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Norm Inequalities and Geometry of Banach Spaces

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We consider some generalizations of Clarkson type inequalities in the general Banach space setting, and investigate several geometrical properties of Banach spaces in terms of these inequalities. The obtained results complement, unify and generalize several classical and some recent results of this type.

バナッハ空間の幾何学的性質の起源は，J.A.Clarkson [7] による一様凸性の概念にまで遡る．L_p 空間の一様凸性が Clarkson 不等式を用いて証明されたように，空間の幾何学的性質の多くはノルム不等式を用いて記述される．その後，M.M.Day [11], R.C.James [20,21] などにより様々な幾何学的性質が導入され，それらの特徴づけや相互の関係などが活発に研究された (e.g.,cf.[2,6,12,14,34,35,36,39,43,49,68])．バナッハ空間の幾何学的性質の研究は，解析学の様々な分野において基本的な概念や道具を提供し応用面からも重要である．他方，ベクトル値確率変数の研究 (大数の強法則，中心極限定理等) に関連して導入された概念である Beck の凸性や Rademacher (or stable) type-cotype などは，バナッハ空間上の確率測度の研究のみならず，バナッハ空間の一般論の研究においても不可欠の道具となっている (cf.[2,3,34,37,44,48])．1973 年，P.Enflo [13] は近似の性質 (approximation property) をもたない可分なバナッハ空間の存在を示し，Grothendieck 予想と Banach の問題を (否定的に) 解決したが，その後，任意の部分空間が近似の性質をもつようなバナッハ空間は，type-cotype の観点から眺めると極めてヒルベルト空間に近い性質をもつことが示された (cf.Szankowski [52])．最近では，W.T.Gowers(1998 年フィールズ賞受賞) によりバナッハ空間論の一連の歴史的な問題 (無条件基底列問題，超平面問題，Schroeder-Bernstein 問題，等質バナッハ空間問題等) が解決され，バナッハ空間論の研究は急速な進歩を遂げた (cf.[58,59])．ところで，Clarkson 不等式 (CI) の研究には，簡単な別証明を与えたり，(CI) が成立する具体的な関数空間の例を与えたようなものが多かったが，一般のバナッハ空間 X において (CI) を定式化し，それを type-cotype 不等式との関係で研究したのは M.Milman [39] が最初のようである．(CI) を作用素論的観点から眺めれば 2 次の Littlewood 行列のノルム評価であり，そこから空間の幾何学的性質 (一様凸性等) が導かれる．また，(CI) を確率論的観点から眺めれば 2 要素の type-cotype 不等式であり，そこから空間の type-cotype が導かれる．本小論では，一般のバナッハ空間 X において Clarkson 不等式 (CI) を定式化し，要素数やパラメータに関する (CI) の一般化の考察，それらの不等式を用いたバナッハ空間 X の幾何学的性質の特徴づけ，及び，X の諸性質の Lebesgue-Bochner 空間 $L_p(X)$ への遺伝性などを紹介する (cf.[26,28,29,30,53,54,56,57,63])．また，L_p 空間で知られている一般 Clarkson 不等式 (Kato [23]), Random Clarkson 不等式 (Tonge [67]), type-cotype 不等式などが，一般のバナッハ空間において (CI) から導かれることを示す．これらの考察により，具体的な関数空間 (Sobolev 空間，Besov 空間，Triebel-Sobolev 空間等; e.g., cf.[9,10]) で知られている様々な結果が統一的に導かれると共に，様々な幾何学的性質の新たな特徴づけが得られ，諸性質の相互の関係が明らかになった．なお，ここで紹介する結果の大部分は加藤幹雄氏 (九州工業大学) との共同研究によるものである．

1. Clarkson inequalities *Let $1 < p \leq 2$ and $1/p + 1/p' = 1$. Then for $X = L_p$ or $L_{p'}$ the following inequalities hold:*

$$(1) \qquad \left(\frac{\|x+y\|^{p'} + \|x-y\|^{p'}}{2}\right)^{1/p'} \leq (\|x\|^p + \|y\|^p)^{1/p} \qquad \forall x, y \in X,$$

$$(2) \qquad \left(\frac{\|x+y\|^{p} + \|x-y\|^{p}}{2}\right)^{1/p} \geq (\|x\|^{p'} + \|y\|^{p'})^{1/p'} \qquad \forall x, y \in X.$$

