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Kyoto University
On non-arithmetic discontinuous groups

佐武 一郎 (Ichiro Satake)

In this talk, we will give a survey on arithmetic and non-arithmetic lattices in a semisimple algebraic group. After giving some basic results on the subject, we'll focus our attention to more recent results, mainly due to Mostow and Deligne, on non-arithmetic lattices in the (projective) unitary group $PU(n,1)$ ($n \geq 2$). (For more details on these topics as well as the closely related rigidities of lattices, see [S 04]).

1. To begin with, we first fix our settings, giving basic definitions and notations. Let $X$ denote a symmetric Riemannian space of non-compact type (with no flat or compact factors) and let $G = I(X)^{o}$ be the identity connected component of the isometry group of $X$. Then, as is well known, $G$ is a connected semisimple Lie group of non-compact type, which is of adjoint type, i.e., with the center reduced to the identity 1. This implies that, denoting by $g$ the Lie algebra of $G$, one has $G = (\text{Aut } g)^{o}$ ($^{o}$ denoting always the identity connected component). The group $G$ acts transitively on $X$ and for any $x_{0} \in X$ the stabilizer $K = Gx_{0}$ is a maximal compact subgroup; thus one has $X \cong G/K$. In this manner, $G$ and $X$ determine one another uniquely (up to isomorphisms).

More generally, let $G'$ denote a connected semisimple linear Lie group, which becomes automatically "real algebraic" in the sense that there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbb{R}$ (uniquely determined up to $\mathbb{R}$-isomorphisms) such that $G' = \mathcal{G}(\mathbb{R})^{o}$. As typical examples, one has $G' = SL(n, \mathbb{R}), SO(p, q)^{o}$, etc. Let $K'$ be a maximal compact subgroup of $G'$, and $K'_{0}$ the maximal compact normal subgroup of $G'$. Then one has

$$G' \supset K' \supset K'_{0} \supset (\text{center of } G').$$

Therefore, setting

$$G = G'/K'_{0}, \quad K = K'/K'_{0}, \quad X = G/K = G'/K',$$

one obtains a pair $(G, X)$ as described in the beginning; in particular, one has $G = G'$ if $K'_{0}$ reduces to the identity group $\{1\}$. We keep these notations throughout the paper.

When $G' = \mathcal{G}(\mathbb{R})^{o}$, the common dimension $r$ of the maximal $\mathbb{R}$-split tori in $\mathcal{G}$ is called the $\mathbb{R}$-rank of $G'$ and written as $r = \mathbb{R}$-rank $G'$. It is well known that, if $g' = k' + p'$ is a Cartan decomposition of $g' = \text{Lie } G'$, then
$r$ coincides with the maximal dimension of the (abelian) subalgebras of $g'$ contained in $\mathfrak{p}'$. Thus one has $R$-rank $G' = R$-rank $G$.

When the algebraic group $G$ is defined over $\mathbb{Q}$, $G'$ is said to have a $\mathbb{Q}$-structure and the $\mathbb{Q}$-rank of $G'$ (with this $\mathbb{Q}$-structure) is the common dimension $r_0$ of the maximal $\mathbb{Q}$-split tori in $G$. $G'$ is called $\mathbb{Q}$-\textit{anisotropic} when $r_0 = 0$.

2. A subgroup $\Gamma$ of $G'$ is called a \textit{lattice} in $G'$ if $\Gamma$ is discrete and the covolume $\text{vol}(\Gamma \backslash G')$ (with respect to the Haar measure of $G'$) is finite. A lattice $\Gamma$ is called \textit{uniform} if, in particular, the quotient space $\Gamma \backslash G'$ is compact.

Two subgroups $\Gamma$ and $\Gamma'$ of $G'$ are said to be \textit{commensurable} if the indices $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite, and one then writes $\Gamma \sim \Gamma'$. As is easily seen, this is an equivalence relation.

A lattice $\Gamma$ in $G$ is said to be \textit{reducible} if there exists a non-trivial direct decomposition $G = G_1 \times G_2$ such that $\Gamma \sim (\Gamma \cap G_1) \times (\Gamma \cap G_2)$; otherwise, $\Gamma$ is called \textit{irreducible}. Every lattice in $G$ is commensurable to the direct product of irreducible ones in the direct factors of $G$.

