<table>
<thead>
<tr>
<th>Title</th>
<th>On non-arithmetic discontinuous groups (Construction of Automorphic Forms and Its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Satake, Ichiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1398: 241-250</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/26021">http://hdl.handle.net/2433/26021</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On non-arithmetic discontinuous groups

佐武一郎 (Ichiro Satake)

In this talk, we will give a survey on arithmetic and non-arithmetic lattices in a semisimple algebraic group. After giving some basic results on the subject, we'll focus our attention to more recent results, mainly due to Mostow and Deligne, on non-arithmetic lattices in the (projective) unitary group $PU(n, 1)$ ($n \geq 2$). (For more details on these topics as well as the closely related rigidities of lattices, see [S 04]).

1. To begin with, we first fix our settings, giving basic definitions and notations. Let $X$ denote a symmetric Riemannian space of non-compact type (with no flat or compact factors) and let $G = I(X)^0$ be the identity connected component of the isometry group of $X$. Then, as is well known, $G$ is a connected semisimple Lie group of non-compact type, which is of adjoint type, i.e., with the center reduced to the identity 1. This implies that, denoting by $\mathfrak{g}$ the Lie algebra of $G$, one has $G = (\text{Aut } \mathfrak{g})^0$ ($^0$ denoting always the identity connected component). The group $G$ acts transitively on $X$ and for any $x_0 \in X$ the stabilizer $K = G_{x_0}$ is a maximal compact subgroup; thus one has $X \cong G/K$. In this manner, $G$ and $X$ determine one another uniquely (up to isomorphisms).

More generally, let $G'$ denote a connected semisimple linear Lie group, which becomes automatically "real algebraic" in the sense that there exists a linear algebraic group $\mathcal{G}$ defined over $\mathbb{R}$ (uniquely determined up to $\mathbb{R}$-isomorphisms) such that $G' = \mathcal{G}(\mathbb{R})^0$. As typical examples, one has $G' = SL(n, \mathbb{R}), SO(p, q)^0$, etc. Let $K'$ be a maximal compact subgroup of $G'$, and $K'_0$ the maximal compact normal subgroup of $G'$. Then one has

$$G' \supset K' \supset K'_0 \supset \text{(center of } G').$$

Therefore, setting

$$G = G'/K'_0, \quad K = K'/K'_0, \quad X = G/K = G'/K',$$

one obtains a pair $(G, X)$ as described in the beginning; in particular, one has $G = G'$ if $K'_0$ reduces to the identity group $\{1\}$. We keep these notations throughout the paper.

When $G' = \mathcal{G}(\mathbb{R})^0$, the common dimension $r$ of the maximal $\mathbb{R}$-split tori in $\mathcal{G}$ is called the $\mathbb{R}$-rank of $G'$ and written as $r = \mathbb{R}$-rank $G'$. It is well known that, if $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ is a Cartan decomposition of $\mathfrak{g}' = \text{Lie } G'$, then
$r$ coincides with the maximal dimension of the (abelian) subalgebras of $g'$ contained in $p'$. Thus one has $R$-rank $G' = R$-rank $G$.

When the algebraic group $G$ is defined over $Q$, $G'$ is said to have a $Q$-structure and the $Q$-rank of $G'$ (with this $Q$-structure) is the common dimension $r_0$ of the maximal $Q$-split tori in $G$. $G'$ is called $Q$-anisotropic when $r_0 = 0$.

2. A subgroup $\Gamma$ of $G'$ is called a lattice in $G'$ if $\Gamma$ is discrete and the covolume $\text{vol}(\Gamma \backslash G')$ (with respect to the Haar measure of $G'$) is finite. A lattice $\Gamma$ is called uniform if, in particular, the quotient space $\Gamma \backslash G'$ is compact.

Two subgroups $\Gamma$ and $\Gamma'$ of $G'$ are said to be commensurable if the indices $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are both finite, and one then writes $\Gamma \sim \Gamma'$. As is easily seen, this is an equivalence relation.

