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# MODIFIED WAVE OPERATORS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH STARK EFFECTS

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## 1. INTRODUCTION

We study the theory of scattering for the nonlinear Schrödinger equation with the Stark effect in one or two space dimensions:

$$i\partial_t u = -\frac{1}{2}\Delta u + (E\cdot x)u + \widetilde{F}_n(u), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

where $n = 1$ or 2, and $u$ is a complex valued unknown function of $(t,x)$. Here $\widetilde{F}_n(u)$ and $E\cdot x$ are a nonlinearity and a linear potential, respectively. The nonlinearity is given by

$$\begin{aligned} \widetilde{F}_n(u) &= G_n(u) + \widetilde{N}_n(u),\\ G_n(u) &= \lambda_0|u|^{2/n}u, \\ \widetilde{N}_1(u) &= \lambda_1 u^3 + \lambda_2\bar{u}^3, \quad \text{when } n=1,\\ \widetilde{N}_2(u) &= \lambda_1 u^2 + \lambda_2\bar{u}^2 + \lambda_3 u\bar{u}, \quad \text{when } n=2, \end{aligned} \tag{1.2}$$

where $\lambda_0\in\mathbb{R}$, $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$ and $E\in\mathbb{R}^n\setminus\{0\}$. We remark that the cubic nonlinearity $u\bar{u}^2$ is excluded in one dimensional case. $\widetilde{F}_n$ is a summation of the gauge invariant nonlinearity $G_n(u)$ and the non-gauge invariant one $\widetilde{N}_n(u)$, and it is a critical power nonlinearity between the short range case and the long range one in $n$ space dimensions $(n=1,2)$. The above potential $E\cdot x$ is called the Stark potential with a constant electric field $E$. Following [9], in this article, we prove the existence of modified wave operators to the equation (1.1) for small final states.

Let $U(t)$ be the free Schrödinger group, that is,

$$U(t) = e^{it\Delta/2}.$$

The Schrödinger operator $-(1/2)\Delta + E\cdot x$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. $H_E$ denotes the self-adjoint realization of that operator defined on $C_0^\infty(\mathbb{R}^n)$ and we define the unitary group $U_E$ generated by $H_E$:

$$U_E(t) = e^{-itH_E}.$$

$\widetilde{F}_n(u)$ is a critical power nonlinearity between the short range scattering and the long range one. The modified wave operator $\widetilde{W}_+$ for the equation (1.1) is defined as follows. Let $\phi$ be a final state. Modifying the solution $U_E(t)\phi$ for the linear Schrödinger equation with the Stark potential, we construct a suitable modified free dynamics $A$, which depends on $\phi$, and we show the existence of a unique solution $u$ for the equation (1.1) which approaches $A$ in $L^2$ as $t \to \infty$. The mapping

$$\widetilde{W}_+ : \phi \mapsto u(0)$$

is called a modified wave operator. In this article, we prove the existence of modified wave operators for the equation (1.1).

The theory of scattering for the ordinary nonlinear Schrödinger equations with critical power nonlinearities was studied, e.g., in [3, 4, 5, 6, 7, 8].

Before stating our main results, we introduce several notations.

**Notation.** We denote the Schwartz space on $\mathbb{R}^n$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^n$. For $w \in \mathcal{S}'$, we denote the Fourier transform of $w$ by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi}\, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space $L^2$ as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' : \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

and put $H^s = H^{s,0}$.

$C$ denotes a constant and so forth. They may differ from line to line, when it does not cause any confusion.

Our result is as follows.

**Theorem 1.1.** *Let $n = 1$ or $2$. Assume that $\phi \in H^2 \cap H^{0,2}$ and that $\|\phi\|_{H^2\cap H^{0,2}}$ is sufficiently small. Then the equation* (1.1) *has a unique solution $u$ satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 1}(t^d \|u(t) - U_E(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi\|_{L^2}) < \infty,$$

$$\sup_{t\geq 1}\left[t^d\left(\int_t^\infty \|U(s)(U_E(-s)u(s) - e^{-i|\cdot|^2/2s}e^{-iS(s,-i\nabla)}\phi)\|_{Y_n}^4\, ds\right)^{1/4}\right] < \infty,$$

*where*

$$S(t,x) = \lambda_0 |\phi(x)|^{2/n} \log t \tag{1.3}$$

*and $d$ is a constant satisfying $n/4 < d < 1$, $Y_1 = L_x^\infty$ and $Y_2 = L_x^4$.*