When $G' = G(\mathbb{R})^o$ is given a $\mathbb{Q}$-structure, a subgroup $\Gamma$ of $G'$ commensurable with $G(\mathbb{Z})$ is called \textit{arithmeti}c; the projection of an arithmetic subgroup of $G'$ in $G = G'/K_0'$ is called \textit{arithmeti}c in a \textit{wider sense}. It is clear that arithmetic subgroups (in a wider sense) are discrete.

The following theorem is fundamental.

\textbf{Theorem 1 (Borel–Harish-Chandra [BHC 62], Mostow–Tamagawa [MT 62])} If $\Gamma$ is an arithmetic subgroup of $G$ in a wider sense, then $\Gamma$ is a lattice in $G$. Moreover, $\Gamma$ is uniform (i.e., cocompact in $G$) if and only if $G'$ is $\mathbb{Q}$-\textit{anisotropic} (i.e., $\mathbb{Q}$-rank $G' = 0$).

Note that, when $\Gamma$ in $G$ is arithmetic only in a wider sense, the $\mathbb{Q}$-rank of $G'$ being $= 0$, $\Gamma$ is uniform. In the early 1960s it was conjectured by Selberg and others that the converse of Theorem 1 would also be true, if the $R$-rank of $G$ is high. Actually, we now have

\textbf{Theorem 2 (Margulis, 1973, [Ma 81])} Suppose that the $R$-rank of $G$ is $\geq 2$. Then any irreducible lattice $\Gamma$ in $G$ is arithmetic in a wider sense (for a certain choice of $G'$ with a $\mathbb{Q}$-structure).

3. Thanks to the above result of Margulis, in order to study the arithmeticity of a lattice $\Gamma$, we may restrict ourselves to the case $R$-rank $G = 1$, which
naturally implies that \( G \) is \( \mathbb{R} \)-simple. According to the classification of \( \mathbb{R} \)-simple Lie groups (due to E. Cartan), we have only the following possibilities for \((G, X)\):

\[
G = PU(D; n, 1)^o = U(D; n, 1)^o/(\text{center}), \quad n \geq 2, \quad (n = 2 \text{ for } D = \mathbb{O}),
\]

\[
X = \mathbb{H}_D^n \text{ (the hyperbolic } n\text{-space over } D),
\]

\( D \) denoting a division composition algebra over \( \mathbb{R} \), i.e.,

\( D = \mathbb{R}, \mathbb{C}, \mathbb{H} \) (Hamilton's quaternions), \( \mathbb{O} \) (Cayley's octonions),

and \( U(D; n, 1) \) denoting the unitary group of the standard \( D \)-hermitian form of signature \((n, 1)\). In the case \( D = \mathbb{O} \), which is non-associative, the projective unitary group is defined to be the automorphism group of the (split) exceptional Jordan algebra \( \text{Her}_3(\mathbb{O}; 2, 1) \); hence \( G \) is of type \( F_{4,1} \).

For \( D = \mathbb{R} \), one has \( G = SO(n, 1)^o \) (Lorentz group) and \( X = \mathbb{H}_R^n \) is the "Lobachevsky space", i.e., the Riemannian \( n \)-space of constant curvature \( \kappa = -1 \), which can be realized by the hyperbolic hypersurface in \( \mathbb{R}^{n+1} \) (with the Lorentz metric):

\[
\{(x_i) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, \quad x_{n+1} > 0 \}.
\]

In particular, \( \mathbb{H}_R^2 (= \mathbb{H}_C^1) \) can be identified with the upper half-plane in \( \mathbb{C} \) and the lattices in \( G = SO(2, 1)^o(\cong SL(2, \mathbb{R})/\{\pm 1\}) \) are so-called Fuchsian groups. In this case, it is classical that there are continuous families of non-arithmetic lattices.

For \( X = \mathbb{H}_R^n \), \( n \geq 3 \), non-arithmetic lattices, especially reflection groups, have been studied intensively by E. B. Vinberg and his school since 1965 (see e.g., [V 85], [V 90]). More recently, it was shown by Gromov and Piatetski-Shapiro [GPS 88] that for any \( n \geq 2 \) one can construct infinitely many non-arithmetic (uniform) lattices as the fundamental group of the "hybrid" of two quotient spaces \( \Gamma_1 \backslash X \) and \( \Gamma_2 \backslash X \) for non-commensurable arithmetic subgroups \( \Gamma_1 \) and \( \Gamma_2 \) of \( G \).