A lattice $\Gamma$ in $G$ is said to be reducible if there exists a non-trivial direct decomposition $G = G_1 \times G_2$ such that $\Gamma \sim (\Gamma \cap G_1) \times (\Gamma \cap G_2)$; otherwise, $\Gamma$ is called irreducible. Every lattice in $G$ is commensurable to the direct product of irreducible ones in the direct factors of $G$.

When $G' = G(R)^o$ is given a $Q$-structure, a subgroup $\Gamma$ of $G'$ commensurable with $G(Z)$ is called arithmetic; the projection of an arithmetic subgroup of $G'$ in $G = G'/K_0'$ is called arithmetic in a wider sense. It is clear that arithmetic subgroups (in a wider sense) are discrete.

The following theorem is fundamental.

Theorem 1 (Borel–Harish-Chandra [BHC 62], Mostow–Tamagawa [MT 62])

If $\Gamma$ is an arithmetic subgroup of $G$ in a wider sense, then $\Gamma$ is a lattice in $G$. Moreover, $\Gamma$ is uniform (i.e., cocompact in $G$) if and only if $G'$ is $Q$-anisotropic (i.e., $Q$-rank $G' = 0$).

Note that, when $\Gamma$ in $G$ is arithmetic only in a wider sense, the $Q$-rank of $G'$ being $= 0$, $\Gamma$ is uniform. In the early 1960s it was conjectured by Selberg and others that the converse of Theorem 1 would also be true, if the $R$-rank of $G$ is high. Actually, we now have

Theorem 2 (Margulis, 1973, [Ma 81]) Suppose that the $R$-rank of $G$ is $\geq 2$. Then any irreducible lattice $\Gamma$ in $G$ is arithmetic in a wider sense (for a certain choice of $G'$ with a $Q$-structure).

3. Thanks to the above result of Margulis, in order to study the arithmeticity of a lattice $\Gamma$, we may restrict ourselves to the case $R$-rank $G = 1$, which
naturally implies that $G$ is $\mathbb{R}$-simple. According to the classification of $\mathbb{R}$-simple Lie groups (due to E. Cartan), we have only the following possibilities for $(G, X)$:

$$G = PU(D; n, 1)^o = U(D; n, 1)^o/\text{(center)}, \quad n \geq 2, \ (n = 2 \text{ for } D = \mathbb{O}),$$

$$X = \mathbb{H}_D^n \quad \text{(the hyperbolic } n\text{-space over } D),$$

$D$ denoting a division composition algebra over $\mathbb{R}$, i.e.,

$$D = \mathbb{R}, \ \mathbb{C}, \ \mathbb{H} \ (\text{Hamilton's quaternions}), \ \mathbb{O} \ (\text{Cayley's octonions}),$$

and $U(D; n, 1)$ denoting the unitary group of the standard $D$-hermitian form of signature $(n, 1)$. In the case $D = \mathbb{O}$, which is non-associative, the projective unitary group is defined to be the automorphism group of the (split) exceptional Jordan algebra $\text{He}_3(\mathbb{O}; 2, 1)$; hence $G$ is of type $F_{4,1}$.

For $D = \mathbb{R}$, one has $G = SO(n, 1)^o$ (Lorentz group) and $X = \mathbb{H}_R^n$ is the "Lobachevsky space", i.e., the Riemannian $n$-space of constant curvature $\kappa = -1$, which can be realized by the hyperbolic hypersurface in $\mathbb{R}^{n+1}$ (with the Lorentz metric):

$$\{(x_i) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, \ x_{n+1} > 0 \}.$$

In particular, $\mathbb{H}_R^2 (= \mathbb{H}_C^1)$ can be identified with the upper half-plane in $\mathbb{C}$ and the lattices in $G = SO(2, 1)^o(\cong SL(2, \mathbb{R})/\{\pm 1\})$ are so-called Fuchsian groups. In this case, it is classical that there are continuous families of non-arithmetic lattices.