Clarkson [7] のオリジナルな不等式は見かけ上８つあるが，いずれも上記の２つの不等式から導かれる．(1) から X の一様平滑性が導かれ，(2) から X の一様凸性が導かれるのであるが，実は，(1) と (2) は同値である．しかしながら，これら２つの不等式は後述の type-cotype 不等式との関係で，それぞれの意味をもつと考えられる．一般のバナッハ空間 X において，(1) を (p,p')-Clarkson 不等式と言う．特に $p=2$ のとき，$(2,2)$-Clarkson 不等式は中線定理と同値であり，それが成立するのはヒルベルト空間に限る (cf.[22])．なお，$p=1$ のとき，(1) は任意のバナッハ空間で成立する．

以下，X は１次元以上のバナッハ空間とし，X^* はその双対空間 (dual space) を表すものとする．X の単位球面と閉単位球を

$$S_X = \{x \in X; \|x\| = 1\}, \ B_X = \{x \in X; \|x\| \leq 1\}$$

で表す．X の幾何学的性質は，S_X あるいは B_X の形状で決定される．また，$1 \leq p, q, r, \ldots \leq \infty$ に対し，$1/p + 1/p' = 1/q + 1/q' = 1/r + 1/r' = \ldots = 1$ とする．

まず，(p,p')-Clarkson 不等式が成立するバナッハ空間 X に対し，その双対空間 X^*，Lebesgue-Bochner 空間 $L_r(X)$ で同様の不等式を考察する (cf.[29,54])．

2. Proposition *Let $1 < p \leq 2$ and $1/p + 1/p' = 1$. Then, (p,p')-Clarkson inequality holds in X if and only if it holds in X^*.*

この結果から L_p 空間における Clarkson 不等式は，$1 < p \leq 2$ の場合にのみ証明すればよいことが分かる．

3. Theorem *Let $1 \leq p \leq 2$, and let $1 \leq r \leq \infty$. If (p,p')-Clarkson inequality holds in X, then (t,t')-Clarkson inequality holds in any Lebesgue-Bochner space $L_r(X)$, where $t = \min\{p, r, r'\}$.*

4. Corollary *Let $1 \leq t \leq p \leq 2$, and let $p \leq r \leq p'$. If (p,p')-Clarkson inequality holds in X, then (t,t')-Clarkson inequality holds in any Lebesgue-Bochner space $L_r(X)$.*

これらの結果から，L_p 空間，Sobolev 空間 $W_p^k(\Omega)$, Besov 空間 $B_{p,q}$, Triebel-Sobolev 空間 $F_{p,q}$, L_q 値 L_p 空間 $L_p(L_q)$ などで知られている Clarkson 不等式は統一的に導かれる (cf.[7,9,10,40])． 次に，n 要素への拡張である type-cotype 不等式を考察する．

5. Type p and cotype q inequalities Let $1 < p \leq 2$ and $1/p + 1/q = 1$.

$$(3) \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^q\right)^{1/q} \leq M\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p} \qquad \forall x_1, \cdots, x_n \in X,$$

$$(4) \qquad \left(\frac{1}{2^n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{nq}\theta_j x_j\right\|^p\right)^{1/p} \geq \frac{1}{M}\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{1/q} \qquad \forall x_1, \cdots, x_n \in X.$$

X is called of type p, resp., cotype q if there is $M > 0$ suct that (3), resp., (4) holds in X. It is known that if (3) holds in X, then (4) holds in X^*, but in general the converse is not true. It is also known that if X is B-convex, then X is of type p if and only if X^* is of cotype q (cf.[2,34,44,48]). Note that any Banach space is of type 1 with $M = 1$, and X is B-convex if and only if it is of type r for some $r > 1$.