On the other hand, for the case \( D = \mathbb{H} \) and \( \mathbb{O} \), Corlette [C 92] and Gromov and Schoen [GS 92] have shown that there exist no non-arithmetic lattices in \( G \) by a differential geometric method (harmonic maps), extending the idea of Margulis.

4. In the rest of the paper, we concentrate to the case \( D = \mathbb{C} \), i.e., the case where \( G = PU(n, 1) \) and \( X = \mathbb{H}_C^n \), studied mainly by G. D. Mostow since the early 1970s.
The complex hyperbolic space $\mathbb{H}_c^n$ can be realized by the unit ball in $\mathbb{C}^n$ as follows. The unitary group $U(n, 1)$ acts on $\mathbb{C}^{n+1}$ and hence on the projective space $\mathbb{P}^n(\mathbb{C})=(\mathbb{C}^{n+1}-\{0\})/\mathbb{C}^\times$ in a natural manner. The orbit of $e_{n+1}=(0,\ldots,0,1)$ (mod $\mathbb{C}^\times$) in $\mathbb{P}^n(\mathbb{C})$ is

$$\{z = (z_i) \in \mathbb{C}^{n+1} | \sum_{i=1}^n |z_i|^2 - |z_{n+1}|^2 < 0 \} / \mathbb{C}^\times,$$

which, in the inhomogeneous coordinates $z'_i = z_i/z_{n+1}$ ($1 \leq i \leq n$), is expressed by the unit ball

$$\{z' = (z'_i) \in \mathbb{C}^n | \sum_{i=1}^n |z'_i|^2 < 1 \}.$$

The stabilizer of $e_{n+1}$ in $U(n, 1)$ is $U(n) \times U(1)$. Hence $\mathbb{H}_c^n = U(n, 1)/U(n) \times U(1)$ is identified with the unit ball in $\mathbb{C}^n$, on which $G = PU(n, 1)$ acts as linear fractional transformations.

We denote by $<>$ the standard hermitian inner product of signature $(n,1)$ on $\mathbb{C}^{n+1}$. For $a \in \mathbb{C}^{n+1}$, $<a,a> > 0$ and $\xi \in \mathbb{C}$, $|\xi|=1$, we define (after Mostow) a "complex reflection" on $\mathbb{C}^{n+1}$ by

$$R'_{a,\xi} : z \mapsto z + (\xi - 1) \frac{<a,z>}{<a,a>} a \quad (z \in \mathbb{C}^{n+1}).$$

Then, for $\xi, \eta \in \mathbb{C}$, $|\xi| = |\eta| = 1$, one has

$$R'_{a,\xi} \circ R'_{a,\eta} = R'_{a,\xi\eta};$$

in particular, if $\xi$ is a root of unity: $\xi^m = 1$, then one has $(R'_{a,\xi})^m = 1$. We denote the image of $R'_{a,\xi}$ in $G = PU(n, 1)$ by $R_{a,\xi}$.

In [M 80] Mostow studied the groups

$$\Gamma = < R_{e_i,\zeta^p} \ (i = 1, 2, 3) >$$

generated by 3 reflections, where $\zeta_p = e^{2\pi i/p}$ with $p = 3$ or 4 or 5 and

$$e_i \in \mathbb{C}^{n+1}, \ <e_i, e_i> = 1, \ <e_1, e_2> = <e_2, e_3> = <e_3, e_1> = -\alpha \varphi,$$

$$\alpha = (2 \sin \frac{\pi}{p})^{-1}, \ \varphi = e^{\pi i/3}$$

with $t \in \mathbb{R}$. Mostow gave a criterion for $\Gamma$ to be a lattice in $G$, and found 17 cases, showing that 7 among them are non-arithmetic (i.e., not arithmetic in a wider sense). The non-arithmetic cases are given by

$$[p, t] = [3, 1/12], [3, 1/30], [3, 5/42], [4, 1/12], [4, 3/20],$$
(It has turned out that actually the $\Gamma$ corresponding to [5, 11/30] is arithmetic.)

