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On non-arithmetic discontinuous groups

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In this talk, we will give a survey on arithmetic and non-arithmetic lattices in a semisimple algebraic group. After giving some basic results on the subject, we'll focus our attention to more recent results, mainly due to Mostow and Deligne, on non-arithmetic lattices in the (projective) unitary group $PU(n,1)$ ($n \geq 2$). (For more details on these topics as well as the closely related rigidities of lattices, see [S 04]).

1. To begin with, we first fix our settings, giving basic definitions and notations. Let $X$ denote a symmetric Riemannian space of non-compact type (with no flat or compact factors) and let $G = I(X)^0$ be the identity connected component of the isometry group of $X$. Then, as is well known, $G$ is a connected semisimple Lie group of non-compact type, which is of adjoint type, i.e., with the center reduced to the identity 1. This implies that, denoting by $g$ the Lie algebra of $G$, one has $G = (\text{Aut } g)^0$ ($^0$ denoting always the identity connected component). The group $G$ acts transitively on $X$ and for any $x_0 \in X$ the stabilizer $K = G_{x_0}$ is a maximal compact subgroup; thus one has $X \cong G/K$. In this manner, $G$ and $X$ determine one another uniquely (up to isomorphisms).

More generally, let $G'$ denote a connected semisimple linear Lie group, which becomes automatically "real algebraic" in the sense that there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbb{R}$ (uniquely determined up to $\mathbb{R}$-isomorphisms) such that $G' = \mathcal{G}(\mathbb{R})^0$. As typical examples, one has $G' = SL(n,\mathbb{R}), SO(p,q)^0$, etc. Let $K'$ be a maximal compact subgroup of $G'$, and $K'_0$ the maximal compact normal subgroup of $G'$. Then one has

$$G' \supset K' \supset K'_0 \supset \text{(center of } G').$$

Therefore, setting

$$G = G'/K'_0, \quad K = K'/K'_0, \quad X = G/K = G'/K',$$

one obtains a pair $(G,X)$ as described in the beginning; in particular, one has $G = G'$ if $K'_0$ reduces to the identity group $\{1\}$. We keep these notations throughout the paper.

When $G' = \mathcal{G}(\mathbb{R})^0$, the common dimension $r$ of the maximal $\mathbb{R}$-split tori in $\mathcal{G}$ is called the $\mathbb{R}$-rank of $G'$ and written as $r = \mathbb{R}$-rank $G'$. It is well known that, if $g' = k' + p'$ is a Cartan decomposition of $g' = \text{Lie } G'$, then
$r$ coincides with the maximal dimension of the (abelian) subalgebras of $g'$ contained in $p'$. Thus one has $\mathbb{R}$-rank $G' = \mathbb{R}$-rank $G$.

When the algebraic group $G$ is defined over $\mathbb{Q}$, $G'$ is said to have a $\mathbb{Q}$-structure and the $\mathbb{Q}$-rank of $G'$ (with this $\mathbb{Q}$-structure) is the common dimension $r_0$ of the maximal $\mathbb{Q}$-split tori in $G$. $G'$ is called $\mathbb{Q}$-\textit{anisotropic} when $r_0 = 0$.

2. A subgroup $\Gamma$ of $G'$ is called a \textit{lattice} in $G'$ if $\Gamma$ is discrete and the covolume $\text{vol}(\Gamma \backslash G')$ (with respect to the Haar measure of $G'$) is finite. A lattice $\Gamma$ is called \textit{uniform} if, in particular, the quotient space $\Gamma \backslash G'$ is compact.

Two subgroups $\Gamma$ and $\Gamma'$ of $G'$ are said to be \textit{commensurable} if the indices $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite, and one then writes $\Gamma \sim \Gamma'$. As is easily seen, this is an equivalence relation.

A lattice $\Gamma$ in $G$ is said to be \textit{reducible} if there exists a non-trivial direct decomposition $G = G_1 \times G_2$ such that $\Gamma \sim (\Gamma \cap G_1) \times (\Gamma \cap G_2)$; otherwise, $\Gamma$ is called \textit{irreducible}. Every lattice in $G$ is commensurable to the direct product of irreducible ones in the direct factors of $G$.

