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<tr>
<td>Citation</td>
<td>数理解析研究所講究録  (2004), 1398: 112-122</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/26008">http://hdl.handle.net/2433/26008</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>publisher</td>
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Tamely ramified factors of zeta integrals for the
Standard $L$-function of $U(2, 1)$

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In the past three decades, integral expressions of many automorphic $L$-functions have been discovered and utilized for study of analytic properties of $L$-functions. But unfortunately, not so much investigation on ramified factors of these integrals has been accumulated, though it is indispensable to arithmetic study of $L$-functions. Recently there seems to be a movement of renovation of zeta integral method toward deeper arithmetic investigation beginning with low rank groups, say $GSp(4), U(3)$. It is reasonable to begin with the Standard $L$-function of $U(3)$. So far we have four different zeta integral expression for this $L$-function. That is a Rankin-Selberg integral [Ge-PS], a Shimura type integral [Shi], [Ge-PS], Murase-Sugano’s integral by using their Shintani functions [Mu-Su] and the doubling integral [PS-Ra], [Tak].

The archimedean factor of the first integral was calculated by Koseki and Oda [K-O], where it is shown that the GCD of the integrals for all $K$-finite vectors turns out to be a product of three $\Gamma_C$’s. Note that this type of zeta integral works only for generic cusp forms. As for the third integral, Tsuzuki calculated the archimedean component in a broader setting [Tsu].

In this note, we report some results on ramified factors of the first and second zeta integrals, which are recalled in §1. In §2, we calculate the archimedean component of Shimura type zeta integral. After normalization of Eisenstein series, we show that it is, up to elementary factors, a product of three $\Gamma_C$’s for any discrete series $\pi_\infty$ and satisfies a symmetric functional equation. In §3, we proceed into study of tamely ramified finite local factors. We begin with the case of Steinberg representation. By using Li’s explicit formula [Li] of Whittaker function for Iwahori spherical vector, we compute the local component of Rankin-Selberg integral of Gelbart- Piatetski-Shapiro.

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*The main part of this work was done during author’s stay in The University of Maryland. He express his gratitude to the Department of Mathematics in UMD for its hospitality.
1 Zeta integrals for the standard $L$-function

Note that we can obtain the same result without any loss of generality, even if we formulate the problem over an arbitrary totally real algebraic number field. So we take $\mathbb{Q}$ for our ground field.

Let $E$ be an imaginary quadratic extension of $\mathbb{Q}$ and denote the non-trivial element of its Galois group by $\gamma$. Put

$$ G := \{ g \in GL(3, E) | {}^t \overline{g} \begin{pmatrix} 1/\kappa & 1/\kappa \\ -1/\kappa & 1 \end{pmatrix} g = \begin{pmatrix} 1/\kappa & 1/\kappa \\ -1/\kappa & 1 \end{pmatrix} \}, $$

where $\kappa$ is an element of $E$ such that $\text{Tr}_{E/\mathbb{Q}} \kappa = 0$. This defines a quasi-split unitary group of three variables over $\mathbb{Q}$. We need a subgroup

$$ H := \text{Im} \{( \iota : U(1, 1) \ni (\begin{array}{ll} * & * \\ * & * \end{array}) \mapsto (\begin{array}{ll} * & * \\ * & 1 \end{array}) \in G) \} $$

as the Euler subgroup for a Rankin-Selberg integral.

<Zeta integrals>

For a cusp form $\varphi$ belonging to a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}) = U(3)_\mathbb{A}$, Gelbart and Piatetski-Shapiro introduced the following zeta integral

$$ Z(s; \varphi, \xi) := \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \theta(g) E^\xi(s; g) dg. $$

Here $E^\xi$ and $\theta$ are an Eisenstein series and a theta series on $G(\mathbb{A})$ respectively. And $\xi$ is a Hecke character of $E$.

