmatrix} N_2^{red} \\ \mathbf{0} \\ \mathbf{0}\end{pmatrix}.$$

Remark We have a relation

$$\frac{1}{(d+1)(d+2)}\cdot\{(N_0+2N_1+N_2\}\cdot\begin{pmatrix}1\\ \cdot\\ \cdot\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \cdot\\ \cdot\\ 1\end{pmatrix}$$

Replacing the elements $X_{+,11}, X_{+,12}$ and $X_{+,22}$ by the elements $X_{-,22}, X_{-,12}$ and $X_{-,11}$ in $\mathfrak{p}_-$, we define three matrices with entries in $\mathfrak{p}_-$ by

$$\begin{aligned} \mathcal{C}_{-;(-2)} &:= L_0\otimes X_{-,11} + 2L_1\otimes X_{-,12} + L_2\otimes X_{-,22} \\ \mathcal{C}_{-;(0)} &:= -\tfrac{1}{d}M_0\otimes X_{-,11} + \tfrac{1}{d}M_1\otimes X_{-,12} + \tfrac{1}{d}M_2\otimes X_{-,22} \\ \mathcal{C}_{-;(+2)} &:= \tfrac{1}{(d+1)(d+2)}\{N_0\otimes X_{-,11} - 2N_1\otimes X_{-,12} + N_2\otimes X_{-,22}\}. \end{aligned}$$

5.3.2 Another description of the operators

The above definition of the matrix operators $\mathcal{C}_{*;(*)}$ is a bit difficult to grasp. We give here another row-wise desrition to undertand these matrices and for the later use in the proofs.

Observation

(i) For each a $(1 \le a \le d-1)$, the a-the row of the $(d+1) \times (d-1)$ matrix $\mathcal{C}_{+;(-2)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-1}, X_{+22}, -2X_{+12}, X_{+11}, \underbrace{0,\cdots,0}_{(d-1)-a}).$$

(ii) The a-th row $(1 \le a \le d+1)$ of the $(d+1) \times (d+1)$ matrix $\mathcal{C}_{+;(0)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-2}, -\frac{a-1}{d}X_{+22}, -\frac{d-2a+2}{d}X_{+12}, \frac{d-a+1}{d}X_{+11}, \underbrace{0,\cdots,0}_{(d-a)}).$$

Here the segment $\underbrace{0,\cdots,0}_{-1}$ with the negative length means that it erases the fist subsquent entry or the last proceeding entry of the middle segment of the length three of the row vector given above.

(iii) The a-th row $(1 \le a \le d+3)$ of the $(d+1) \times (d+1)$ matrix $\mathcal{C}_{+;(0)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-3}, \tfrac{(a-2)(a-1)}{(d+1)(d+2)}X_{+22}, \quad 2\tfrac{(d+3-a)(a-1)}{(d+1)(d+2)}X_{+12},$$
$$\tfrac{(d+2-a)(d+3-a)}{(d+1)(d+2)}X_{+11}, \underbrace{0,\cdots,0}_{(d+1-a)}).$$

Here the segment $\underbrace{0,\cdots,0}_{-m}$ with the negative length $-m$ means that it erases the fist m subsquent entries or the last m proceeding entries of the middle segment of the length three of the row vector given above to get a row vector of length $(d+1)$.

(i) For each a $(1 \le a \le d-1)$, the a-the row of the $(d+1) \times (d-1)$ matrix $\mathcal{C}_{-;(-2)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-1}, X_{-11}, 2X_{-12}, X_{-22}, \underbrace{0,\cdots,0}_{(d-1)-a}).$$

(ii) The a-th row $(1 \le a \le d+1)$ of the $(d+1) \times (d+1)$ matrix $\mathcal{C}_{-;(0)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-2}, -\frac{a-1}{d}X_{-11}, \frac{d-2a+2}{d}X_{-12}, \frac{d-a+1}{d}X_{-22}, \underbrace{0,\cdots,0}_{(d-a)}).$$

Here the segment $\underbrace{0,\cdots,0}_{-1}$ with the negative length means that it erases the fist subsquent entry or the last proceeding entry of the middle segment of the length three of the row vector given above.

(iii) The a-th row $(1 \leq a \leq d+3)$ of the $(d+1)\times(d+1)$ matrix $C_{-;(0)}$ is given by

$$(\underbrace{0,\cdots,0}_{a-3}, \frac{(a-2)(a-1)}{(d+1)(d+2)}X_{-11}, \quad -2\frac{(d+3-a)(a-1)}{(d+1)(d+2)}X_{-12},$$
$$\frac{(d+2-a)(d+3-a)}{(d+1)(d+2)}X_{-22}, \underbrace{0,\cdots,0}_{(d+1-a)}).$$

Here the segment $\underbrace{0,\cdots,0}_{-m}$ with the negative length $-m$ means that it erases the fist m subsquent entries or the last m proceeding entries of the middle segment of the length three of the row vector given above to get a row vector of length $(d+1)$.

5.4 Preparation for contiguous relations

Now we can introduce the constant matrices which represent the homomorphisms Γ_a of Problem (5.B).

The case of even principal series

Lemma 5.1.A (even principal series, $\mathfrak{p}_+$-side)

(i) (even,+): We have an equation with some constant matrix $\Gamma_{+;(-2);00}(d,m)$ of size $\frac{d}{2}\times(\frac{d}{2}+1)$

$$C_{+;(-2)}\{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[0,\cdots,d-2]}\Delta^{l_2+2}\}\cdot\Gamma_{+;(-2);00}(d,m).$$

Similarly we have an equation

$$C_{+;(-2)}\{\mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[1,\cdots,d-3]}\Delta^{l_2+2}\}\cdot\Gamma_{+;(-2);11}(d,m).$$

with some constant matrix $\Gamma_{+;(-2);11}$ of size $(\frac{d}{2}-1)\times\frac{d}{2}$.