For $X = \mathbb{H}_R^n, \ n \geq 3$, non-arithmetic lattices, especially reflection groups, have been studied intensively by E. B. Vinberg and his school since 1965 (see e.g., [V 85], [V 90]). More recently, it was shown by Gromov and Piatetski-Shapiro [GPS 88] that for any $n \geq 2$ one can construct infinitely many non-arithmetic (uniform) lattices as the fundamental group of the "hybrid" of two quotient spaces $\Gamma_1 \backslash X$ and $\Gamma_2 \backslash X$ for non-commensurable arithmetic subgroups $\Gamma_1$ and $\Gamma_2$ of $G$.

On the other hand, for the case $D = \mathbb{H}$ and $\mathbb{O}$, Corlette [C 92] and Gromov and Schoen [GS 92] have shown that there exist no non-arithmetic lattices in $G$ by a differential geometric method (harmonic maps), extending the idea of Margulis.

4. In the rest of the paper, we concentrate to the case $D = \mathbb{C}$, i.e., the case where $G = PU(n, 1)$ and $X = \mathbb{H}_C^n$, studied mainly by G. D. Mostow since the early 1970s.
The complex hyperbolic space $\mathbb{H}_C^n$ can be realized by the unit ball in $\mathbb{C}^n$ as follows. The unitary group $U(n, 1)$ acts on $\mathbb{C}^{n+1}$ and hence on the projective space $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$ in a natural manner. The orbit of $e_{n+1} = (0, ..., 0, 1) \pmod{\mathbb{C}^\times}$ in $\mathbb{P}^n(\mathbb{C})$ is

$$\{z = (z_i) \in \mathbb{C}^{n+1} | \sum_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2 < 0 \}/\mathbb{C}^\times,$$

which, in the inhomogeneous coordinates $z'_i = z_i/z_{n+1}$ ($1 \leq i \leq n$), is expressed by the unit ball

$$\{z' = (z'_i) \in \mathbb{C}^n | \sum_{i=1}^{n} |z'_i|^2 < 1 \}.$$

The stabilizer of $e_{n+1}$ in $U(n, 1)$ is $U(n) \times U(1)$. Hence $\mathbb{H}_C^n = U(n, 1)/U(n) \times U(1)$ is identified with the unit ball in $\mathbb{C}^n$, on which $G = PU(n, 1)$ acts as linear fractional transformations.

We denote by $\langle , \rangle$ the standard hermitian inner product of signature $(n, 1)$ on $\mathbb{C}^{n+1}$. For $a \in \mathbb{C}^{n+1}$, $\langle a, a \rangle > 0$ and $\xi \in \mathbb{C}$, $|\xi| = 1$, we define (after Mostow) a "complex reflection" on $\mathbb{C}^{n+1}$ by

$$R'_{a,\xi} : z \mapsto z + (\xi - 1)\frac{\langle a, z \rangle}{\langle a, a \rangle}a \quad (z \in \mathbb{C}^{n+1}).$$

Then, for $\xi, \eta \in \mathbb{C}$, $|\xi| = |\eta| = 1$, one has

$$R'_{a,\xi} \circ R'_{a,\eta} = R'_{a,\xi \eta};$$

in particular, if $\xi$ is a root of unity: $\xi^m = 1$, then one has $(R'_{a,\xi})^m = 1$. We denote the image of $R'_{a,\xi}$ in $G = PU(n, 1)$ by $R_{a,\xi}$.

In [M 80] Mostow studied the groups

$$\Gamma = \langle R_{e_i, \zeta_p} \mid (i = 1, 2, 3) \rangle$$

generated by 3 reflections, where $\zeta_p = e^{2\pi i/p}$ with $p = 3$ or 4 or 5 and

$$e_i \in \mathbb{C}^{n+1}, \quad \langle e_i, e_i \rangle = 1, \quad \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = -\alpha \varphi,$$

$$\alpha = (2 \sin \frac{\pi}{p})^{-1}, \quad \varphi = e^{\pi i/3}$$

with $t \in \mathbb{R}$. Mostow gave a criterion for $\Gamma$ to be a lattice in $G$, and found 17 cases, showing that 7 among them are non-arithmetic (i.e., not arithmetic in a wider sense). The non-arithmetic cases are given by

$$[p, t] = [3, 1/12], [3, 1/30], [3, 5/42], [4, 1/12], [4, 3/20],$$
[5, 1/5], [5, 11/30].