6. Theorem *Let $1 < p \leq 2$ and $1/p + 1/p' = 1$. If (p,p')-Clarkson inequality holds in X, then type p and cotype q inequalities (3), (4) holds in X with $M = 1$, where $q = p'$. (In this case, (3) and (4) are equivalent.)*

定理 6 をより厳密に言うと，$M = 1$, $q = p'$ のとき，(1) から (3) が導かれ，(2) から (4) が導かれる．この定理を用いると，具体的な関数空間や数列空間などで type p, cotype q (type-cotype 定数 1) を示したいときは，Clarkson 不等式を証明すればよいことになり，具体的な関数空間や数列空間で調査された type-cotype に関する多くの結果は Clarkson 不等式から導かれることになる．

7. Theorem (Clarkson-Boas-Koskela type inequality) *Let $1 \leq p \leq 2$ and let $1 \leq r, s \leq \infty$. If (p,p')-Clarkson inequality holds in X, then for all $x, y \in X$*

$$(\mathrm{CBK}_X) \qquad (\|x+y\|^s + \|x-y\|^s)^{1/s} \leq 2^{c(r,s;p)}(\|x\|^r + \|y\|^r)^{1/r}$$

holds, where

$$c(r,s;p) = \begin{cases} 1/r' + 1/s - 1/p & \text{if (i) } p \leq r \leq \infty,\ 1 \leq s \leq p', \\ 1/s & \text{if (ii) } 1 \leq r \leq p,\ 1 \leq s \leq r', \\ 1/r' & \text{if (iii) } s' \leq r \leq \infty,\ p' \leq s \leq \infty. \end{cases}$$

定理 7 は Clarkson 不等式のパラメータに関する一般化である．$X = L_p$ or $L_{p'}$ $(1 \leq p \leq 2)$ のとき，部分的な結果は Boas [5], 上記の結果は Koskela [32] が示した．また，パラメータに関する更なる一般化が Maligranda-Persson [35] により得られている．一般のバナッハ空間において，パラメータの更なる一般化も可能である (cf.Kato-Persson-Takahashi [26])．

8. Definition Let $A_n = \{\varepsilon_{ij}\}$ be the *Littlewood matrices,* that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix}.$$

9. Theorem (Generalized Clarkson's inequality) *Let $1 \leq p \leq 2$ and $1 \leq r, s \leq \infty$. If (p,p')-Clarkson inequality holds in X, then for any positive integer n and for all $x_1, x_2, ..., x_{2^n} \in X$,*

$$\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{1/s} \leq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r},$$

where $c(r,s;p)$ is the same as in Theorem 7.

定理 9 は Clarkson 不等式のパラメータ及び要素数に関する究極の一般化で，一般 Clarkson 不等式 (GCI) と呼ばれている．$n=1$ のとき，(GCI) は (CBK_X) と同値である．$X = L_p$ or $L_{p'}$ $(1 \le p \le 2)$ のとき，(GCI) は Kato [23] によって示された．定理 9 は自然数 n に関する帰納法で証明され，$(r,s)=(p,p')$ のときが本質的である (cf.[26])．

定理 9 は，行列のテンソル積のノルム評価に関する次の結果からも証明される (cf.Takahashi-Kato [61])．

10. Theorem *Let X be any Banach space and $1 \le p \le q \le \infty$. Then for any $m \times n$ matrix A and $r \times s$ matrix B*

$$\|A \otimes B : \ell_p^{ns}(X) \to \ell_q^{mr}(X)\| \le \|A : \ell_p^n(X) \to \ell_q^m(X)\| \|B : \ell_p^s(X) \to \ell_q^r(X)\|.$$

Here $A \otimes B$ is a tensor product of matrices A and B, that is,

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \quad (mr \times ns \text{ 行列})$$

G. Bennett [4] はスカラー・ケースの場合に定理 10 を証明した．定理 10 に Littlewood 行列を適用すると定理 9 が証明できる．

11. Theorem (Random Clarkson inequality) *Let $1 \le p \le 2$ and $1 \le r, s \le \infty$. Let n be any positive integer and let $A = (a_{ij})$ be an $n \times n$ matrix whose coefficients are independent identically distributed random variables taking the values ± 1 with equal probability. If (p,p')-Clarkson inequality holds in X, then for all $x_1, x_2, \ldots, x_n inX$*

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \le n^{c(r,s;p)}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r},$$

where $c(r,s;p)$ is the same as in Theorem 7.