(ii) (even,+): For some constant matrix $\Gamma_{+;(0);01}(d,m)$ and $\Gamma^{+;(0);10}(d,m)$ of sizes $\frac{d}{2}\times)\frac{d}{2}+1)$ and $(\frac{d}{2}+1)\times\frac{d}{2}$ respectively, we have

$$C_{+;(0)}\{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{d}_{[1,\cdots,d-1]}\Delta^{l_2+1}\}\Gamma_{+;(0);01}(d,m), \quad \text{if } (-1)^d = \mathrm{sgn}(\sigma),$$

and

$$C_{+;(0)}\{\mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2+1}\}\Gamma_{+;(0);10}(d,m), \quad \text{if } (-1) \neq \mathrm{sgn}(\sigma).$$

(iii) (even,+): For some constant matrix $\Gamma_{+;(+2);00}(d,m)$ of size $(\frac{d}{2}+2)\times(\frac{d}{2}+1)$, we have

$$C_{+;(+2)}\ \mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[0,\cdots,d+2]}\Delta^{l_2}\}\Gamma_{+;(+2);00}(d,m).$$

Moreover for some constant matrix $\Gamma_{+;(+2);11}(d,m)$ of size $(\frac{d}{2}+1)\times\frac{d}{2}$, we have

$$\mathcal{C}_{+;(+2)}\ \mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[1,\cdots,d+1]}\Delta^{l_2}\}\Gamma_{+;(+2);11}(d,m).$$

(even case, $\mathfrak{p}_-$-side)
(d even, $\mathfrak{p}_-$)
(i) (even,-): With some contant matrix $\Gamma_{-;(-2);00}(d,m)$ of size $\frac{1}{2}d\times\frac{1}{2}(d+2)$, we have

$$\mathcal{C}^-_{-2}\{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[0,\cdots,d-2]}\Delta^{l_2-2}\}\cdot\Gamma_{-;(-2)}(d,m).$$

and with some contant matrix $\Gamma_{-;(-2);11}(d,m)$ of size $\frac{1}{2}(d-2)\times\frac{1}{2}d$, we have

$$\mathcal{C}^-_{-2}\{\mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[1,\cdots,d-3]}\Delta^{l_2-2}\}\cdot\Gamma_{-;(-2);11}(d,m).$$

(ii) (even,-): For some constant matrix $\Gamma_{-;(0);00}(d,m)$ and $\Gamma_{-;(0);01}(d,m)$ of sizes $\frac{1}{2}d\times\frac{1}{2}(d+2)$ and $\frac{1}{2}(d+2)\times\frac{1}{2}d$ respectively, we have

$$\mathcal{C}^-_0\{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{d}_{[1,\cdots,d-1]}\Delta^{l_2-1}\}\Gamma_{-;(0);00}(d,m), \quad \text{if } (-1)^d=\mathrm{sgn}(\sigma(\varepsilon_i)),$$

and

$$\mathcal{C}^-_0\{\mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2-1}\}\Gamma_{-;(0);01}(d,m), \quad \text{if } (-1)\neq\mathrm{sgn}(\sigma(\varepsilon_i)).$$

(iii) (even,-): For some constant matrix $\Gamma_{-;(+2);00}(d,m)$ of size $\frac{1}{2}(d+4)\times\frac{1}{2}(d+2)$, we have

$$\mathcal{C}^-_{+2}\ \mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[0,\cdots,d+2]}\Delta^{l_2}\}\Gamma_{-;(+2);00}(d,m).$$

and for some constant matrix $\Gamma_{-;(+2);11}(d,m)$ of size $\frac{1}{2}(d+2)\times\frac{1}{2}d$, we have

$$\mathcal{C}^-_{+2}\ \mathbf{S}^{(d)}_{[0,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[0,\cdots,d+2]}\Delta^{l_2}\}\Gamma_{-;(+2);11}(d,m).$$

Remark If the size of the matrix consider above is impossible, say, if $d=$ then $\frac{d}{2}-1=-1$, the correponding matrices do not exist and the equation also do not exist.

We have similar formulation for the case of odd principal series.

the case of odd principal series

Lemma 5.1.B (odd principal series, $\mathfrak{p}_+$-side)
(i) (odd,+): We have an equation with some constant matrix $\Gamma_{+;(-2);01}(d,m)$ of size $\frac{1}{2}(d-1)\times\frac{1}{2}(d+1)$

$$\mathcal{C}_{+;(-2)}\{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[0,\cdots,d-3]}\Delta^{l_2+2}\}\cdot\Gamma_{+;(-2);01}(d,m).$$

Similarly we have an equation

$$\mathcal{C}_{+;(-2)}\{\mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[1,\cdots,d-2]}\Delta^{l_2+2}\}\cdot\Gamma_{+;(-2);10}(d,m).$$

with some constant matrix $\Gamma_{+;(-2);10}$ of size $\frac{1}{2}(d+1)\times\frac{1}{2}(d+1)$.
(ii),(odd,+): For some constant matrix $\Gamma_{+;(0);01}(d,m)$ and $\Gamma^{+;(0);10}(d,m)$ of size $\frac{1}{2}(d+1)\times\frac{1}{2}(d+1)$ respectively, we have

$$\mathcal{C}_{+;(0)}\{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{d}_{[1,\cdots,d]}\Delta^{l_2+1}\}\Gamma_{+;(0);01}(d,m), \quad \text{if } (-1)^d=\mathrm{sgn}(\sigma),$$

and

$$\mathcal{C}_{+;(0)}\{\mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2+1}\}\Gamma_{+;(0);10}(d,m), \quad \text{if } (-1) \neq \mathrm{sgn}(\sigma).$$

(iii),(odd,+): For some constant matrix $\Gamma_{+;(+2);01}(d,m)$ of size $\frac{1}{2}(d+3) \times \frac{(}{d}+1)$, we have

$$\mathcal{C}_{+;(+2)}\ \mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[0,\cdots,d+1]}\Delta^{l_2}\}\Gamma_{+;(+2);01}(d,m).$$

Moreover for some constant matrix $\Gamma_{+;(+2);10}(d,m)$ of size $\frac{1}{2}(d+3) \times \frac{1}{2}(d+1)$, we have

$$\mathcal{C}_{+;(+2)}\ \mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[1,\cdots,d+2]}\Delta^{l_2}\}\Gamma_{+;(+2);10}(d,m).$$

(the case of odd principal series, i.e., d even, $\mathfrak{p}_-$-side)
(i) (odd,-): With some contant matrix $\Gamma_{-;(-2);01}(d,m)$ of size $\frac{1}{2}(d-1) \times \frac{1}{2}(d+1)$, we have

$$\mathcal{C}^-_{-2}\{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[0,\cdots,d-3]}\Delta^{l_2-2}\} \cdot \Gamma_{-;(-2)}(d,m).$$

and with some contant matrix $\Gamma_{-;(-2);10}(d,m)$ of size $\frac{1}{2}(d-1) \times \frac{1}{2}(d+1)$, we have

$$\mathcal{C}^-_{-2}\{\mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d-2)}_{[1,\cdots,d-2]}\Delta^{l_2-2}\} \cdot \Gamma_{-;(-2);10}(d,m).$$