When $G' = \mathcal{G}(\mathbb{R})^0$ is given a $\mathbb{Q}$-structure, a subgroup $\Gamma$ of $G'$ commensurable with $\mathcal{G}(\mathbb{Z})$ is called \textit{arithmetic}; the projection of an arithmetic subgroup of $G'$ in $G = G'/K_0$ is called \textit{arithmetic in a wider sense}. It is clear that arithmetic subgroups (in a wider sense) are discrete.

The following theorem is fundamental.

**Theorem 1** (Borel–Harish-Chandra [BHC 62], Mostow–Tamagawa [MT 62]) If $\Gamma$ is an arithmetic subgroup of $G$ in a wider sense, then $\Gamma$ is a lattice in $G$. Moreover, $\Gamma$ is uniform (i.e., cocompact in $G$) if and only if $G'$ is $\mathbb{Q}$-\textit{anisotropic} (i.e., $\mathbb{Q}$-rank $G' = 0$).

Note that, when $\Gamma$ in $G$ is arithmetic only in a wider sense, the $\mathbb{Q}$-rank of $G'$ being $= 0$, $\Gamma$ is uniform. In the early 1960s it was conjectured by Selberg and others that the converse of Theorem 1 would also be true, if the $\mathbb{R}$-rank of $G$ is high. Actually, we now have

**Theorem 2** (Margulis, 1973, [Ma 81]) Suppose that the $\mathbb{R}$-rank of $G$ is $\geq 2$. Then any irreducible lattice $\Gamma$ in $G$ is arithmetic in a wider sense (for a certain choice of $G'$ with a $\mathbb{Q}$-structure).

3. Thanks to the above result of Margulis, in order to study the arithmeticity of a lattice $\Gamma$, we may restrict ourselves to the case $\mathbb{R}$-rank $G = 1$, which
naturally implies that $G$ is $\mathbb{R}$-simple. According to the classification of $\mathbb{R}$-simple Lie groups (due to E. Cartan), we have only the following possibilities for $(G, X)$:

\[ G = PU(D; n, 1)^o = U(D; n, 1)^o/(\text{center}), \quad n \geq 2, \quad (n = 2 \text{ for } D = \mathbb{O}), \]

\[ X = \mathbb{H}^D \] (the hyperbolic $n$-space over $D$),

$D$ denoting a division composition algebra over $\mathbb{R}$, i.e.,

$D = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (Hamilton's quaternions), $\mathbb{O}$ (Cayley's octonions),

and $U(D; n, 1)$ denoting the unitary group of the standard $D$-hermitian form of signature $(n, 1)$. In the case $D = \mathbb{O}$, which is non-associative, the projective unitary group is defined to be the automorphism group of the (split) exceptional Jordan algebra $\text{Her}_3(\mathbb{O}; 2, 1)$; hence $G$ is of type $F_{4,1}$.

For $D = \mathbb{R}$, one has $G = SO(n, 1)^o$ (Lorentz group) and $X = \mathbb{H}_R^n$ is the "Lobachevsky space", i.e., the Riemannian $n$-space of constant curvature $\kappa = -1$, which can be realized by the hyperbolic hypersurface in $\mathbb{R}^{n+1}$ (with the Lorentz metric):

\[ \{(x_i) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, \ x_{n+1} > 0 \}. \]

In particular, $\mathbb{H}_R^2 (= \mathbb{H}_C^1)$ can be identified with the upper half-plane in $\mathbb{C}$ and the lattices in $G = SO(2, 1)^o(\cong SL(2, \mathbb{R})/\{\pm 1\})$ are so-called Fuchsian groups. In this case, it is classical that there are continuous families of non-arithmetic lattices.

For $X = \mathbb{H}_R^n$, $n \geq 3$, non-arithmetic lattices, especially reflection groups, have been studied intensively by E. B. Vinberg and his school since 1965 (see e.g., [V 85], [V 90]). More recently, it was shown by Gromov and Piatetski-Shapiro [GPS 88] that for any $n \geq 2$ one can construct infinitely many non-arithmetic (uniform) lattices as the fundamental group of the "hybrid" of two quotient spaces $\Gamma_1 \backslash X$ and $\Gamma_2 \backslash X$ for non-commensurable arithmetic subgroups $\Gamma_1$ and $\Gamma_2$ of $G$.