<Unfolding and local integrals>

By using the multiplicity one result on Whittaker models and an unfolding procedure, the Rankin-Selberg integral decomposes into a product of local integrals:

$$ Z(s; \varphi, \xi) = \prod_v Z_v(s; W^\pi_{\psi}, \Phi_{\xi}^{(s)}), $$

with

$$ Z_v(s; W^\pi_{\psi}, \Phi_{\xi}^{(s)}) := \int_{Z_{N,v} \backslash H_v} W^\pi_{\psi}(s; h_v) \Phi_{\xi}^{(s)}(h_v) dh_v. $$

Here $Z_{N,v}$ is the center of the maximal nilpotent subgroup $N_v$ of $G_v$, $W^\pi_{\psi}$ is a Whittaker vector for $\pi_v$ and $\Phi_{\xi}^{(s)}$ is a special section of the principal series $\text{Ind}_{P_v}^H(\mu| \cdot|^s)$ of $H$ induced
up from its Borel subgroup $i((\cdot, \cdot))$. Note that this integral vanishes unless $\varphi$ is a generic cusp form.

Similarly, by using the multiplicity one result on Fourier-Jacobi models (cf. [B-PS-R]) and an unfolding procedure, the Shimura type integral decomposes into a product of local integrals:

$$Z(s; \varphi, \theta, \xi) = \prod_v Z_v(s; W_\eta^\pi, W_{(\xi\eta)^\Theta}^{\sigma_\xi}, \Phi_{\sigma_\xi}^{(s)})$$

with

$$Z_v(s; W_\eta^\pi, W_{(\xi\eta)^\Theta}^{\sigma_\xi}, \Phi_{\sigma_\xi}^{(s)}) := \int_{R_v \setminus G_v} \langle W_{(\xi\eta)^\Theta}(g_v) | W_\eta^\pi(g_v) \rangle_{\eta} \Phi_{\sigma_\xi}^{(s)}(g_v) \mathrm{d}g_v.$$

Here $R_v$ modulo the center is the stabilizer subgroup of $Z_{N,v}$ in a Borel subgroup $B$ of $G$ for the adjoint action. And $W^X$ means a generalized Whittaker vector in Fourier-Jacobi model for $X$ (see §2) and $\Phi_{\sigma_\xi}^{(s)}$ is a section of the principal series $\text{Ind}_{B(G_c \xi| |)}^G (\sigma_\xi : s)$ of $G$.

<Unramified components>

Over the places where everything is unramified, the local components $Z_v(s; W_\psi^\pi, \Phi_{\xi}^{(s)})$ and $Z_v(s; W_\eta^\pi, W_{(\xi\eta)^\Theta}^{\sigma_\xi}, \Phi_{\sigma_\xi}^{(s)})$ of these zeta integrals were computed by Gelbart-Piatetski-Shapiro, Shintani and Gelbart-Rogawski respectively.

**Proposition 1 ([Ge-PS] §4)**

$$Z_v(s; W_\psi^\pi, \Phi_{\xi}^{(s)}) = L_v(s; \pi_v \otimes \xi_v)$$

**Proof.** Use Casselman-Shalika formula. 

**Proposition 2 ([Ge-Ro] §8, [Shi])**

$$Z_v(s; W_\eta^\pi, W_{(\xi\eta)^\Theta}^{\sigma_\xi}, \Phi_{\sigma_\xi}^{(s)}) = \frac{L_v(s + \frac{1}{2}; \pi_v \otimes \xi_v)}{L_{E,v}(s + 1; \xi_v)L_v(2s + 1; (\xi|_v \gamma_E|_v)v)}$$

**Proof.** Use recursion relations, coming from the Hecke action, for unramified generalized Whittaker vector in Fourier-Jacobi model for $\pi_p$. 

Note the local factor $L_v(s; \pi_v \otimes \xi_v)$ is given by

$$L_v(s; \pi_v \otimes \xi_v) = L_{E,v}(s; \xi_v)L_v(2s; \xi_v\nu)L_v(2s; \xi_v/\nu).$$

Here $\nu$ is the unramified character to define the unramified principal series $\pi_v$ (see §3).

## 2 The archimedean factors

In this section, we calculate the archimedean component $Z_\infty(s; W_\eta^\pi, W_{(\xi\eta)^\Theta}^{\sigma_\xi}, \Phi_{\sigma_\xi}^{(s)})$ of a Shimura type integral to have a local functional equation and a nice expression, after recalling a result of Koseki and Oda [K-O].