(ii) (odd,-): For some constant matrix $\Gamma_{-;(0);01}(d,m)$ and $\Gamma_{-;(0);10}(d,m)$ of size $\frac{1}{2}(d+1) \times \frac{1}{2}(d+1)$ and $\frac{1}{2}(d+1) \times \frac{1}{2}(d+1)$ respectively, we have

$$\mathcal{C}^-_0\{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{d}_{[1,\cdots,d]}\Delta^{l_2-1}\}\Gamma_{-;(0);01}(d,m), \quad \text{if } (-1)^d = \mathrm{sgn}(\sigma(\varepsilon_i)),$$

and

$$\mathcal{C}^-_0\{\mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2-1}\}\Gamma_{-;(0);10}(d,m), \quad \text{if } (-1) \neq \mathrm{sgn}(\sigma(\varepsilon_i)).$$

(iii) (odd,-): For some constant matrix $\Gamma_{-;(+2);01}(d,m)$ of size $\frac{1}{2}(d+3) \times \frac{1}{2}(d+1)$, we have

$$\mathcal{C}^-_{+2}\ \mathbf{S}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[0,\cdots,d+1]}\Delta^{l_2}\}\Gamma_{-;(+2);01}(d,m).$$

and for some constant matrix $\Gamma_{-;(+2);10}(d,m)$ of size $\frac{1}{2}(d+3) \times \frac{1}{2}(d+1)$, we have

$$\mathcal{C}^-_{+2}\ \mathbf{S}^{(d)}_{[1,\cdots,d]}\Delta^{l_2}\} = \{\mathbf{S}^{(d+2)}_{[1,\cdots,d+2]}\Delta^{l_2}\}\Gamma_{-;(+2);10}(d,m).$$

5.5 Contiguous equations: Determination of intertwining constants

Now we can decide the homomorphism Γ_a of Problem 5.B. We have to compute the matrices $\Gamma_{**;(*)}(d,m)$ of intertwining constants explicitly. These are basically generalized di-diagonal matrices, which are expressed as a sum of two blocks of square diagonal matrices; the sizes of two blocks are the same or different up to ± 1. Each diagonal square blocks of size $q-p+1$ is written in the forms: $\mathbf{diag}_{p\le a\le q}(l(a))$, where the diagonal entries $l(a)$ are linear functions in the variable integer a.

This is the main result of this paper.

The case of even principal series

Theorem 5.2.A The matrices of constants of the 12 equalities in Lemma (5.1.A) are given as follows:

($\mathfrak{p}_+$-side)

(i) (even, +):

$$\Gamma_{+;(-2);00}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-2)/2}(\nu_2+\rho_2+m+2a), \mathbf{0}\right] + \left[\mathbf{0}, \mathbf{diag}_{0\le a\le (d-2)/2}(\nu_1+\rho_1+m-d+2a)\right].$$

$$\Gamma_{+;(-2);11}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-4)/2}(\nu_2+\rho_2+m+2a+1), \mathbf{0}\right] + \left[\mathbf{0}, \mathbf{diag}_{0\le a\le (d-4)/2}(\nu_1+\rho_1+m-d+2a+1)\right].$$

(ii) (even,+):

$$\Gamma_{+;(0);01}(d,m) = [\mathbf{diag}_{1\le a\le d/2}(-\frac{2a-1}{d}\{(\nu_2+\rho_2)+m+2(a-1)\}), \mathbf{0}] + [\mathbf{0}, \mathbf{diag}_{1\le a\le d/2}(\frac{d+1-2a}{d}\{(\nu_1+\rho_1)+m+2(a-1)\})]$$

$$\Gamma_{+;(0);10}(d,m) = \begin{bmatrix}\mathbf{diag}_{1\le a\le d/2-1}(\frac{d+2-2a}{d}\{(\nu_1+\rho_1)+m+2a-3\}) \\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \mathbf{diag}_{1\le a\le d/2-1}(-\frac{2a}{d}\{(\nu_2+\rho_2)+m+2a-1\})\end{bmatrix}$$

(iii) (even,+):

$$\Gamma_{+;(+2);00}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le d/2}(\frac{(d+1-2a)(d+2-2a)}{(d+1)(d+2)}(\nu_1+\rho_1+m+d+2a)) \\ \mathbf{0}_{1\times(d/2+1)}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{1\times(d/2+1)} \\ \mathbf{diag}_{0\le a\le d/2}(\frac{2a+1)(2a+2)}{(d+1)(d+2)}(\nu_2+\rho_2+m+2a))\end{bmatrix}$$

$$\Gamma_{+;(+2);11}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-2}{2}}(\frac{(d-2a)(d+1-2a)}{(d+1)(d+2)}(\nu_1+\rho_1+m+d+2a+1)) \\ \mathbf{0}_{1\times(d/2)}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{1\times(d/2)} \\ \mathbf{diag}_{0\le a\le \frac{d-2}{2}}(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(\nu_2+\rho_2+m+2a+1))\end{bmatrix}$$

($\mathfrak{p}_-$-side)
(i) (even,-): We have

$$\Gamma_{-;(-2);00}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-2)/2}((\nu_1+\rho_1)-(m+d+2a+2)),\mathbf{0}\right] + \left[\mathbf{0},\mathbf{diag}_{0\le a\le (d-2)/2}((\nu_2+\rho_2)-(m+2a+2))\right].$$

$$\Gamma_{-;(-2);11}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-4)/2}((\nu_1+\rho_1)-(m+d+2a+3)),\mathbf{0}\right] + \left[\mathbf{0},\mathbf{diag}_{0\le a\le (d-4)/2}((\nu_2+\rho_2)-(m+2a+3))\right].$$

(ii) (even,-): We have

$$\Gamma_{-;(0);01}(d,m) = \left[\mathbf{diag}_{1\le a\le d/2}(\tfrac{2a-1}{d}\{(\nu_1+\rho_1)-m-2a\}),\mathbf{0}_{d/2\pm *\times 1}\right] + \left[\mathbf{0}_{d/2\pm *\times 1},\mathbf{diag}_{1\le a\le d/2}(\tfrac{d+1-2a}{d}\{(\nu_2+\rho_2)-m-2a\})\right]]$$

$$\Gamma_{-;(0);10}(d,m) = \begin{bmatrix}\mathbf{diag}_{1\le a\le d/2}(\frac{d+2-2a}{d}\{\nu_2+\rho_2)-(m+2a-1)\}) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ \mathbf{diag}_{1\le a\le d/2}(-\frac{2a}{d}\{(\nu_1+\rho_1)-(m+2a+1)\})\end{bmatrix}.$$