(It has turned out that actually the $\Gamma$ corresponding to [5, 11/30] is arithmetic.)

5. Mostow then studied, in collaboration with Deligne, the analytic construction of lattices in $PU(n, 1)$. They consider a system of differential equations of Fuchsian type in $n$ variables, studied for $n = 2$ by Picard and in general by Lauricella (1893). The solution space of such equations is $\cong \mathbb{C}^{n+1}$, spanned by the period integrals generalizing the classical Euler integral:

$$F_{g,h}(x_1, \ldots, x_n) = \int_{g}^{h} \prod_{i=1}^{n} (u - x_i)^{-\mu_i} \cdot u^{-\mu_{n+1}} (u - 1)^{-\mu_{n+2}} \, du,$$

where

$$\mu = (\mu_1, \ldots, \mu_{n+3}) \in \mathbb{C}^{n+3}, \quad \mu_{n+3} = 2 - \sum_{i=1}^{n+2} \mu_i$$

is the parameter, which we will restrict to the so-called "disc $(n+3)$-tuple" satisfying the condition $0 < \mu_i < 1$ ($1 \leq i \leq n + 3$), and

$$g, h \in M = \{x = (x_1, \ldots, x_n, 0, 1, \infty) | x_i \in \mathbb{C} - \{0, 1\}, \, x_i \neq x_j \text{ for } i \neq j\}.$$ 

Let $\hat{M}$ be the universal covering space of $M$. Then there exists a natural map from $\hat{M}$ to $\mathbb{P}^n(\mathbb{C})$, the space of non-zero solutions modulo $\mathbb{C}^\times$, which is equivariant with respect to the actions of the fundamental group on $\hat{M}$ and the projective monodromy group, denoted by $\Gamma_\mu$, on $\mathbb{P}^n(\mathbb{C})$. It is also shown that there exists a hermitian inner product of signature $(n, 1)$ on the solution space such that $\Gamma_\mu$ is in $PU(n, 1)$.

In [DM 86] it was shown that the following condition (INT) is sufficient for $\Gamma_\mu$ to be a lattice in $G = PU(n, 1)$.

**(INT)** If $\mu_i + \mu_j < 1$ with $i \neq j$, then one has $(1 - \mu_i - \mu_j)^{-1} \in \mathbb{Z}$.

Actually, for $n = 2$, this condition is equivalent to the one given by Picard in 1885, so that the 27 lattices obtained in this manner are called "Picard lattices". (In counting the lattices $\Gamma_\mu$ we disregard the order of $\mu_i$'s because it is not essential.) There are 9 more $\mu$'s satisfying the condition (INT) for $3 \leq n \leq 5$, the longest one being \(14\) \((1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\).

In [M 86] Mostow showed that the following weaker condition ($\Sigma$INT) is sufficient to yield the same conclusion.

**($\Sigma$INT)** One can choose a subset $S_1$ of $\{1, \ldots, n+3\}$ such that $\mu_i = \mu_j$ for $i, j \in S_1$ and that, if $\mu_i + \mu_j < 1$ with $i \neq j$, one has $(1 - \mu_i - \mu_j)^{-1} \in \frac{1}{2}\mathbb{Z}$ when $i, j \in S_1$ and $\in \mathbb{Z}$ otherwise.
In particular, taking $S_1$ with $|S_1| = 3$, one obtains $\Gamma_\mu$ commensurable to a lattice generated by 3 reflections, including all lattices constructed in [M 80].

In [M88] Mostow showed further that the converse of the above result is also true in the following sense. First, all $\Gamma_\mu$ which is discrete is a lattice in $PU(n, 1)$ (Prop. 5.3) and if $n > 3$ the condition (SINT) is necessarily satisfied (Th. 4.13). For $n = 2, 3$ there are 10 exceptional lattices $\Gamma_\mu$ with $\mu$ not satisfying (SINT). The list of all 94 $\mu$'s satisfying the condition (SINT) is given in [M88], in which the longest one is $\frac{1}{6}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with $n = 9$.