定理 11 は定理 6 を用いて証明される (cf.Takahashi-Kato [53], see also [26])．$X = L_p$ or $L_{p'}$ $(1 \le p \le 2)$ のとき，類似の結果が Tonge [67] によって示されたが，その証明には Bennett [4] の結果を用いたため不定の絶対定数が含まれていて不完全なものであった．本定理では，より一般のバナッハ空間で最良の結果を示した．なお，X が type p の仮定のみで，Tonge [67] と同様の結果を示すことが可能である．

12. p-uniformly smooth and q-uniformly convex spaces

A Banach space X is called *q-uniformly convex* $(2 \le q < \infty)$ if there is $C > 0$ such that $\delta_X(\varepsilon) \ge C\varepsilon^q$ for all $0 < \varepsilon < 2$. X is called *p-uniformly smooth* $(1 \le p \le 2)$ if there is $K > 0$ such that $\rho_X(\tau) \le K\tau^p$ for all $\tau > 0$. It is well-known that *X is p-uniformly smooth if and only if X^* is q-uniformly convex*, where $1 < p \le 2$ and $1/p + 1/q = 1$. Note that any Banach space is 1-*uniformly smooth* with $K = 1$. X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$, *uniformly smooth* if $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$, and *uniformly non-square* if $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$. Here, $\delta_X(\varepsilon)$ resp. $\rho_X(\tau)$ is the modulus of convexity, resp., smoothness of X, that is,

$$\delta_X(\varepsilon) := \inf\{1 - \|(x+y)/2\|;\ \|x-y\| \geq \varepsilon,\ x,\ y \in B_X\} \quad (0 < \varepsilon < 2)$$

$$\rho_X(\tau) := \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1;\ \|x\| = 1,\ \|y\| \leq \tau\right\} \quad (\tau > 0)$$

13. Theorem (p-uniform smoothness) *Let $1 < p \leq 2$ and $1 \leq s < \infty$. The following are equivalent.*

(i) X is p-uniformly smooth.

(ii) There exists $K \geq 1$ such that

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|Ky\|^p)^{1/p} \quad \forall x,\ y \in X. \tag{5}$$

If $p \leq s < \infty$, in addition:

(iii) There exists $K \geq 1$ such that

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^n \theta_j x_j\right\|^s\right)^{1/s} \leq \left(\|x_1\|^p + \sum_{j=2}^n \|Kx_j\|^p\right)^{1/p} \quad \forall x_1, \cdots, x_n \in X. \tag{6}$$

Note that if (5) holds in X with $K \geq 1$, then (6) holds in X with the same constant $K \geq 1$.

定理13において，$s = p'$とすると(5)と(6)は同じ定数Kで成立することになり，特に$K = 1$の場合，(5)は(p, p')-Clarkson不等式であり，(6)はtype p不等式(type定数1）である．なお，$1 \leq s < p$であっても(i)と(iii)は同値であるが，このときは(ii)の定数Kと同じものとは限らない(cf.Takahashi-Hashimoto-Kato [63]).

14. Corollary *Let $1 \leq p \leq 2$. If (p, p')-Clarkson inequality holds in X, then (3) holds in X with $M = 1$, that is, X is of type p with type p constant 1.*

15. Theorem (q-uniform convexity) *Let $2 \leq q < \infty$ and $1 < t \leq \infty$. The following are equivalent.*

(i) X is q-uniformly convex.

(ii) There exists $0 < C \leq 1$ such that

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q} \quad \forall x,\ y \in X. \tag{7}$$

If $1 < t \leq q$, in addition:

(iii) There exists $0 < C \leq 1$ such that

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^n \theta_j x_j\right\|^t\right)^{1/t} \geq \left(\|x_1\|^q + \sum_{j=2}^n \|Cx_j\|^q\right)^{1/q} \quad \forall x_1, \cdots, x_n \in X. \tag{8}$$

Note that if (7) holds in X with $0 < C \leq 1$, then (8) holds in X with the same constant $0 < C \leq 1$.