(iii) (even,-):

$$\Gamma_{-;(+2);00}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le d/2}(\frac{(d+1-2a)(d+2-2a)}{(d+1)(d+2)}\{(\nu_2+\rho_2)-(d+m)+d-2a\}) \\ \mathbf{0}_{1\times(d/2+1)}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{1\times(d/2+1)}, \\ \mathbf{diag}_{0\le a\le d/2}(\frac{(2a+1)(2a+2)}{(d+1)(d+2)}\{(\nu_1+\rho_1)-(d+m)+2d-2a\})\end{bmatrix}$$

$$\Gamma_{-;(+2);11}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-2}{2}}(\frac{(d-2a)(d+1-2a)}{(d+1)(d+2)}\{(\nu_2+\rho_2)-(m+2a+1)\}) \\ \mathbf{0}_{1\times(d/2)}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{1\times(d/2)}, \\ \mathbf{diag}_{0\le a\le \frac{d-2}{2}}(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}\{(\nu_1+\rho_1)-(m+2a+1)+d\})\end{bmatrix}$$

The case of odd principal series

Theorem 5.2.B The matrices of constants of the 12 equalities in Lemma (5.1.B) are given as follows:

($\mathfrak{p}_+$-side)

(i) (odd,+):

$$\Gamma_{+;(-2);01}(d,m) = \left[(\mathbf{diag}_{0\le a\le (d-3)/2}(\nu_2+\rho_2+m+2a),\mathbf{0})\right] + \left[\mathbf{0},\mathbf{diag}_{0\le a\le (d-3)/2}(\nu_1+\rho_1+m-d+2a)\right].$$

$$\Gamma_{+;(-2);10}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-3)/2}(\nu_2+\rho_2+m+1+2a),\mathbf{0}\right] + \left[\mathbf{0},\mathbf{diag}_{0\le a\le (d-3)/2}(\nu_1+\rho_1+m-d+1+2a)\right].$$

(ii) (odd,+):

$$\Gamma_{+;(0);01}(d,m) = \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(-\frac{2a+1}{d}(\nu_2+\rho_2+m+2a)) + \begin{bmatrix}\mathbf{0}, & \mathbf{diag}_{0\le a\le \frac{d-3}{2}}(\frac{d-2a-1}{d}(\nu_1+\rho_1+m+2a))\\ 0, & \mathbf{0}\end{bmatrix}$$

$$\Gamma_{+;(0);10}(d,m) = \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{d-2a}{d}(\nu_1+\rho_1+m+2a-1)) + \begin{bmatrix}\mathbf{0}, & 0\\ \mathbf{diag}_{1\le a\le \frac{d-1}{2}}(-\frac{2a}{d}(\nu_2+\rho_2+m+2a-1)), & \mathbf{0}\end{bmatrix}$$

(iii) (odd,+) :

$$\Gamma_{+;(+2);01}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(d+1-2a)(d+2-2a)}{d+1)(d+2)}(\nu_1+\rho_1+m+d+2a))\\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(2a+1)(2a+2)}{(d+1)(d+2)}(\nu_2+\rho_2+m+2a))\end{bmatrix}$$

$$\Gamma_{+;(+2);10}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(d-2a)(d+1-2a)}{d+1)(d+2)}(\nu_1+\rho_1+m+d+2a+1))\\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(\nu_2+\rho_2+m+2a+1))\end{bmatrix}$$

(odd, $\mathfrak{p}_-$-side)
(i) (odd,-):

$$\Gamma_{-;(-2);01}(d,m) = \left[(\mathbf{diag}_{0\le a\le (d-3)/2}\{(\nu_1+\rho_1)-(m+d+2a+2)\},\quad \mathbf{0})\right] + \left[\mathbf{0},\quad \mathbf{diag}_{0\le a\le (d-3)/2}\{(\nu_2+\rho_2)-(m+2a+2)\}\right].$$

$$\Gamma_{-;(-2);10}(d,m) = \left[\mathbf{diag}_{0\le a\le (d-3)/2}\{(\nu_1+\rho_1)-(m+d+2a+3)\},\quad \mathbf{0}\right] + \left[\mathbf{0},\quad \mathbf{diag}_{0\le a\le (d-3)/2}\{(\nu_2+\rho_2)-(m+2a+3)\}\right].$$

(ii) (odd, -) : We have

$$\Gamma_{-;(0);01}(d,m) = \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(-\frac{2a+1}{d}\{(\nu_1+\rho_1)-(m+2a+2)\} + \begin{bmatrix}\mathbf{0}, & \mathbf{diag}_{0\le a\le \frac{d-3}{2}}(\frac{d-2a-1}{d}\{(\nu_2+\rho_2)-(m+2a;2)\} \\ 0, & \mathbf{0}\end{bmatrix}$$

$$\Gamma_{-;(0);10}(d,m) = \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{d-2a}{d}\{(\nu_2+\rho_2)-(m+2a+1)\}) + \begin{bmatrix} \mathbf{0}, & 0 \\ \mathbf{diag}_{1\le a\le \frac{d-1}{2}}(-\frac{2a;2}{d}\{(\nu_1+\rho_1)-(m+2a+3)\}) & \mathbf{0}\end{bmatrix}$$

(iii) (odd,-) :

$$\Gamma_{-;(+2);01}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(d+1-2a)(d+2-2a)}{d+1)(d+2)}(\nu_2+\rho_2-m-2a)) \\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(2a+1)(2a+2)}{(d+1)(d+2)}(\nu_1+\rho_1-m+d-2a))\end{bmatrix}$$

$$\Gamma_{+;(+2);10}(d,m) = \begin{bmatrix}\mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(d-2a)(d+1-2a)}{d+1)(d+2)}(\nu_2+\rho_2-m-2a-1)) \\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \mathbf{diag}_{0\le a\le \frac{d-1}{2}}(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(\nu_1+\rho_1-m+d-2a-1))\end{bmatrix}$$

Proof (of the contiguous relations) We have to determine the constant matrices $\Gamma_{\pm;(*);**}$ of the contiguous equations. For this purpose, it suffices to evaluate the both sides of the equations in question at $e \in K$.