6. As for the arithmeticity of $\Gamma_\mu$, the following criterion was first given in [DM 86] under the assumption (INT):

(A) Let $d$ be the least common denominator of the $\mu_i$'s. Then, for all $A \in \mathbb{Z}$, $1 < A < d - 1$, $(A, d) = 1$, one has

$$\sum_{i=1}^{n+3} < A\mu_i > = 1 \text{ or } n + 2,$$

where $< x > = x - [x]$ for $x \in \mathbb{R}$, $[x]$ being the symbol of Gauss.

It was finally established in [M 88] (Prop. 5.4) that, without any additional assumption, the condition (A) is necessary and sufficient for $\Gamma_\mu$ to be an arithmetic lattice in $PU(n, 1)$.

Summing up the above results, we obtain the following

**Theorem 3 (Mostow, 1988)** The projective monodromy group $\Gamma_\mu$ is a lattice in $PU(n, 1)$ if and only if the condition (SINT) is satisfied, except for the 10 exceptional lattices $\Gamma_\mu$ with $n = 2, 3$ not satisfying the condition (SINT). The group $\Gamma_\mu$ is an arithmetic lattice (in a wider sense) if and only if the condition (A) is satisfied.

In the list of the $\mu$'s satisfying (SINT) in [M 88], those giving non-arithmetic lattices are marked as NA. (However, this list still seems containing some misprints and erroneous markings.) We give below a (corrected) list of non-arithmetic lattices $\Gamma_\mu$ in $PU(n, 1)$, in which the numbering of the $\mu$'s is the one given in [M 88].
List of non-arithmetic lattices \( \Gamma_\mu \) in \( PU(n, 1) \)

\( n = 3 \)

39\( P \) \( \frac{1}{12}(3, 3, 3, 5, 7) \)

\( n = 2 \)

69\( P \) \( \frac{1}{12}(3, 3, 3, 7, 8) \)\[4, 1/12\] NA1

71\( P \) \( \frac{1}{12}(3, 3, 5, 6, 7) \) (not uniform) NA2

73\( P \) \( \frac{1}{12}(4, 4, 4, 5, 7) \)\[6, 1/6\] NA3

74\( P \) \( \frac{1}{12}(4, 4, 5, 5, 6) \) NA1

78\( P \) \( \frac{1}{18}(4, 4, 4, 5, 7) \)\[6, 1/6\] NA1

80 \( \frac{1}{18}(2, 7, 7, 7, 13) \)\[9, 11/18\] NA5

D7 \( \frac{1}{18}(4, 5, 5, 11, 11) \) NA5

84 \( \frac{1}{18}(7, 7, 7, 7, 8) \) NA5

85\( P \) \( \frac{1}{20}(5, 5, 5, 11, 14) \)\[4, 3/20\] NA6

86 \( \frac{1}{20}(6, 6, 6, 9, 13) \)\[5, 1/5\] NA7

87 \( \frac{1}{20}(6, 6, 9, 9, 10) \) NA6

D8 \( \frac{1}{21}(4, 8, 10, 10, 10) \) NA9

88 \( \frac{1}{24}(4, 4, 4, 17, 19) \)\[3, 1/12\] NA8

D9 \( \frac{1}{24}(5, 10, 11, 11, 11) \) NA8

89\( P \) \( \frac{1}{24}(7, 9, 9, 9, 14) \)\[8, 7/24\] NA8

91 \( \frac{1}{30}(5, 5, 5, 22, 23) \)\[3, 1/30\] NA4

D10 \( \frac{1}{30}(7, 13, 13, 13, 14) \) NA4

93 \( \frac{1}{42}(7, 7, 7, 29, 34) \)\[3, 5/42\] NA9

94 \( \frac{1}{42}(13, 15, 15, 15, 26) \)\[7, 13/42\] NA9

Remark 1. "\( P \)" indicates a Picard lattice, i.e. a lattice satisfying (INT). "\( D \)" indicates an exceptional lattice, i.e. a lattice not satisfying (\( \Sigma \)INT). For \( n = 2 \), there are 54 lattices (41–94) satisfying (\( \Sigma \)INT) (including 27 Picard lattices) and 9 exceptional lattices (D2–D10).