<Rankin-Selberg integral >

Koseki and Oda used their explicit formula for Whittaker functions on $SU(2, 1)$. Their result looks quite complicated. But after rewriting the result by our coordinate, it turns out that their GCD of archimedean integral $Z_\infty^\xi(s; W, \Phi)$'s can be written as follows.
Proposition 3 ([K-O] Theorem 6.8) Let \( \mathcal{L}^\xi(s; D^{(1,1)}_\Lambda) \) be the GCD of the family
\[
\{ \mathcal{Z}^\xi_N(s; W, \Phi^{(s)}_{\xi,\phi}) \mid W : K_\infty\text{-finite Whittaker vector}, \phi \in \mathcal{S}(\mathbb{C}^2) \}.
\]
See §3 for \( \Phi^{(s)}_{\xi,\phi} \). Then the GCD \( \mathcal{L}^\xi(s; D^{(1,1)}_\Lambda) \) is given as follows.

1. When \( \Lambda_1 + \Lambda_3 \geq m \geq \Lambda_2 + \Lambda_3 \),
\[
2^{-s} \Gamma(s + t + \Lambda_1 - \frac{m}{2}) \Gamma(s + t - \Lambda_2 + \frac{m}{2})
\[
\begin{array}{ll}
\Gamma(s + t - \Lambda_3 + \frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_3 \text{ or } m \geq \Lambda_3 \\
\Gamma(s + t + \frac{m}{2}) & \text{when } m \geq \Lambda_3 > 0 \\
\Gamma(s + t - \frac{m}{2}) & \text{when } 0 \geq \Lambda_3 > m \\
\Gamma(s + t + \Lambda_3 - \frac{m}{2}) & \text{when } m \geq 0, \Lambda_3 > 0 \text{ or } m \geq 0 \\
\end{array}
\]

2. When \( \Lambda_2 + \Lambda_3 \geq m \),
\[
2^{-s} \Gamma(s + t + \Lambda_1 - \frac{m}{2}) \Gamma(s + t - \Lambda_3 - \frac{m}{2})
\[
\begin{array}{ll}
\Gamma(s + t - \Lambda_2 + \frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_2 \text{ or } m \geq \Lambda_2 \\
\Gamma(s + t + \frac{m}{2}) & \text{when } m \geq \Lambda_2 > 0 \\
\Gamma(s + t - \frac{m}{2}) & \text{when } 0 \geq \Lambda_2 > m \\
\Gamma(s + t + \Lambda_2 - \frac{m}{2}) & \text{when } m \geq 0, \Lambda_2 > 0 \text{ or } m \geq 0 \\
\end{array}
\]

3. When \( m \geq \Lambda_1 + \Lambda_3 \),
\[
2^{-s} \Gamma(s + t - \Lambda_2 + \frac{m}{2}) \Gamma(s + t - \Lambda_3 + \frac{m}{2})
\[
\begin{array}{ll}
\Gamma(s + t - \Lambda_1 + \frac{m}{2}) & \text{when } m \geq 0, 0 \geq \Lambda_1 \text{ or } m \geq \Lambda_1 \\
\Gamma(s + t + \frac{m}{2}) & \text{when } m \geq \Lambda_1 > 0 \\
\Gamma(s + t - \frac{m}{2}) & \text{when } 0 \geq \Lambda_1 > m \\
\Gamma(s + t + \Lambda_1 - \frac{m}{2}) & \text{when } m \geq 0, \Lambda_1 > 0 \text{ or } m \geq 0 \\
\end{array}
\]

Note that in some cases the GCD in the above list may vanish by virtue of \( K_\infty\)-type compatibility. A natural question arises here. Is it possible to regain the third missing Harish-Chandra parameter \( \Lambda_i \) and to obtain a local functional equation by normalizing the Eisenstein series \( E^{\xi, H}_\xi \) on \( H \)? We will study this problem in the near future.