We can compute the matrices $\Gamma_{+;(-2),00}(d,m)$ and $\Gamma_{+;(-2),11}(d,m)$ in the same time by the following merging procedure.

Merging: The matrices $\mathbf{S}^{(d)}_{[0,\cdots,d}\Delta^{l_2}$ and $\mathbf{S}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2}$ are derived from a single matrix of elemnatry functions $\mathbf{S}^{(d)}_{[0,1,\cdots,d-1.d]}$ of size $(d+1)\times(d+1)$; the former is collection of column vectos at odd row indices and the latter of column vectors with even row indices, respectively. In the same way, we can consider that the two matrices $\Gamma_{+;(-2),00}(d,m)$ and $\Gamma_{+;(-2),11}(d,m)$ are derived from a single "merged" matrix $\tilde{\Gamma}_{+;(-2)}(d,m)$.

Therefore the real task of the proof is to compute the left sides

$$\{\mathcal{C}_{+;(-2)}\mathbf{S}^{(d)}_{[0,\cdots,d]}\}(e)$$

which are equal to $\tilde{\Gamma}_{+;(-2)}(d,m)$. To compute each colum of this matrix, we have to compute the colum vector

$$\{\mathcal{C}_{\pm;(*)}\mathbf{s}^{(d)}_i\Delta^m\}(e)$$

for each i, utilizing Iwasawa decomposition of $X_{\pm;i,j}$. Each row vector of $\mathcal{C}$ have entries which are constant multiple of $X_{\pm;i,j}$, and each entries of the vector $\mathbf{s}^{(d)}_i$ is of the form $\sum_\alpha \mu_\alpha$ with α runs over the shuffles of certain type, and aech μ_α a monimial in s_{ij} associated with some shuffle α.

Firstly we prepare the computation of the values

$$\{E_{2e_i}(\mu_\alpha\Delta^m\}(e), \{E_{e_1\pm e_2}(\mu_\alpha\Delta^m\}(e), \{H_i(\mu_\alpha\Delta^m)\}(e),$$
$$\text{and } \{\kappa(e_{ij})(\mu_\alpha\Delta^m)\}(e) \quad (i=1,2).$$

Claim 1

(i) $\{E_{2e_i}(\mu_\alpha\Delta^m)\}(e) = \{E_{e_1\pm e_2}(\mu_\alpha\Delta^m)\}(e) = 0.$

(ii) $\{H_i(\mu_\alpha\Delta^m)\}(e) = (\nu_i+\rho_i)(\mu_\alpha\Delta^m)(e)$ with

$$(\mu_\alpha\Delta^m)(e) = \begin{cases} 1, & \text{if } w_{12}() + w_{21}() = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(iii)

$$\kappa(e_{ii})(\mu_\alpha\Delta^m)(e) = \begin{cases} m + w_{ii}(\mu_\alpha), & \text{if } w_{12}(\mu_\alpha) + w_{21}(\mu_\alpha) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(iv)

$$\kappa(e_{21})(\mu_\alpha\Delta^m)(e) = \begin{cases} 1, & \text{if } w_{21}(\mu_\alpha) = 1 \text{ and } w_{12}(\mu_\alpha) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\kappa(e_{12})(\mu_\alpha\Delta^m)(e) = \begin{cases} 1, & \text{if } w_{12}(\mu_\alpha) = 1 \text{ and } w_{21}(\mu_\alpha) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here for each $(i,j)\in\{(1,1),(1,2),(2,1),(2,2)\}$, we set

$$w_{ij}(\mu_\alpha) = \text{the number of the factor } s_{ij} \text{ in the monimial } \mu_\alpha.$$

Proof of Claim 1 Direct compuation. □

Claim 2 Here is the formula of the $(a+1)$-th entry of the vector $\Delta^m \mathbf{s}_i^{(d)}$:

$$\Delta^m \sum_{b=0}^{a} \binom{d-a}{i-b}\binom{a}{b} s_{11}^{(d-a)-(i-b)} s_{21}^{i-b} s_{12}^{a-b} s_{22}^{b}.$$

Proof of Claim 2 This follows almost immediately from the definition of the shuffle product. □

After the above preparation, let us start the substantial computation. The (a_1+1)-th row vector of $C_{+;(-2)}$ is

$$(\underbrace{0,\cdots,0}_{a_1}, X_{+22}.-2X_{+12}, X_{+,11}, \underbrace{0,\cdots,0}_{d-a_1}).$$

The value at $e\in K$ of the inner product of this row vector of operators and the colum vector $\Delta^m \mathbf{s}_i^{(d)}$ of elementary functions is the sum

$$\{X_{+,22}s_{a_1,i}^{(d)}\Delta^m\}(e) - 2\{X_{+,12}s_{a_1+1,i}^{(d)}\Delta^m\}(e) + \{X_{+,11}s_{a_1+2,i}^{(d)}\Delta^m\}(e).$$

Claim 3 (i) We have the following:

$$\begin{aligned}
\{X_{+22}s_{a_1,i}^{(d)}\Delta^m\}(e) &= (\nu_2+\rho_2+m+i)\delta_{a_1,i}\\
\{X_{+11}s_{a_1+2,i}^{(d)}\Delta^m\}(e) &= (\nu_1+\rho_1+m+d-i)\delta_{i,a_1+2}\\
-2\{X_{+,12}s_{a_1+1,i}^{(d)}\Delta^m\}(e) &= -2\{d-(a_1+1)\}\delta_{a_1+1,i-1}
\end{aligned}$$

Proof of Claim 3 By Iwasawa decompostion, Claim 1 (ii), (iii) and Claim 2, the left side of the first formula reads

$$\sum_{b=0}^{a_1}\binom{d-a_1}{i-}\binom{a_1}{b}(\nu_2+\rho_2+m+b)\cdot\{s_{11}^{(d-a)-(i-b)}s_{21}^{(i-b)}s_{12}^{(a_1-b)}s_{22}^{b}\Delta^m\}(e)$$

Since

$$\{s_{11}^{(d-a)-(i-b)}s_{21}^{(i-b)}s_{12}^{(a_1-b)}s_{22}^{b}\Delta^m\}(e) = \delta_{i,b}\delta_{a_1,b} = \delta_{a_1,i}$$

with Kronecker delta's $\delta_{*,*}$, this last formula gives the value $(\nu_2+\rho_2+m+i)\delta_{a_1,i}$.