定理 15 において，$t=p, 1/p+1/q=1$ とすると (7) と (8) は同じ定数 C で成立することになり，特に $C=1$ の場合，(7) は (2) と同値であり，(8) はcotype q 不等式 (cotype 定数 1) である．なお，$t>q$ であっても (i) と (iii) は同値であるが，このときは (ii) の定数 C と同じものとは限らない (cf.[63]).

16. Corollary *Let $1 \le p \le 2$. If (p,p')-Clarkson inequality holds in X, then (4) holds in X with $M=1$, where $q=p'$, that is, X is of cotype q with cotype q constant 1.*

17. Theorem (p-uniform smoothness and q-uniform convexity for $L_r(X)$)
(i) Let $1<p\le 2$ and $p \le r \le s$. Then p-uniform smoothness inequality

$$\left(\frac{\|x+y\|^s+\|x-y\|^s}{2}\right)^{1/s} \le \left(\|x\|^p+\|Ky\|^p\right)^{1/p} \quad \forall x,\, y\in X \tag{9}$$

holds in X if and only if it holds in $L_r(X)$ with the same constant $K \ge 1$.
(ii) Let $2\le q<\infty$ and $1<t\le r\le q$. Then q-uniform convexity inequality

$$\left(\frac{\|x+y\|^t+\|x-y\|^t}{2}\right)^{1/t} \ge \left(\|x\|^q+\|Cy\|^q\right)^{1/q} \quad \forall x,\, y\in X \tag{10}$$

holds in X if and only if it holds in $L_r(X)$ with the same constant $0<C\le 1$.

定理 17 において，$s=p', K=1$ とすると，(i) は (p,p')-Clarkson 不等式の Lebesgue-Bochner 空間 $L_r(X)$ への遺伝性を示す (cf. 系 4).

18. Theorem (duality) *Let $1\le p\le 2$, $1<s<\infty$ and $1/p+1/q=1/s+1/t=1$. Let $1\le K<\infty$. Then*

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^s\right)^{1/s} \le \left(\|x_1\|^p+\sum_{j=2}^{n}\|Kx_j\|^p\right)^{1/p} \quad \text{for } X \tag{11}$$

implies

$$\left(\frac{1}{2^n}\sum_{\theta_j=\pm1}\left\|\sum_{j=1}^{n}\theta_j x_j^*\right\|^t\right)^{1/t} \ge \left(\|x_1^*\|^q+\sum_{j=2}^{n}\|K^{-1}x_j^*\|^q\right)^{1/q} \quad \text{for } X^*. \tag{12}$$

If $p\le s<\infty$ the converse is true.

定理 18 において，$p\le s<\infty$ ならば (11), (12) は同じ定数 K で同値である．また，$n=2$ のときは $p>s$ であっても，(11), (12) は同じ定数 K で同値である．なお，(11) は X の p-uniform smoothness, (12) は X^* の q-uniform convexity を特徴づける不等式である．

19. B-convexity and B_n-convexity X is said to be *uniformly non-ℓ_1^n* (or *B_n-convex*) provided there exists ε $(0<\varepsilon<1)$ such that for all $x_1,\dots,x_n\in B_X$ there exist ε_j $(\varepsilon_j=\pm1)$ satisfying

$$\|\varepsilon_1x_1+\dots+\varepsilon_nx_n\|\le n(1-\varepsilon),$$

where B_X denotes the closed unit ball of X. X is called *B-convex* if X is B_n-convex for some n. It is well-known that X is *B-convex* if and only if it is of *type p* for some $p>1$.

20. Theorem *Let R_n be a $2^n \times n$ Rademacher matrix and $1 < p < \infty$. For a Banach space X the following are equivalent:*

(i) *X is B_n-convex.*

(ii) $\|R_n : \ell_p^n(X) \to \ell_p^{2^n}(X)\| < 2^{n/p} n^{1/p'}$.