On the other hand, for the case $D = \mathbb{H}$ and $\mathbb{O}$, Corlette [C 92] and Gromov and Schoen [GS 92] have shown that there exist no non-arithmetic lattices in $G$ by a differential geometric method (harmonic maps), extending the idea of Margulis.

4. In the rest of the paper, we concentrate to the case $D = \mathbb{C}$, i.e., the case where $G = PU(n, 1)$ and $X = \mathbb{H}_C^n$, studied mainly by G. D. Mostow since the early 1970s.
The complex hyperbolic space $\mathbb{H}^n_C$ can be realized by the unit ball in $\mathbb{C}^n$ as follows. The unitary group $U(n, 1)$ acts on $\mathbb{C}^{n+1}$ and hence on the projective space $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ in a natural manner. The orbit of $e_{n+1} = (0, ..., 0, 1) \pmod{\mathbb{C}^*}$ in $\mathbb{P}^n(\mathbb{C})$ is

$$\{z = (z_i) \in \mathbb{C}^{n+1} | \sum_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2 < 0 \}/\mathbb{C}^*,$$

which, in the inhomogeneous coordinates $z'_i = z_i/z_{n+1}$ (1 ≤ $i$ ≤ $n$), is expressed by the unit ball

$$\{z' = (z'_i) \in \mathbb{C}^n | \sum_{i=1}^{n} |z'_i|^2 < 1 \}.$$

The stabilizer of $e_{n+1}$ in $U(n, 1)$ is $U(n) \times U(1)$. Hence $\mathbb{H}^n_C = U(n, 1)/U(n) \times U(1)$ is identified with the unit ball in $\mathbb{C}^n$, on which $G = PU(n, 1)$ acts as linear fractional transformations.

We denote by $<>$ the standard hermitian inner product of signature $(n,1)$ on $\mathbb{C}^{n+1}$. For $a \in \mathbb{C}^{n+1}$, $<a,a> > 0$ and $\xi \in \mathbb{C}$, $|\xi| = 1$, we define (after Mostow) a "complex reflection" on $\mathbb{C}^{n+1}$ by

$$R_{a,\xi}' : z \mapsto z + (\xi - 1)\frac{<a,z>}{<a,a>} a \quad (z \in \mathbb{C}^{n+1}).$$

Then, for $\xi, \eta \in \mathbb{C}$, $|\xi| = |\eta| = 1$, one has

$$R'_{a,\xi} \circ R'_{a,\eta} = R'_{a,\xi\eta};$$

in particular, if $\xi$ is a root of unity: $\xi^m = 1$, then one has $(R'_{a,\xi})^m = 1$. We denote the image of $R'_{a,\xi}$ in $G = PU(n, 1)$ by $R_{a,\xi}$.

In [M 80] Mostow studied the groups

$$\Gamma = <R_{e_i,\zeta_p} (i = 1, 2, 3)>$$

generated by 3 reflections, where $\zeta_p = e^{2\pi i/p}$ with $p = 3$ or 4 or 5 and

$$e_i \in \mathbb{C}^{n+1}, \quad <e_i, e_i> = 1, \quad <e_1, e_2> = <e_2, e_3> = <e_3, e_1> = -\alpha \varphi,$$

$$\alpha = (2 \sin \frac{\pi}{p})^{-1}, \quad \varphi = e^{\pi i/3};$$

with $t \in \mathbb{R}$. Mostow gave a criterion for $\Gamma$ to be a lattice in $G$, and found 17 cases, showing that 7 among them are non-arithmetic (i.e., not arithmetic in a wider sense). The non-arithmetic cases are given by

$$[p, t] = [3, 1/12], [3, 1/30], [3, 5/42], [4, 1/12], [4, 3/20],$$
(It has turned out that actually the $\Gamma$ corresponding to $[5,11/30]$ is arithmetic.)