<generalized Whittaker vector>
Different from the \( GL_2 \cong U(1, 1) \) case, the maximal nilpotent subgroup \( N_v \) of our \( G_v \cong U(3) \) is not abelian, is isomorphic to the Heisenberg group. The unitary dual \( N_v^\Lambda \) consists not only of unitary characters \( \psi \) but also of infinite dimensional irreducible unitary representations \( \rho \). So considering \( \text{Hom}_{N_v}(\pi_v|N_v, \rho) \) seems to be natural. But this intertwining space is infinite dimensional. This is the reason why the bigger group \( R_v \cong U(1)^k \ltimes N_v \) is introduced. Then we have the correct intertwining space \( \text{Hom}_{G_v}(\pi_v, \text{Ind}_{R_v}^{G_v}\eta) \), i.e. the Fourier-Jacobi model of \( \pi_v \). Multiplicity one result \( \text{dim}_C \text{Hom}_{G_v}(\pi_v, \text{Ind}_{R_v}^{G_v}\eta) \leq 1 \) has last
been published in [B-PS-R]. Here $\eta := \bar{\chi} \otimes \omega \rho$ is an irreducible unitary representation of $R_v$ induced from $\rho$. When $v = \infty$, $\eta$ is parameterized by $(\tilde{\mu}, \ell) \in (\frac{1}{2}\mathbb{Z}\backslash \mathbb{Z}) \times (\mathbb{Z}\backslash \{0\})$. We give an explicit formula for the generalized Whittaker vectors coming from Fourier-Jacobi model of $\pi_\infty$, when $\pi_\infty$ is a cohomological unitarizable representation of $U(2, 1)$.

**Proposition 4** The moderate growth generalized Whittaker vector belonging to the minimal $K_\infty$-type of $\pi_\infty$ is given as follows by expanding by the Fock and the Gel’fand-Zetlin basis.

$$W_{\eta}^{(\pi, \gamma)}|_{A}(a) = \sum_{k=-\lambda_2}^{-\lambda_1} c_k^\pi(a) \left( \left[ \begin{array}{c} \bar{\mu}, \ell \\ j_k \end{array} \right] \otimes \begin{array}{c} -\lambda_2 \\ -\lambda_1 \\ -\lambda_0 \end{array} \right).$$

Here $j_k^\mu$ is $k + \lambda_1 + \lambda_2 - \tilde{\mu} - (\text{sgn} \ell)\frac{1}{2}$.

i) When $\pi$ is a discrete series representation $D_{\Lambda}^{p,q}$ with Blattner parameter $\lambda = [\lambda_1, \lambda_2; \lambda_0]$.

i-1) The case of large discrete series $D_{\Lambda}^{1,1}$ i.e. contributes to $H^{(1,1)}$.

The generalized Whittaker model exists exactly when $\ell > 0$ and $\lambda_1 \geq \tilde{\mu} + \frac{1}{2}$, or when $\ell < 0$ and $\lambda_2 \leq \tilde{\mu} - \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\text{La}, \text{sgn} \ell} \cdot y^{-\lambda_2 + \lambda_1 - 1} W_{\kappa, \mu}(2\pi|\ell|y^2)$$

with

$$\kappa = (\text{sgn} \ell)\frac{-k + 2\tilde{\mu} - c^\pi}{2}, \quad \mu = \frac{-k + 2\lambda_0 - c^\pi}{2}.$$ 

Here $c^\pi = \lambda_1 + \lambda_2 + \lambda_0$.

i-2) The case of holomorphic discrete series $D_{\Lambda}^{2,0}$ i.e. contributes to $H^{(2,0)}$.

The generalized Whittaker model exists exactly when $\ell > 0$ and $\lambda_1 \geq \tilde{\mu} + \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\text{Hol}}(\lambda) \cdot y^{-2 - \lambda_0 - k} e^{-\pi t y^2}.$$ 

i-3) The case of anti-holomorphic discrete series $D_{\Lambda}^{0,2}$ which contributes to $H^{(0,2)}$.

The generalized Whittaker model exists exactly when $\ell < 0$ and $\lambda_2 \leq \tilde{\mu} - \frac{1}{2}$.

$$c_k^\pi(a_y) = \gamma_k^{\text{AnH}}(\lambda) \cdot y^{-2 + \lambda_0 + k} e^{\pi t y^2}.$$ 

ii) When $\pi$ is a cohomological unitarizable representation $A_\delta(\lambda)$ which contributes to $H^1$.