5. Mostow then studied, in collaboration with Deligne, the analytic construction of lattices in $PU(n, 1)$. They consider a system of differential equations of Fuchsian type in $n$ variables, studied for $n=2$ by Picard and in general by Lauricella (1893). The solution space of such equations is $\cong C^{n+1}$, spanned by the period integrals generalizing the classical Euler integral:

$$F_{g,h}(x_1, ..., x_n) = \int_{g}^{h} \prod_{i=1}^{n} (u-x_i)^{-\mu_i} \cdot (u-1)^{-\mu_{n+1} + \mu_{n+2}} du,$$

where

$$\mu = (\mu_1, ..., \mu_{n+3}) \in C^{n+3}, \quad \mu_{n+3} = 2 - \sum_{i=1}^{n+2} \mu_i$$

is the parameter, which we will restrict to the so-called "disc $(n+3)$-tuple" satisfying the condition $0 < \mu_i < 1$ ($1 \leq i \leq n+3$), and

$$g, h \in M = \{ x = (x_1, ..., x_n, 0, 1, \infty) \mid \} x_i \in \mathbb{C} - \{0, 1\}, \quad x_i \neq x_j \text{ for } i \neq j \}.$$ 

Let $\hat{M}$ be the universal covering space of $M$. Then there exists a natural map from $\hat{M}$ to $\mathbb{P}^n(\mathbb{C})$, the space of non-zero solutions modulo $\mathbb{C}^\times$, which is equivariant with respect to the actions of the fundamental group on $\hat{M}$ and the projective monodromy group, denoted by $\Gamma_\mu$, on $\mathbb{P}^n(\mathbb{C})$. It is also shown that there exists a hermitian inner product of signature $(n, 1)$ on the solution space such that $\Gamma_\mu$ is in $PU(n, 1)$.

In [DM 86] it was shown that the following condition (INT) is sufficient for $\Gamma_\mu$ to be a lattice in $G = PU(n, 1)$.

(INT) If $\mu_i + \mu_j < 1$ with $i \neq j$, then one has $(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z}$.

Actually, for $n = 2$, this condition is equivalent to the one given by Picard in 1885, so that the 27 lattices obtained in this manner are called "Picard lattices". (In counting the lattices $\Gamma_\mu$ we disregard the order of $\mu_i$'s because it is not essential.) There are 9 more $\mu$'s satisfying the condition (INT) for $3 \leq n \leq 5$, the longest one being $\frac{1}{4}$ $(1, 1, 1, 1, 1, 1, 1)$.

In [M 86] Mostow showed that the following weaker condition ($\Sigma$INT) is sufficient to yield the same conclusion.

($\Sigma$INT) One can choose a subset $S_1$ of $\{1, ..., n+3\}$ such that $\mu_i = \mu_j$ for $i, j \in S_1$ and that, if $\mu_i + \mu_j < 1$ with $i \neq j$, one has $(1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2} \mathbb{Z}$ when $i, j \in S_1$ and $\in \mathbb{Z}$ otherwise.
In particular, taking $S_1$ with $|S_1| = 3$, one obtains $\Gamma_\mu$ commensurable to a lattice generated by 3 reflections, including all lattices constructed in [M 80].

In [M88] Mostow showed further that the converse of the above result is also true in the following sense. First, all $\Gamma_\mu$ which is discrete is a lattice in $PU(n,1)$ (Prop. 5.3) and if $n > 3$ the condition (SINT) is necessarily satisfied (Th. 4.13). For $n = 2, 3$ there are 10 exceptional lattices $\Gamma_\mu$ with $\mu$ not satisfying (SINT). The list of all 94 $\mu$'s satisfying the condition (SINT) is given in [M88], in which the longest one is $\frac{1}{6}(1,1,1,1,1,1,1,1,1,1,1,1)$ with $n = 9$.

6. As for the arithmeticity of $\Gamma_\mu$, the following criterion was first given in [DM 86] under the assumption (INT):

(A) Let $d$ be the least common denominator of the $\mu_i$'s. Then, for all $A \in \mathbb{Z}$, $1 < A < d - 1$, $(A,d) = 1$, one has

$$\sum_{i=1}^{n+3} < A\mu_i > = 1 \text{ or } n + 2,$$

where $< x > = x - [x]$ for $x \in \mathbb{R}$, $[x]$ being the symbol of Gauss.

It was finally established in [M 88] (Prop. 5.4) that, without any additional assumption, the condition (A) is necessary and sufficient for $\Gamma_\mu$ to be an arithmetic lattice in $PU(n,1)$.