5. Mostow then studied, in collaboration with Deligne, the analytic construction of lattices in $PU(n,1)$. They consider a system of differential equations of Fuchsian type in $n$ variables, studied for $n=2$ by Picard and in general by Lauricella (1893). The solution space of such equations is $\cong \mathbb{C}^{n+1}$, spanned by the period integrals generalizing the classical Euler integral:

$$F_{g,h}(x_1, ..., x_n) = \int_{g}^{h} \prod_{i=1}^{n} (u-x_i)^{-\mu_i} \cdot u^{-\mu_{n+1}}(u-1)^{-\mu_{n+2}} \, du,$$

where

$$\mu = (\mu_1, ..., \mu_{n+3}) \in \mathbb{C}^{n+3}, \quad \mu_{n+3} = 2 - \sum_{i=1}^{n+2} \mu_i$$

is the parameter, which we will restrict to the so-called "disc (n+3)-tuple" satisfying the condition $0 < \mu_i < 1$ ($1 \leq i \leq n+3$), and

$$g, h \in M = \{x = (x_1, ..., x_n, 0, 1, \infty) | x_i \in \mathbb{C} - \{0,1\}, \ x_i \neq x_j \text{ for } i \neq j\}.$$

Let $\hat{M}$ be the universal covering space of $M$. Then there exists a natural map from $\hat{M}$ to $\mathbb{P}^n(\mathbb{C})$, the space of non-zero solutions modulo $\mathbb{C}^\times$, which is equivariant with respect to the actions of the fundamental group on $\hat{M}$ and the projective monodromy group, denoted by $\Gamma_\mu$, on $\mathbb{P}^n(\mathbb{C})$. It is also shown that there exists a hermitian inner product of signature $(n,1)$ on the solution space such that $\Gamma_\mu$ is in $PU(n,1)$.

In [DM 86] it was shown that the following condition (INT) is sufficient for $\Gamma_\mu$ to be a lattice in $G = PU(n,1)$.

**(INT)** If $\mu_i + \mu_j < 1$ with $i \neq j$, then one has $(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z}$.

Actually, for $n=2$, this condition is equivalent to the one given by Picard in 1885, so that the 27 lattices obtained in this manner are called "Picard lattices". (In counting the lattices $\Gamma_\mu$ we disregard the order of $\mu_i$'s because it is not essential.) There are 9 more $\mu$'s satisfying the condition (INT) for $3 \leq n \leq 5$, the longest one being $\frac{1}{4}$ (1,1,1,1,1,1,1,1,1).

In [M 86] Mostow showed that the following weaker condition ($\Sigma$INT) is sufficient to yield the same conclusion.

**($\Sigma$INT)** One can choose a subset $S_1$ of $\{1, ..., n+3\}$ such that $\mu_i = \mu_j$ for $i,j \in S_1$ and that, if $\mu_i + \mu_j < 1$ with $i \neq j$, one has $(1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2}\mathbb{Z}$ when $i,j \in S_1$ and $\in \mathbb{Z}$ otherwise.
In particular, taking $S_1$ with $|S_1| = 3$, one obtains $\Gamma_\mu$ commensurable to a lattice generated by 3 reflections, including all lattices constructed in [M 80].

In [M 88] Mostow showed further that the converse of the above result is also true in the following sense. First, all $\Gamma_\mu$ which is discrete is a lattice in $PU(n, 1)$ (Prop. 5.3) and if $n > 3$ the condition (SINT) is necessarily satisfied (Th. 4.13). For $n = 2, 3$ there are 10 exceptional lattices $\Gamma_\mu$ with $\mu$ not satisfying (SINT). The list of all 94 $\mu$'s satisfying the condition (SINT) is given in [M 88], in which the longest one is $\frac{1}{6}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with $n = 9$.

6. As for the arithmeticity of $\Gamma_\mu$, the following criterion was first given in [DM 86] under the assumption (INT):

(A) Let $d$ be the least common denominator of the $\mu_i$'s. Then, for all $A \in \mathbb{Z}$, $1 < A < d - 1$, $(A, d) = 1$, one has

$$\sum_{i=1}^{n+3} < A\mu_i > = 1 \text{ or } n + 2,$$

where $< x > = x - [x]$ for $x \in \mathbb{R}$, $[x]$ being the symbol of Gauss.