Similarly the left side of the second formula reads

$$\sum_{b=0}^{a_1+2}\{\nu_1+\rho_1+m+d-a_1-2-(i-b)\}\binom{d-(a_1+1)}{i-b}\binom{a_1+2}{b}\delta_{i,b}\delta_{a_1+2,b}$$
$$= (\nu_1+\rho_1+m+d-i)\delta_{i,a_1+2}.$$

Finally the left side of the last formulae reads

$$\begin{aligned}
-2\kappa(e_{21})(\mathbf{s}_{a_1+}^{(d)}\Delta^m)(e) &= \textstyle\sum_{b=0}^{a_1+1}\binom{d-a_1}{i-b}\binom{a}{b}\delta_{i-b,1}\delta_{a_1+1,b}\\
&= -2\{d-(a_1+1)\}\delta_{i,(a_1+)+}.
\end{aligned}$$

□

Thus

$$\mathfrak{c}_{a_1+1}\{\mathbf{s}_i^{(d)}\}(e) = \begin{cases} 0, & \text{unless either } a_1 = i \text{ or } a_1 = i-2. \\ \nu_2 + \rho_2 + m + i, & \text{if } a_1 = i. \\ \nu_1 + \rho_1 + m - d + i + 2, & \text{if } a_1 = i-2. \end{cases}$$

This implies the following.

Claim 4 (i)

$$\{\mathcal{C}_{+;(-2)}\mathbf{s}_i^{(d)}\Delta^m\}(e) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \nu_1 + \rho_1 + m - d + i + 2 \\ 0 \\ \nu_2 + \rho_2 + m + i \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The remaining cases are treated similarly. □

6 Examples of contiguous relations and of thier composites

Here are some examples of the contiguous relations at the peripheral K-types.

6.1 the case of even principal series:the $\mathfrak{p}_+$-side

We assume that $sgn(\sigma) = (+1,+1)$ or $= (-1,-1)$ for $\sigma \in \hat{M}$. Then there is a unique injective K-homomorphism $\tau_{(m,m)} \subset \pi_{P_{min};\sigma,\nu}$ when $(-1)^m = sgn(\sigma)$. The generator of $\tau_{(m,m)}$ is given by Δ^m. The multiplicity $[\pi, \tau_{(m+2,m)}]$ equals to 2. The $\tau_{(m+2,m)}$-isotypic component $\pi([\tau_{(m+2,m)}])$ is realized by the subspace generated by two normalized set of basis $\{s_{11}^2\Delta^m, s_{11}s_{12}\Delta^m, s_{12}^2\Delta^m\}$ and $\{s_{21}^2\Delta^m, s_{21}s_{22}\Delta^m, s_{22}^2\Delta^m\}$. Therefore, by Dirac-Schmid operator

$$pr_{(2,0)} \cdot \nabla_+ : \pi([\tau_{(m,m)}]) \to \pi([\tau_{(m+2,m)}])$$

the normalized set of basis

$$\{\, X_{+,11}(\Delta^m),\ X_{+,12}(\Delta^m),\ X_{+,22}(\Delta^m)\}$$

is mapped to a linear combination of these two sets.

Formula 6.1 *(horizontal to the right) We have*

$$\begin{pmatrix} X_{+,11}(\Delta^m) \\ X_{+,12}(\Delta^m) \\ X_{+,22}(\Delta^m) \end{pmatrix} = \begin{pmatrix} s_{11}^2\Delta^m & s_{21}^2\Delta^m \\ s_{11}s_{12}\Delta^m & s_{21}s_{22}\Delta^m \\ s_{12}^2\Delta^m & s_{22}^2\Delta^m \end{pmatrix} \cdot \begin{pmatrix} \nu_1+\rho_1+m \\ \nu_2+\rho_2+m \end{pmatrix}$$

Formula 6.2 *(vertical, up) We have*

$$(X_{+,22},\quad -2X_{+,12},\quad X_{+,11}) \cdot \begin{pmatrix} s_{11}^2\Delta^m & s_{21}^2\Delta^m \\ s_{11}s_{12}\Delta^m & s_{21}s_{22}\Delta^m \\ s_{12}^2\Delta^m & s_{22}^2\Delta^m \end{pmatrix}$$

$$= \Delta^{m+2}\left(\nu_2+\rho_2+m,\quad \nu_1+\rho_1+m-2\right)$$

Formula 6.3 *(Composite of the above two operators) We have*

$$\{X_{+,11}X_{+,22} - X_{+,12}^2\}(\Delta^m) = (\nu_1+m+1)(\nu_2+m+1)\Delta^{m+2}.$$

6.2 the case of even principal series:the $\mathfrak{p}_-$-side

The $\tau_{(m,m-2)}$-isotypic component $\pi([\tau_{(m,m-2)}])$, i.e., the image of the evaluation map:

$$\tau_{(m,m-2)} \otimes \mathrm{Hom}_K(\tau_{(m,m-2)}, \pi) \to \pi$$

is realized by the subsapce generated by two normalized set of basis

$$\{\bar{s}_{11}^2\Delta^m, \bar{s}_{11}\bar{s}_{12}\Delta^m, \bar{s}_{12}^2\Delta^m\} \text{ and } \{\bar{s}_{21}^2\Delta^m, \bar{s}_{21}\bar{s}_{22}\Delta^m, \bar{s}_{22}^2\Delta^m\}.$$

Note here the relations:

$$\begin{pmatrix} \bar{s}_{12}^2 \\ -\bar{s}_{11}\bar{s}_{12} \\ \bar{s}_{11}^2 \end{pmatrix} \Delta^m = \begin{pmatrix} s_{21}^2 \\ s_{21}s_{22} \\ s_{22}^2 \end{pmatrix} \Delta^{m-2}, \quad \begin{pmatrix} \bar{s}_{22}^2 \\ -\bar{s}_{21}\bar{s}_{22} \\ \bar{s}_{21}^2 \end{pmatrix} \Delta^m = \begin{pmatrix} s_{11}^2 \\ s_{11}s_{12} \\ s_{12}^2 \end{pmatrix} \Delta^{m-2}$$

Formula 6.4 *(vertical, down) We have*

$$\begin{pmatrix} X_{-,22}(\Delta^m) \\ X_{-,12}(\Delta^m) \\ X_{-,11}(\Delta^m) \end{pmatrix} = (\nu_1 + \rho_1 - m) \begin{pmatrix} \bar{s}_{12}^2\Delta^m \\ -\bar{s}_{11}\bar{s}_{12}\Delta^m \\ \bar{s}_{11}^2\Delta^m \end{pmatrix} + (\nu_2 + \rho_2 - m) \begin{pmatrix} \bar{s}_{22}^2\Delta^m \\ -\bar{s}_{21}\bar{s}_{22}\Delta^m \\ \bar{s}_{21}^2\Delta^m \end{pmatrix}$$

Proof This is quite similar to the case of $\mathfrak{p}_-$.