For these representations the indices $j_k^\mu$ of Fock basis are always zero.

ii-1) The case of lowest weight module, i.e. contributes to $H^{(1,0)}$.

The generalized Whittaker model exists exactly when $\ell > 0$.

$$c_k^\pi(a_y) = \gamma_k^{\text{low}}(\lambda) \cdot y^{-2 - \alpha(\gamma) - k} e^{-\pi t y^2}.$$ 

ii-2) The case of highest weight module, i.e. contributes to $H^{(0,1)}$.

The generalized Whittaker model exists exactly when $\ell < 0$.

$$c_k^\pi(a_y) = \gamma_k^{\text{last}}(\lambda) \cdot y^{-2 + \alpha(\gamma) + k} e^{\pi t y^2}.$$ 

For normalizing constants $\gamma_k^\pi(\lambda)$, see [I].
Theorem 4. Assume the archimedean component $\pi_\infty$ of cuspidal representation generated by $\varphi$ is discrete series representation $D_\Lambda^{p,q}$ with Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$. The archimedean zeta integral vanishes unless

$$\lambda_2 \leq \mu + \frac{(\text{sgn} \ell)}{2} \leq \lambda_1.$$ 

When $\alpha(\gamma) = 1$, the archimedean zeta integral is given as follows, if it does not vanish.

1) When $\pi$ is a holomorphic discrete series $D_\Lambda^{2,0}$ and the parameter $\ell$ of $\eta = \hat{X} \otimes (\omega \rho \psi)$ is positive,

$$Z_\infty(s; W_\eta, W_{(\xi)}^\Theta; \Phi_{(s)}^\xi) = \frac{(-1)^{\beta+\frac{1}{2}+\Lambda_2}((\dim \tau_\lambda - 1)!}{2\ell^s+\frac{1}{2}+\Lambda_2+\Lambda_1+\ell} \Gamma_c(s+\frac{1}{2}+\frac{1}{2}+\Lambda_1+\frac{m}{2}).$$

2) When $\pi$ is an anti-holomorphic discrete series $D_\Lambda^{0,2}$ and $\eta$ has the negative parameter $\ell < 0$,

$$Z_\infty(s; W_\eta, W_{(\xi)}^\Theta; \Phi_{(s)}^\xi) = \frac{(-1)^{\beta+\frac{1}{2}+\Lambda_2}((\dim \tau_\lambda - 1)!}{2(-\ell)^s+\frac{1}{2}+\Lambda_2+\Lambda_1+\frac{m}{2}).$$

3) When $\pi$ is a large discrete series $D_\Lambda^{1,1}$, there are two subcases.

3+) If the parameter $\ell$ of $\eta$ is positive,

$$Z_\infty(s; W_\eta, W_{(\xi)}^\Theta; \Phi_{(s)}^\xi) = \frac{(-1)^{\beta+\frac{1}{2}+\Lambda_2}((\dim \tau_\lambda - 1)!}{2\ell^s+\frac{1}{2}+\Lambda_2+\Lambda_1+\frac{m}{2}).$$

3-) If the parameter $\ell$ of $\eta$ is negative,

$$Z_\infty(s; W_\eta, W_{(\xi)}^\Theta; \Phi_{(s)}^\xi) = \frac{(-1)^{\beta+\frac{1}{2}+\Lambda_2}((\dim \tau_\lambda - 1)!}{2(-\ell)^s+\frac{1}{2}+\Lambda_2+\Lambda_1+\frac{m}{2}).$$

Here $P_\pm$ are polynomials in $s$ and $c_\pm$ are constants, see [Is].

Normalization and Local functional equation

In order to have a local functional equation in a symmetric form, we normalize the intertwining operator $A_\infty(s)$ and a section $\Phi_{(s)}^\xi$. The intertwining operator is defined by

$$(A_v(s), \Phi_{v}^{(s)})(g) := \int_{N_v} \Phi_{v}^{(s)}(w^{-1}ng)dn.$$
where $w$ is the longest Weyl element. Normalize this $A_v(s)$ as
\[
A_v(s) := \epsilon_v(s;\xi,\psi) \frac{L_v(1-s;\xi)}{L_v(s;\xi)} \cdot \epsilon_v(2s;\xi|q,\psi) \frac{L_v(1-2s;\xi|q,\gamma)}{L_v(2s;\xi|q,\gamma)} A_v(s),
\]
then $A_v(-s)A_v(s) = \text{Id}$, i.e. self-adjoint.