Summing up the above results, we obtain the following

**Theorem 3 (Mostow, 1988)** The projective monodromy group $\Gamma_\mu$ is a lattice in $PU(n,1)$ if and only if the condition (SINT) is satisfied, except for the 10 exceptional lattices $\Gamma_\mu$ with $n = 2, 3$ not satisfying the condition (SINT). The group $\Gamma_\mu$ is an arithmetic lattice (in a wider sense) if and only if the condition (A) is satisfied.

In the list of the $\mu$'s satisfying (SINT) in [M 88], those giving non-arithmetic lattices are marked as NA. (However, this list still seems containing some misprints and erroneous markings.) We give below a (corrected) list of non-arithmetic lattices $\Gamma_\mu$ in $PU(n,1)$, in which the numbering of the $\mu$'s is the one given in [M 88].
List of non-arithmetic lattices $\Gamma_\mu$ in $PU(n, 1)$

$n = 3$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\frac{1}{12}(3, 3, 3, 3, 5, 7)$</th>
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<tbody>
<tr>
<td>$39P$</td>
<td>[4, 1/12] NA1</td>
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$n = 2$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\frac{1}{12}(3, 3, 3, 7, 8)$</th>
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<tbody>
<tr>
<td>$69P$</td>
<td>[4, 1/12] (not uniform) NA2</td>
<td></td>
</tr>
<tr>
<td>$71P$</td>
<td>$\frac{1}{12}(4, 4, 4, 5, 7)$</td>
<td>[6, 1/6] NA3</td>
</tr>
<tr>
<td>$73P$</td>
<td>$\frac{1}{12}(4, 4, 5, 5, 6)$</td>
<td>NA1</td>
</tr>
<tr>
<td>$74P$</td>
<td>$\frac{1}{12}(4, 4, 5, 5, 6)$</td>
<td>NA1</td>
</tr>
<tr>
<td>$78P$</td>
<td>$\frac{1}{12}(4, 6, 6, 6, 8)$</td>
<td>[10, 4/15] NA4</td>
</tr>
<tr>
<td>$80$</td>
<td>$\frac{1}{12}(2, 7, 7, 7, 13)$</td>
<td>[9, 11/18] NA5</td>
</tr>
<tr>
<td>$D7$</td>
<td>$\frac{1}{12}(4, 5, 5, 11, 11)$</td>
<td>NA5</td>
</tr>
<tr>
<td>$84$</td>
<td>$\frac{1}{18}(7, 7, 7, 7, 8)$</td>
<td>NA5</td>
</tr>
<tr>
<td>$85P$</td>
<td>$\frac{1}{18}(5, 5, 5, 11, 14)$</td>
<td>[4, 3/20] NA6</td>
</tr>
<tr>
<td>$86$</td>
<td>$\frac{1}{20}(6, 6, 6, 9, 13)$</td>
<td>[5, 1/5] NA7</td>
</tr>
<tr>
<td>$87$</td>
<td>$\frac{1}{20}(6, 6, 9, 9, 10)$</td>
<td>NA6</td>
</tr>
<tr>
<td>$D8$</td>
<td>$\frac{1}{21}(4, 8, 10, 10, 10)$</td>
<td>NA9</td>
</tr>
<tr>
<td>$88$</td>
<td>$\frac{1}{24}(4, 4, 4, 17, 19)$</td>
<td>[3, 1/12] NA8</td>
</tr>
<tr>
<td>$D9$</td>
<td>$\frac{1}{24}(5, 10, 11, 11, 11)$</td>
<td>NA8</td>
</tr>
<tr>
<td>$89P$</td>
<td>$\frac{1}{24}(7, 9, 9, 9, 14)$</td>
<td>[8, 7/24] NA8</td>
</tr>
<tr>
<td>$91$</td>
<td>$\frac{1}{30}(5, 5, 5, 22, 23)$</td>
<td>[3, 1/30] NA4</td>
</tr>
<tr>
<td>$D10$</td>
<td>$\frac{1}{30}(7, 13, 13, 13, 14)$</td>
<td>NA4</td>
</tr>
<tr>
<td>$93$</td>
<td>$\frac{1}{42}(7, 7, 7, 29, 34)$</td>
<td>[3, 5/42] NA9</td>
</tr>
<tr>
<td>$94$</td>
<td>$\frac{1}{42}(13, 15, 15, 15, 26)$</td>
<td>[7, 13/42] NA9</td>
</tr>
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Remark 1. "P" indicates a Picard lattice, i.e. a lattice satisfying (INT). "D" indicates an exceptional lattice, i.e. a lattice not satisfying (ΣINT). For $n = 2$, there are 54 lattices (41–94) satisfying (ΣINT) (including 27 Picard lattices) and 9 exceptional lattices ($D2$–$D10$).