It was finally established in [M 88] (Prop. 5.4) that, without any additional assumption, the condition (A) is necessary and sufficient for $\Gamma_\mu$ to be an arithmetic lattice in $PU(n, 1)$.

Summing up the above results, we obtain the following

**Theorem 3 (Mostow, 1988)** The projective monodromy group $\Gamma_\mu$ is a lattice in $PU(n, 1)$ if and only if the condition (SINT) is satisfied, except for the 10 exceptional lattices $\Gamma_\mu$ with $n = 2, 3$ not satisfying the condition (SINT).

The group $\Gamma_\mu$ is an arithmetic lattice (in a wider sense) if and only if the condition (A) is satisfied.

In the list of the $\mu$'s satisfying (SINT) in [M 88], those giving non-arithmetic lattices are marked as NA. (However, this list still seems containing some misprints and erroneous markings.) We give below a (corrected) list of non-arithmetic lattices $\Gamma_\mu$ in $PU(n, 1)$, in which the numbering of the $\mu$'s is the one given in [M 88].
List of non-arithmetic lattices $\Gamma_{\mu}$ in $PU(n, 1)$

$n = 3$

| $39P$ | $\frac{1}{12}(3, 3, 3, 5, 7)$ |

$n = 2$

| $69P$ | $\frac{1}{12}(3, 3, 3, 7, 8)$ | [4, 1/12] NA1 |
| $71P$ | $\frac{1}{12}(3, 3, 5, 6, 7)$ | (not uniform) NA2 |
| $73P$ | $\frac{1}{12}(4, 4, 4, 5, 7)$ | [6, 1/6] NA3 |
| $74P$ | $\frac{1}{12}(4, 4, 5, 5, 6)$ | NA1 |
| $78P$ | $\frac{1}{12}(4, 6, 6, 6, 8)$ | [10, 4/15] NA4 |
| $80$ | $\frac{1}{12}(2, 7, 7, 7, 13)$ | [9, 11/18] NA5 |
| $D7$ | $\frac{1}{12}(4, 5, 5, 11, 11)$ | NA5 |
| $84$ | $\frac{1}{12}(7, 7, 7, 7, 8)$ | NA5 |
| $85P$ | $\frac{1}{20}(5, 5, 5, 11, 14)$ | [4, 3/20] NA6 |
| $86$ | $\frac{1}{20}(6, 6, 6, 9, 13)$ | [5, 1/5] NA7 |
| $87$ | $\frac{1}{20}(6, 6, 9, 9, 10)$ | NA6 |
| $D8$ | $\frac{1}{24}(4, 8, 10, 10, 10)$ | NA9 |
| $88$ | $\frac{1}{24}(4, 4, 4, 17, 19)$ | [3, 1/12] NA8 |
| $D9$ | $\frac{1}{24}(5, 10, 11, 11, 11)$ | NA8 |
| $89P$ | $\frac{1}{24}(7, 9, 9, 9, 14)$ | [8, 7/24] NA8 |
| $91$ | $\frac{1}{30}(5, 5, 5, 22, 23)$ | [3, 1/30] NA4 |
| $D10$ | $\frac{1}{30}(7, 13, 13, 13, 14)$ | NA4 |
| $93$ | $\frac{1}{42}(7, 7, 7, 29, 34)$ | [3, 5/42] NA9 |
| $94$ | $\frac{1}{42}(13, 15, 15, 15, 26)$ | [7, 13/42] NA9 |

**Remark 1.** "$P$" indicates a **Picard lattice**, i.e. a lattice satisfying (INT). "$D$" indicates an exceptional lattice, i.e. a lattice not satisfying (EINT). For $n = 2$, there are 54 lattices (41–94) satisfying (EINT) (including 27 Picard lattices) and 9 exceptional lattices ($D2–D10$).

**Remark 2.** $\Gamma_{\mu}$ with $\mu = (\mu_1, ..., \mu_5)$, $S_1 = \{\mu_1, \mu_2, \mu_3\}$, $\mu_4 \leq \mu_5$ is commensurable with a reflection group with $[p,t]$, where $p = 2(1 - 2\mu_1)^{-1}$, $t = \mu_5 - \mu_4$.