**Lemma 6** Assume $\mu = m < 0$ and $\alpha(\gamma) = 1$, then we have
\[
A_\infty(s). \left[ d\left(s + \frac{t}{2}\right) \Phi^\text{(s)} \right] = \epsilon_{\mu,\gamma} \cdot \left[ d\left(-s - \frac{t}{2}\right) \Phi^{-\text{(s)}} \right]
\]
for the section $\Phi^\text{(s)}$ belonging to the corner $K_{\infty}$-type. Here
\[
\epsilon_{\mu,\gamma} = (-1)^{\mu + \frac{\alpha(\gamma)}{2}} = (-1)^m \sqrt{-1}
\]
and
\[
d\left(s + \frac{t}{2}\right) := 2^{s + \frac{t}{2}} \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \mu - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \frac{|m|}{2}).\]

Moreover if we normalize as
\[
\hat{Z}_\infty(s; W_\eta^\pi, W_{(\xi_\eta)^*}^\Theta, \Phi_{\sigma_\xi}^{(s)}) := Z_\infty(s; W_\eta^\pi, W_{(\xi_\eta)^*}^\Theta, d(s') \Phi_{\sigma_\xi}^{(s)}),
\]
where $s' = s + \frac{t}{2}$, then we have a clean functional equation and a nice expression of the archimedean zeta integral.

**Theorem 7** Assume $\mu = m < 0$ and $\alpha(\gamma) = 1$. The archimedean component of normalized zeta integral satisfies a local functional equation
\[
\hat{Z}_\infty(-s; W_\eta^\pi, W_{(\xi_\eta)^*}^\Theta, A_\infty(s), \Phi^\text{(s)}_{\sigma_\xi}) = \epsilon_{\mu,\gamma} \cdot \hat{Z}_\infty(s; W_\eta^\pi, W_{(\xi_\eta)^*}^\Theta, \Phi^\text{(s)}_{\sigma_\xi}),
\]
and is, up to simple factors appearing in Proposition 5, of the following form.

1) When $\pi$ is a holomorphic discrete series $D_\Lambda^{2,0}$ and the parameter $\ell$ is positive,
\[
\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \frac{t}{2} - \Lambda_3 + \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \Lambda_1 - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \frac{|m|}{2}).
\]

2) When $\pi$ is an anti-holomorphic discrete series $D_\Lambda^{0,2}$ and the negative parameter $\ell < 0$,
\[
\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \frac{t}{2} + \Lambda_3 + 1 + \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \Lambda_2 - 1 - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \frac{|m|}{2}).
\]

3) When $\pi$ is a large discrete series $D_\Lambda^{1,1}$, there are two subcases.
3+) If the parameter $\ell$ of $\eta$ is positive,
\[
\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \frac{t}{2} - \Lambda_2 - 1 + \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \Lambda_1 - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \frac{|m|}{2}).
\]

3-) If the parameter $\ell$ of $\eta$ is negative,
\[
\Gamma_{\mathbb{C}}(s + \frac{1}{2} + \frac{t}{2} + \Lambda_1 - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \Lambda_2 - 1 - \frac{m}{2}) \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \frac{|m|}{2}).
\]

Here $(t, m) \in \mathbb{C} \times \mathbb{Z}$ is the parameter of $\xi$. \hfill \Box
The Langlands parameterization of irreducible admissible representations of real reductive group says the archimedean factor should be

\[ L_{\infty}(s; \pi_{\infty} \otimes \xi_{\infty}) := \prod_{i=1}^{3} \Gamma_{\mathbb{C}}(s + \frac{t}{2} + \Lambda_{i} + \frac{|m|}{2}), \]

with the Harish-Chandra parameter \((\Lambda_{1}, \Lambda_{2}, \Lambda_{3})\) for \(\pi_{\infty} \cong D_{\Lambda}^{p,q}\).