(iii) *For any (resp. some) r and s with $1 < r \le \infty$, $1 \le s < \infty$*

$$\|R_n : \ell_r^n(X) \to \ell_s^{2^n}(X)\| < 2^{n/s} n^{1/r'}.$$

Here R_n is defined by

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & -R_n \\ -1 & \end{array}\right) \qquad (n = 1, 2 \ldots).$$

21. *J*-convexity, J_n-convexity and super-reflexivity X is called *J_n-convex* provided there exists ε $(0 < \varepsilon < 1)$ such that for all $x_1, \ldots, x_n \in B_X$ there exists k $(1 \le k \le n)$ satisfying

$$\|(x_1 + \ldots + x_k) - (x_{k+1} + \ldots + x_n)\| \le n(1 - \varepsilon)$$

X is called *J-convex* if X is *J_n-convex* for some n. It is clear that X is *uniformly non-squre* if and only if it is *J_2-convex* (*B_2-convex*). It is known that X is *J-convex* if and only if it is *super-reflexive* (James [21]). It is also known that X is *super-reflexive* if and only if X is *uniformly convexifiable*, that is, it admits an equivalent uniformly convex norm (cf. Enflo [12]). Recall that X is *super-reflexive* provided every Banach space Y which is finitely representable in X is reflexive. Here, Y is said to be *finitely representable* (f.r.) *in* X if for any $\lambda > 1$ and for each finite dimensional subspace Y_1 of Y, there exists an isomorphism T of Y_1 into X for which

$$\lambda^{-1}\|x\| \le \|Tx\| \le \lambda\|x\| \quad \forall\, x \in Y_1$$

holds.

22. Theorem *Let A_n be an $n \times n$ admissible matrix and $1 < p < \infty$. For a Banach space X the following are equivalent:*

(i) *X is J_n-convex.*

(ii) $\|A_n : \ell_p^n(X) \to \ell_p^n(X)\| < n$.

(iii) *For any (resp. some) r and s with $1 < r \le \infty$, $1 \le s < \infty$*

$$\|A_n : \ell_r^n(X) \to \ell_s^n(X)\| < n^{1/s + 1/r'}.$$

Here A_n is defined by

$$A_n = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & -1 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

23. Theorem *Let X be a Banach space and $1 < r \le p \le s < \infty$. Then for any Lebesgue-Bochner space $L_p(X)$ and for any $m \times n$ matrix $A = (a_{ij})$, it holds*

$$\|A : \ell_r^n(L_p(X)) \to \ell_s^m(L_p(X))\| \le \|A : \ell_r^n(X) \to \ell_s^m(X)\|.$$

定理 23 はバナッハ空間 X の幾何学的性質の Lebesgue-Bochner 空間 $L_r(X)$ への遺伝性を考察する際極めて有用である．定理 20，定理 22 を用いて次の結果が得られる．

24. Corollary *Let $1 < p < \infty$. Then:*
(i) *The Lebesgue-Bochner space $L_p(X)$ is uniformly non-square if and only if X is.*
(ii) *$L_p(X)$ is B_n-convex if and only if X is.*
(iii) *$L_p(X)$ is J_n-convex if and only if X is.*
(iv) *$L_p(X)$ is B-convex if and only if X is.*
(v) *$L_p(X)$ is super-reflexive if and only if X is .*

ここで紹介した結果は，Clarkson 型不等式の一般化とそれらを用いたバナッハ空間の幾何学的性質の特徴づけに関するものである．最近，加藤幹雄氏（九州工業大学），斎藤吉助氏（新潟大学），高橋眞映氏（山形大学）等との共同研究では，Hanner 型不等式の一般化とそれらを用いた空間の幾何学的性質の特徴づけ，von Neumann-Jordan 定数，James 定数，Schäffer 定数などの一般化とそれらを用いた空間の幾何学的性質の特徴づけ，更には，Hlawka 不等式の一般化や absolute norm などに関する研究も行っている．これらは別の機会に紹介させていただきたい（文献参照）．

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