7. We say that two subgroups $\Gamma$ and $\Gamma'$ of $G$ are **conjugate commensurable** if $\Gamma$ is commensurable with a conjugate of $\Gamma'$. This kind of relations between the $\Gamma_{\mu}$'s was studied in [M 88], [DM 93]. Some of their results are listed
below, where we write $\mu \approx \mu'$ if $\Gamma_\mu$ is conjugate commensurable with $\Gamma_{\mu'}$. It turns out that the 19 non-arithmetic lattices $\Gamma_\mu$ for $n = 2$ are divided into 9 conjugate commensurability classes (NA1–NA9).

It is still an open problem to decide whether or not there exist non-arithmetic lattices not conjugate commensurable to any of $\Gamma_\mu$, especially such lattices for $n \geq 4$. It would also be interesting to study the arithmetic properties of the non-arithmetic lattices $\Gamma_\mu$, e.g., the corresponding automorphic representations.

(A) ([DM 93], §10) For $a, b > 0$, $1/2 < a + b < 1$, one has

$$(a, a, b, b, 2 - 2a - 2b) \approx (1 - b, 1 - a, a + b - \frac{1}{2}, a + b - \frac{1}{2}, 1 - a - b).$$

In particular, for $a = b$,

$$(a, a, a, 2 - 4a) \approx (1 - a, 1 - a, 2a - \frac{1}{2}, 2a - \frac{1}{2}, 1 - 2a)$$

$$(\frac{3}{2} - 2a, a, a, \frac{1}{2} - a).$$

Example. 

$$\frac{1}{18}(7, 7, 7, 7, 8) \approx \frac{1}{18}(11, 11, 5, 5, 4) \approx \frac{1}{18}(13, 7, 7, 2)$$

(i.e., $84 \approx D7 \approx 80$).

For $a + b = 3/4$,

$$(a, a, b, b, \frac{1}{2}) \approx (1 - b, 1 - a, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).$$

Examples.

$$\frac{1}{12}(4, 4, 5, 5, 6) \approx \frac{1}{12}(7, 8, 3, 3, 3)$$

(i.e., $74 \approx 69$),

$$\frac{1}{20}(6, 6, 9, 9, 10) \approx \frac{1}{20}(11, 14, 5, 5, 5)$$

(i.e., $87 \approx 85$).

(B) For $\pi, \rho, \sigma$ with $1/\pi + 1/\rho + 1/\sigma = 1/2$, set

$$\mu(\pi, \rho, \sigma) = \left(\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\rho}, \frac{1}{2} + \frac{1}{\rho}, \frac{1}{2} - \frac{1}{\sigma}, \frac{1}{2} - \frac{1}{\sigma}, \frac{1}{2} + \frac{1}{\sigma}, \frac{1}{2} - \frac{1}{\sigma}\right).$$
Then ([M 88], Th. 5.6) for $1/\rho + 1/\sigma = 1/6$, one has

$$
\mu(3, \rho, \sigma) \approx \mu(\rho, 3, \sigma) \approx \mu(\sigma, 3, \rho).
$$

**Examples.**

- $\rho = 10, \sigma = 15$:
  
  $$
  \frac{1}{30}(5, 5, 5, 22, 23) \approx \frac{1}{15}(6, 6, 6, 4, 8) \approx \frac{1}{30}(13, 13, 13, 7, 14)
  $$
  
  (i.e., $91 \approx 78 \approx D10$),

- $\rho = 8, \sigma = 24$:
  
  $$
  \frac{1}{24}(4, 4, 4, 17, 19) \approx \frac{1}{24}(9, 9, 9, 7, 14) \approx \frac{1}{24}(11, 11, 11, 5, 10)
  $$
  
  (i.e., $88 \approx 89 \approx D9$),

- $\rho = 7, \sigma = 42$:
  
  $$
  \frac{1}{42}(7, 7, 7, 29, 34) \approx \frac{1}{42}(15, 15, 15, 13, 26) \approx \frac{1}{21}(10, 10, 10, 4, 8)
  $$
  
  (i.e., $93 \approx 94 \approx D8$).

**References**