So the ratio \( \hat{Z}_{\infty}(s; W_{\eta}^{'\pi}, W_{(\xi\eta)^{*}}^{\ominus}, \Phi_{\sigma\xi}^{(s)})/L_{\infty}(s; \pi_{\infty} \otimes \xi_{\infty}) \) is rational function, sometimes polynomial, in \(s\). Here is a question. Is it possible to find an appropriate \(K_{\infty}\)-finite generalized Whittaker vector by which the zeta integral \( \hat{Z}_{\infty}(s; W_{\eta}^{'\pi}, W_{(\xi\eta)^{*}}^{\ominus}, \Phi_{\sigma\xi}^{(s)}) \) expresses local factor \( L_{\infty}(s; \pi_{\infty} \otimes \xi_{\infty}) \) itself?

## 3 The case of Steinberg representation

In this section, we calculate the local component of Rankin-Selberg integral for the Iwahori spherical Whittaker vector in the Steinberg representation. From now on we denote \(p\) for a fixed finite place \(v\) and \(\mathfrak{p}\) for the place of \(E\) which lies over \(p\). Assume \(E_{\mathfrak{p}}/\mathbb{Q}_{p}\) is unramified extension. Let \(G_{p}\) denote the \(Q_{p}\)-valued points \(G(\mathbb{Q}_{p})\) of \(G\) and \(J\) the Iwahori subgroup of the hyperspecial maximal compact subgroup \(K_{p} := G(\mathbb{Z}_{p})\), defined as \(J \mod p = B(\mathbb{Z}_{p}/p\mathbb{Z}_{p})\), i.e.

\[
J = \begin{pmatrix} O_{p} & O_{p} & O_{p} \\ p & O_{p} & O_{p} \\ p & p & O_{p} \end{pmatrix}.
\]

**<Iwahori spherical vectors>**

We recall Borel's characterization of Iwahori spherical representations.

**Proposition 8 ([Bo])** For an irreducible admissible representation \(\pi_{p}\) of \(G_{p}\), the followings are equivalent.

(i) \(\pi_{p}\) has a non-zero Iwahori spherical vector, i.e. \(\pi_{p}J \neq \{0\}\).

(ii) \(\pi_{p}\) is a subquotient of unramified principal series \(I_{\nu}(s)\).

Here \(I_{\nu}(s) := \text{Ind}_{B_{p}}^{G_{p}} \nu \cdot |^{s} (\nu' \circ \text{det})\) with unramified characters \(\nu, \nu'\).

The reducing points of \(I_{\nu}(s)\) is known for our group \(G_{p}\).

**Proposition 9** The principal series representation \(I_{\nu}(s)\) is reducible exactly when \(s = \pm 1, \pm \frac{1}{2}, 0\).

Especially when \(s = +1\), we have

\[ 0 \rightarrow \text{St}(\nu') \rightarrow I_{\nu}(+1) \rightarrow \nu' \circ \text{det} \rightarrow 0. \]

The representation \(\text{St}(\nu')\) is the Steinberg representation.
Lemma 10 The Steinberg representation $St(\nu')$ has unique Iwahori spherical vector up to constant multiple, i.e. $\dim_{\mathbb{C}} St(\nu')^{J} = 1$. Moreover the vector is given by $\Phi^{St} := -p\Phi_{s} + \Phi_{w}$, i.e. $St(\nu')^{J} = C \Phi^{St}$. Here $\Phi_{s}$ denote the characteristic function of $JsJ$ with $s \in W = \{ e := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, w := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \}$.

Proof. Just trace the argument of Casselman in [Cas].