Remark 2. $\Gamma_\mu$ with $\mu = (\mu_1, ..., \mu_5)$, $S_1 = \{\mu_1, \mu_2, \mu_3\}$, $\mu_4 \leq \mu_5$ is commensurable with a reflection group with $[p, t]$, where $p = 2(1 - 2\mu_1)^{-1}$, $t = \mu_5 - \mu_4$.

7. We say that two subgroups $\Gamma$ and $\Gamma'$ of $G$ are conjugate commensurable if $\Gamma$ is commensurable with a conjugate of $\Gamma'$. This kind of relations between the $\Gamma_\mu$'s was studied in [M 88], [DM 93]. Some of their results are listed
below, where we write $\mu \approx \mu'$ if $\Gamma_\mu$ is conjugate commensurable with $\Gamma_{\mu'}$. It turns out that the 19 non-arithmetic lattices $\Gamma_\mu$ for $n = 2$ are divided into 9 conjugate commensurability classes (NA1–NA9).

It is still an open problem to decide whether or not there exist non-arithmetic lattices not conjugate commensurable to any of $\Gamma_\mu$, especially such lattices for $n \geq 4$. It would also be interesting to study the arithmetic properties of the non-arithmetic lattices $\Gamma_\mu$, e.g., the corresponding automorphic representations.

(A) ([DM 93], §10) For $a, b > 0$, $1/2 < a + b < 1$, one has

$$(a, a, b, b, 2 - 2a - 2b) \approx (1 - b, 1 - a, a + b - \frac{1}{2}, a + b - \frac{1}{2}, 1 - a - b).$$

In particular, for $a = b$,

$$(a, a, a, 2 - 4a) \approx (1 - a, 1 - a, 2a - \frac{1}{2}, 2a - \frac{1}{2}, 1 - 2a) \approx (\frac{3}{2} - 2a, a, a, \frac{1}{2} - a).$$

Example.

$$\frac{1}{18}(7, 7, 7, 7, 8) \approx \frac{1}{18}(11, 11, 5, 5, 4) \approx \frac{1}{18}(13, 7, 7, 2)$$

(i.e., $84 \approx D7 \approx 80$).

For $a + b = 3/4$,

$$(a, a, b, b, \frac{1}{2}) \approx (1 - b, 1 - a, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).$$

Examples.

$$\frac{1}{12}(4, 4, 5, 5, 6) \approx \frac{1}{12}(7, 8, 3, 3, 3) \quad (i.e., 74 \approx 69),$$

$$\frac{1}{20}(6, 6, 9, 9, 10) \approx \frac{1}{20}(11, 14, 5, 5, 5) \quad (i.e., 87 \approx 85).$$

(B) For $\pi, \rho, \sigma$ with $1/\pi + 1/\rho + 1/\sigma = 1/2$, set

$$\mu(\pi, \rho, \sigma) = (\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\rho}, \frac{1}{2} - \frac{1}{\rho}, \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\rho} - \frac{1}{\rho} - \frac{1}{\sigma}).$$
Then ([M 88], Th. 5.6) for $1/\rho + 1/\sigma = 1/6$, one has

$$\mu(3, \rho, \sigma) \approx \mu(\rho, 3, \sigma) \approx \mu(\sigma, 3, \rho).$$

**Examples.**

$\rho = 10, \sigma = 15$:

$$\frac{1}{30}(5, 5, 5, 22, 23) \approx \frac{1}{15}(6, 6, 6, 4, 8) \approx \frac{1}{30}(13, 13, 13, 7, 14)$$

(i.e., $91 \approx 78 \approx D10$),

$\rho = 8, \sigma = 24$:

$$\frac{1}{24}(4, 4, 4, 17, 19) \approx \frac{1}{24}(9, 9, 9, 7, 14) \approx \frac{1}{24}(11, 11, 11, 5, 10)$$

(i.e., $88 \approx 89 \approx D9$),

$\rho = 7, \sigma = 42$:

$$\frac{1}{42}(7, 7, 7, 29, 34) \approx \frac{1}{42}(15, 15, 15, 13, 26) \approx \frac{1}{21}(10, 10, 10, 4, 8)$$

(i.e., $93 \approx 94 \approx D8$).

**References**