Remark 2. \( \Gamma_\mu \) with \( \mu = (\mu_1, \ldots, \mu_5) \), \( S_1 = \{\mu_1, \mu_2, \mu_3\}, \mu_4 \leq \mu_5 \) is commensurable with a reflection group with \([p, t]\), where \( p = 2(1 - 2\mu_1)^{-1}, t = \mu_5 - \mu_4 \).

7. We say that two subgroups \( \Gamma \) and \( \Gamma' \) of \( G \) are conjugate commensurable if \( \Gamma \) is commensurable with a conjugate of \( \Gamma' \). This kind of relations between the \( \Gamma_\mu \)'s was studied in [M 88], [DM 93]. Some of their results are listed
below, where we write $\mu \approx \mu'$ if $\Gamma_\mu$ is conjugate commensurable with $\Gamma_{\mu'}$. It turns out that the 19 non-arithmetic lattices $\Gamma_\mu$ for $n = 2$ are divided into 9 conjugate commensurability classes (NA1–NA9).

It is still an open problem to decide whether or not there exist non-arithmetic lattices not conjugate commensurable to any of $\Gamma_\mu$, especially such lattices for $n \geq 4$. It would also be interesting to study the arithmetic properties of the non-arithmetic lattices $\Gamma_\mu$, e.g., the corresponding automorphic representations.

(A) ([DM 93], §10) For $a, b > 0$, $1/2 < a + b < 1$, one has

$$(a, a, b, b, 2 - 2a - 2b) \approx (1 - b, 1 - a, a + b - \frac{1}{2}, a + b - \frac{1}{2}, 1 - a - b).$$

In particular, for $a = b$,

$$(a, a, a, a, 2 - 4a) \approx (1 - a, 1 - a, 2a - \frac{1}{2}, 2a - \frac{1}{2}, 1 - 2a)$$

$$\approx \left(\frac{3}{2} - 2a, a, a, \frac{1}{2} - a\right).$$

Example.

$$\frac{1}{18}(7, 7, 7, 7, 8) \approx \frac{1}{18}(11, 11, 5, 5, 4) \approx \frac{1}{18}(13, 7, 7, 7, 2)$$

(i.e., $84 \approx D7 \approx 80$).

For $a + b = 3/4$,

$$(a, a, b, b, \frac{1}{2}) \approx (1 - b, 1 - a, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).$$

Examples.

$$\frac{1}{12}(4, 4, 5, 5, 6) \approx \frac{1}{12}(7, 8, 3, 3, 3) \quad (i.e., \; 74 \approx 69),$$

$$\frac{1}{20}(6, 6, 9, 9, 10) \approx \frac{1}{20}(11, 14, 5, 5, 5) \quad (i.e., \; 87 \approx 85).$$

(B) For $\pi, \rho, \sigma$ with $1/\pi + 1/\rho + 1/\sigma = 1/2$, set

$$\mu(\pi, \rho, \sigma) = \left(\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\rho}, \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\rho}, \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\sigma}\right).$$
Then ([M 88], Th. 5.6) for 1/\rho + 1/\sigma = 1/6, one has
\[ \mu(3, \rho, \sigma) \approx \mu(\rho, 3, \sigma) \approx \mu(\sigma, 3, \rho). \]

**Examples.**
\[ \rho = 10, \sigma = 15: \frac{1}{30}(5,5,22,23) \approx \frac{1}{15}(6,6,4,8) \approx \frac{1}{30}(13,13,7,14) \]
(\text{i.e., } 91 \approx 78 \approx D10),
\[ \rho = 8, \sigma = 24: \frac{1}{24}(4,4,17,19) \approx \frac{1}{19}(9,9,7,14) \approx \frac{1}{24}(11,11,5,10) \]
(\text{i.e., } 88 \approx 89 \approx D9),
\[ \rho = 7, \sigma = 42: \frac{1}{42}(7,7,29,34) \approx \frac{1}{21}(15,15,13,26) \approx \frac{1}{21}(10,10,4,8) \]
(\text{i.e., } 93 \approx 94 \approx D8).

**References**