<Whittaker vectors>

Let

$$\Lambda_{\nu} : I_{\nu}(s) \rightarrow \mathbb{C}$$

be the Whittaker functional with respect to a non-degenerate character $\psi_{N}$ of $N_{p}$, i.e.

$$\Lambda_{\nu}(R(n).\phi) = \psi_{N}(n)\Lambda_{\nu}(\phi)$$

for $\forall n \in N_{p}$ and for $\forall \phi \in I_{\nu}(s)$. Here $R(\ast)$ means the left regular representation. Fortunately, it can be easily checked that the Whittaker vector coming from the Iwahori spherical vector $\Phi^{St}$ coincides with Jian-Shu Li's $W_{\nu}^{(2)}$ up to constant, i.e.

$$\Lambda_{\nu}(R(g).\Phi^{St}) = (\text{const.})W_{\nu}^{(2)}(g).$$

Li obtained an explicit formula for four $J$-spherical Whittaker vectors $W_{\nu}^{(i)}$ ($i = 1, \ldots, 4$) on arbitrary quasi-split reductive group $G_{p}$. $W_{\nu}^{(1)}$ is $K_{p}$-spherical and his explicit formula is Casselman-Shalika formula. We need $W_{\nu}^{(2)}$ for our purpose, and write down Li's formula in the case of $G_{p} \cong U(3)$.

Proposition 11 ([Li]) We denote $\#(E_{\mathbb{P}}/O_{\mathfrak{P}})$ by $q_{E}$. For $k \in \mathbb{Z}_{\geq}$,

$$W_{\nu}^{(2)}\left(\begin{pmatrix} p^{k} \\ 1 \\ p^{-k} \end{pmatrix}\right) = \ \{ (1 - q_{E}^{-1}\nu(p))(1 + p^{-1}\nu(p))\nu(p)^{k+1} + (q_{E}^{-1} - \nu(p))(p^{-1} + \nu(p))\nu(p)^{-k-1} \} \frac{|p^{k}|^{2}}{\nu(p) - \nu(p^{-1})}$$

<Zeta integral>

Now we compute the local factor $Z_{\infty}(s; W_{\psi}^{\pi}, \Phi_{\xi}^{(s)})$ of Rankin-Selberg integral for the Steinberg representation $St(\nu')$ by a standard procedure.

Theorem 12 When $\pi_{p} \cong St(\nu')$ and $W_{p}$ is the Iwahori spherical Whittaker vector $W_{\nu}^{(2)}$ in $\pi_{p}$,

$$Z_{p}(s; W_{\psi}^{\pi}, \Phi_{\xi}^{(s)}) \text{ expresses } L_{E,p}(s; \xi_{v})L_{p}(2s; \xi_{v}\nu).$$
Proof. By the Iwasawa decomposition $H_p = Z_{N,p}A_pK_H$ with $K_H := K_p \cap H_p$, the local integral is

$$Z_p(s; W^\pi_\psi, \Phi^{(s)}_\xi) = \int_{A_p} \left( \int_{K_H} W_\nu^{(2)} \left( \begin{array}{cc} a & 1 \\ a^{-1} & 1 \end{array} \right) k \right) \Phi^{(s)}_\xi (k) dk \xi(a)|a|^{2s} \frac{da}{|a|^2}.$$  

For the inner integral, use the decomposition

$$K_H = \iota \left( \Gamma_0(p) \sqcup \prod_{x \equiv 0 \mod p} \left( \begin{array}{cc} 1 & x \\ 1 & 1 \end{array} \right) \right) \Gamma_0(p).$$

For the outer integral, insert Li's explicit formula (Proposition 11) and section of the form,

$$\Phi^{(s)}_{\xi, \phi} := \int_{E^b} (h.\phi)(t(\begin{array}{l} 100 \end{array})) \xi(t)|t|_p^{s} \frac{dt}{t}$$

with $\phi \in S(E_p(\begin{array}{l} 001 \end{array}) \oplus E_p(\begin{array}{l} 100 \end{array}))$. Choose $\phi$ suitably. \hfill \Box

<Problems>
Several problems are remained. First, there are other tamely ramified $\pi_p$, i.e. subquotient of $I_\nu(\pm \frac{1}{2})$ and $I_\nu(0)$. Is it possible to calculate local Rankin-Selberg integral $Z_\infty(s; W^\pi_\eta, \Phi^{(\sigma)}_\xi)$? There is a related result of Watanabe [Wat]. Second, it is also interesting to study ramified local factors $Z_\nu(s; W^\pi_\eta, W^\Theta_{\xi\eta}^\phi, \Phi^{(s)}_{\sigma\xi})$ of Shimura type integral.

References


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