Formula 6.5 *(horizontal to the left) We have*

$$X_{-,11}(\bar{s}_{12}^2\Delta^m) - 2X_{-,12}(\bar{s}_{11}\bar{s}_{12}\Delta^m) + X_{-,22}(\bar{s}_{11}^2\Delta^m) = (\nu_2 + \rho_2 - m)\Delta^{m-2};$$

$$X_{-,11}(\bar{s}_{22}^2\Delta^m) - 2X_{-,12}(\bar{s}_{21}\bar{s}_{22}\Delta^m) + X_{-,22}(\bar{s}_{21}^2\Delta^m) = (\nu_1 + \rho_1 - m - 2)\Delta^{m-2};$$

Formula 6.6 *(Composite of the above two operators) We have*

$$\begin{aligned} \{X_{-,11}X_{-,22} - X_{-,12}^2\}(\Delta^m) &= (\nu_1 + \rho_1 - m - 1)(\nu_2 + \rho_2 - m)\Delta^{m-2} \\ &= (\nu_1 - m + 1)(\nu_2 - m + 1)\Delta^{m-2}. \end{aligned}$$

6.3 the case of odd principal series:the $\mathfrak{p}_+$-side

We investigate the shift operator:

$$pr_{(1,1)} \cdot \nabla_+ : \pi([\tau_{(m+1,m)}]) \to \pi([\tau_{(m+2,m+1)}]).$$

A set of normaized basis in $\pi([\tau_{(m+1,m)})$ is given by

$$\text{either } \{s_{11}\Delta^m, s_{12}\Delta^m\}, \text{ or } \{s_{21}\Delta^m, s_{22}\Delta^m\}$$

depending on the product of the parity of σ and m. Similarly for $\pi([\tau_{(m+2,m+1)}])$, we can take a set of normalized basis by

$$\text{either } \{s_{21}\Delta^{m+1}, s_{22}\Delta^{m+1}\}, \text{ or } \{s_{11}\Delta^{m+1}, s_{12}\Delta^{m+1}\}.$$

Formula 6.7 *(slant up) We have*

(i): $$\begin{pmatrix} -X_{+,12}(s_{11}\Delta^m) + X_{+,11}(s_{12}\Delta^m) \\ -X_{+,22}(s_{11}\Delta^m) + X_{+,12}(s_{12}\Delta^m) \end{pmatrix} = -(\nu_2 + \rho_2 + m) \begin{pmatrix} s_{21}\Delta^{m+1} \\ s_{22}\Delta^{m+1} \end{pmatrix}$$

(ii): $$\begin{pmatrix} -X_{+,12}(s_{21}\Delta^m) + X_{+,11}(s_{22}\Delta^m) \\ -X_{+,22}(s_{21}\Delta^m) + X_{+,12}(s_{22}\Delta^m) \end{pmatrix} = (\nu_1 + \rho_1 + m - 1) \begin{pmatrix} s_{11}\Delta^{m+1} \\ s_{12}\Delta^{m+1} \end{pmatrix}$$

Formula 6.8 *(Successive composition of the above operators) We have*

(i): $\{X_{+,11}X_{+,22} - X^2_{+,12}\}\begin{pmatrix} s_{11}\Delta^m \\ s_{12}\Delta^m \end{pmatrix} = (\nu_1 + m + 2)(\nu_2 + m + 1)\begin{pmatrix} s_{11}\Delta^{m+2} \\ s_{12}\Delta^{m+2} \end{pmatrix}.$

(ii): $\{X_{+,11}X_{+,22} - X^2_{+,12}\}\begin{pmatrix} s_{21}\Delta^m \\ s_{22}\Delta^m \end{pmatrix} = (\nu_1 + m + 1)(\nu_2 + m + 2)\begin{pmatrix} s_{21}\Delta^{m+2} \\ s_{22}\Delta^{m+2} \end{pmatrix}.$

Remark

$$(\nu_1 + m + 2)(\nu_2 + m + 1) = (\nu_1 + \rho_1 + m)(\nu_2 + \rho_2 + m),$$
$$(\nu_1 + m + 1)(\nu_2 + m + 2) = (\nu_1 + \rho_1 + m - 1)(\nu_2 + \rho_2 + m + 2).$$

6.4 the case of odd principal series:the $\mathfrak{p}_-$-side

We investigate the shift operator:

$$pr_{(-1,-1)} \cdot \nabla_+ : \pi([\tau_{(m+1,m)}]) \to \pi([\tau_{(m,m-1)}]).$$

A set of normaized basis in $\pi([\tau_{(m+1,m)})$ is given by

$$\text{either } \{s_{11}\Delta^m, s_{12}\Delta^m\}, \text{ or } \{s_{21}\Delta^m, s_{22}\Delta^m\}$$

depending on the product of the parity of σ and m. Similarly for $\pi([\tau_{(m,m-1)}])$, we can take a set of normalized basis by

$$\text{either } \{s_{21}\Delta^{m-1}, s_{22}\Delta^{m-1}\}, \text{ or } \{s_{11}\Delta^{m-1}, s_{12}\Delta^{m-1}\}.$$

Formula 6.9 *(slant down) We have*

(i): $\begin{pmatrix} X_{-,12}(s_{11}\Delta^m) + X_{-,22}(s_{12}\Delta^m) \\ -X_{-,11}(s_{11}\Delta^m) - X_{-,12}(s_{12}\Delta^m) \end{pmatrix} = -(\nu_1 + \rho_1 - m - 2)\begin{pmatrix} s_{21}\Delta^{m-1} \\ s_{22}\Delta^{m-1} \end{pmatrix}$

(ii): $\begin{pmatrix} X_{-,12}(s_{21}\Delta^m) + X_{-,22}(s_{22}\Delta^m) \\ -X_{-,11}(s_{21}\Delta^m) - X_{-,12}(s_{22}\Delta^m) \end{pmatrix} = (\nu_2 + \rho_2 - m - 1)\begin{pmatrix} s_{11}\Delta^{m-1} \\ s_{12}\Delta^{m-1} \end{pmatrix}$

Formula 6.10 *(Successive composition of the above operators) We have*

(i): $\{X_{-,11}X_{-,22} - X^2_{-,12}\}\begin{pmatrix} s_{11}\Delta^m \\ s_{12}\Delta^m \end{pmatrix} = -(\nu_1 - m + 1)(\nu_2 - m)\begin{pmatrix} s_{11}\Delta^{m-2} \\ s_{12}\Delta^{m-2} \end{pmatrix}.$

(ii): $\{X_{-,11}X_{-,22} - X^2_{-,12}\}\begin{pmatrix} s_{21}\Delta^m \\ s_{22}\Delta^m \end{pmatrix} = -(\nu_1 - m)(\nu_2 - m + 1)\begin{pmatrix} s_{21}\Delta^{m-2} \\ s_{22}\Delta^{m-2} \end{pmatrix}.$

Remark. The formulae in this section for the odd principal series is already obtained in Miyazaki-Oda [3] by using Harish-Chandra's hypergeometric series.

6.5 Casimir operators

The action of Casimir operators are described as the composites of the contiguous relations, at least substancially.

The contiguous equation for $\tau_{(m,m)} \to \tau_{(m+2,m)}$ is given by

$$\begin{pmatrix} X_{+11}(\Delta^m) \\ X_{+12}(\Delta^m) \\ X_{+22}(\Delta^m) \end{pmatrix} = (\nu_1+\rho_1+m)\begin{pmatrix} s_{11}^2\Delta^m \\ s_{11}s_{12}\Delta^m \\ s_{12}^2\Delta^m \end{pmatrix} + (\nu_2+\rho_2+m)\begin{pmatrix} s_{21}^2\Delta^m \\ s_{21}s_{22}\Delta^m \\ s_{22}^2\Delta^m \end{pmatrix}.$$

The cotinupus equations for $\tau_{(m.m)} \leftarrow \tau_{(m+2,m)}$ are given by

$$(X_{-11}, 2X_{-12}, X_{-22})\begin{pmatrix} s_{21}^2\Delta^m \\ s_{21}s_{22}\Delta^m \\ s_{22}^2\Delta^m \end{pmatrix} = \{\nu_2+\rho_2-(m+2)\}\Delta^m = \{\nu_2-(m+1)\}\Delta^m.$$

$$(X_{-11}, 2X_{-12}, X_{-22})\begin{pmatrix} s_{11}^2\Delta^m \\ s_{11}s_{12}\Delta^m \\ s_{12}^2\Delta^m \end{pmatrix} = \{\nu_1+\rho_1-(m+4)\}\Delta^m = \{\nu_1-(m+2)\}\Delta^m.$$

The composition of these equations is

$$(X_{-11}, 2X_{-12}, X_{-22})\begin{pmatrix} X_{+11} \\ X_{+12}X_{+22} \end{pmatrix}\Delta^m = \{\nu_1^2+\nu_2^2-(m+2)^2-(m+1)^2\}\Delta^m.$$

Taking the $*$-conjugate equations of the above contiguous equations, we also have

$$(X_{+11}, -2X_{+12}, X_{+22})\begin{pmatrix} X_{-11} \\ -X_{-12}X_{-22} \end{pmatrix}\Delta^m = \{\nu_1^2+\nu_2^2-(m-2)^2-(m-1)^2\}\Delta^m.$$

The sum of the above two equation is the equation comming from the Casimir operator.

6.6 Generation of the peripheral K-types

6.6.1 Down-shift operator, up-shift operator:$\det(\mathcal{C}_\pm)$

We have

$$(X_{-,11}X_{-,22} - X_{-,12}^2)\Delta^m = (\nu_1+\rho_1-m-1)(\nu_2+\rho_2-m)\Delta^{m-2}$$

$$(X_{+,11}X_{+,22} - X_{+,12}^2)\Delta^m = (\nu_1+\rho_1-+m-1)(\nu_2+\rho_2+m)\Delta^{m+2}$$

$$(X_{+,11}X_{+,22} - X_{+,12}^2)\Delta^{m-2} = (\nu_1+\rho_1+m-3)(\nu_2+\rho_2+m-2)\Delta^m.$$

Consequently we have

$$\begin{aligned} \det(\mathcal{C}_+)\cdot\det(\mathcal{C}_-)\Delta^m &= (\nu_1+m-1)(\nu_2+m-1)(\nu_1+1-m)(\nu_2+1-m)\Delta^m \\ &= \{\nu_1^2-(m-1)^2\}\{\nu_2^2-(m-1)^2\}\Delta^m. \end{aligned}$$

We have

$$\Delta^{m-2} = \frac{1}{(\nu_1+1-m)(\nu_2+1-m)}\det(\mathcal{C}_-)\Delta^m;$$

and

$$\Delta^{m+2} = \frac{1}{(\nu_1+1+m)(\nu_2+1+m)}\det(\mathcal{C}_+)\Delta^m.$$

6.6.2 Generation of the part $d = l_1 - l_2 = 2$

We define two vectors of elements in $\mathfrak{p}_\pm$ by

$$\mathcal{X}_+^{(1)} = \begin{pmatrix} X_{+,11} \\ X_{+,12} \\ X_{+,22} \end{pmatrix}, \quad \mathcal{X}_-^{(1)} = \begin{pmatrix} X_{-,11} \\ X_{-,12} \\ X_{-,22} \end{pmatrix}.$$

We have to define also ${}^*\mathcal{X}_-^{(1)}$ by

$${}^*\mathcal{X}_-^{(1)} = \begin{pmatrix} X_{-,22} \\ \pm X_{-,12} \\ X_{-,11} \end{pmatrix}.$$

Then we have contiguous equations:

$$\mathcal{X}_+^{(1)}\Delta^m = (\nu_1 + \rho_1 + m)\Delta^m \mathbf{s}_0^{(2)} + (\nu_2 + \rho_2 + m)\Delta^m \mathbf{s}_2^{(2)}.$$

$${}^*\mathcal{X}_-^{(1)}\Delta^{m+2} = (\nu_2 + \rho_2 - m - 2)\Delta^{m+2}\,{}^*\bar{\mathbf{s}}_2^{(2)} + (\nu_1 + \rho_1 - m - 2)\Delta^{m+2}\,{}^*\bar{\mathbf{s}}_0^{(2)}.$$

References

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