*Furthermore the modified wave operator $\widetilde{W}_+ \colon \phi \mapsto u(0)$ is well-defined.*

*A similar result holds for negative time.*

**Remark 1.1.** Since the multiplication operator $e^{-i|\cdot|^2/2t}$ converges the identity strongly in $L^2$ as $t \to \infty$, the solution obtained in Theorem 1.1 approaches $U_E(t)e^{-iS(t,-i\nabla)}\phi$ in $L^2$. Noting the phase correction $S$ depends only on the gauge invariant nonlinearity $G_n(u)$, we see that the contribution of the non-gauge invariant term $\widetilde{N}_n(u)$ is a short range interaction, that is, it is negligible as $t \to \infty$, under our assumptions. We also note that the assumption $\phi \in H^2$ is needed only if $\widetilde{N}_n(u) \neq 0$ (see Lemma 3.3 below).

**Remark 1.2.** If we consider the asymptotic behavior of solutions to the Cauchy problem for the equation (1.1) with initial data $u(0,x) = \phi_0(x)$, $x \in \mathbb{R}^n$, then we see from Theorem 1.1 that for any initial data $\phi_0$ belonging to the range of the modified wave operator $\widetilde{W}_+$, there exists a unique global solution $u \in C([0,\infty); L^2)$ of the Cauchy problem for the equation (1.1) which has the modified free profile $U_E(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi$. More precisely, $u$ satisfies the asymptotic formula of Theorem 1.1. However it is not clear how to describe the initial data belonging to the range of the operator $\widetilde{W}_+$.

## 2. The Cauchy Problem at Infinite Initial Time

First we reduce the scattering problem for the equation (1.1) to that of the following non-autonomous nonlinear Schrödinger equation without a potential

$$i\partial_t v = -\frac{1}{2}\Delta v + F_n(t,v), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1}$$

where $n = 1, 2$,

$$F_n(t,v) = G_n(v) + N_n(t,v), \tag{2.2}$$

$$N_1(t,v) = \lambda_1 v^3 e^{-2i(tE\cdot x - t^3|E|^2/3)} + \lambda_3 \bar{v}^3 e^{4i(tE\cdot x - t^3|E|^2/3)}, \tag{2.3}$$

$$\begin{aligned} N_2(t,v) =& \lambda_1 v^2 e^{-i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 \bar{v}^2 e^{3i(tE\cdot x - t^3|E|^2/3)} \\ &+ \lambda_3 v\bar{v} e^{i(tE\cdot x - t^3|E|^2/3)}, \end{aligned} \tag{2.4}$$

$G_n(v)$ is defined by (1.2). By a direct calculation, we obtain the following relation between a solution to the equation (1.1) and that to the equation (2.1). The following proposition is not essentially new but almost well-known (see Cycon, Froese, Kirsch and Simon [2]).

**Proposition 2.1.** *If $v$ solves the equation* (2.1), *then*

$$u(t,x) = v\left(t, x + \frac{t^2}{2}E\right) e^{-i(tE\cdot x + t^3|E|^2/6)}$$

*solves the equation* (1.1).

*Conversely, if $u$ solves the equation* (1.1), *then*

$$v(t,x) = u\left(t, x - \frac{t^2}{2}E\right) e^{i(tE\cdot x - t^3|E|^2/3)}$$

*solves the equation* (2.1).

According to Proposition 2.1, Theorem 1.1 is an immediate consequence of Proposition 2.2 below.

**Proposition 2.2.** *Assume that $\phi$ satisfies all the assumptions of Theorem 1.1. Then there exists a unique solution $v$ for the equation* (2.1) *satisfying*

$$v \in C([0,\infty); L^2),$$

$$\sup_{t\geq 1}\left(t^d \|v(t) - U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi\|_{L^2}\right) < \infty,$$

$$\sup_{t\geq 1}\left[t^d\left(\int_t^\infty \|v(s) - U(s)e^{-i|\cdot|^2/2s}e^{-iS(s,-i\nabla)}\phi\|_{Y_n}\, ds\right)^{1/4}\right] < \infty,$$

*where $S$ is defined by* (1.3), *$d$ is a constant satisfying $n/4 < d < 1$, $Y_1 = L_x^\infty$ and $Y_2 = L_x^4$.*

*A similar result holds for negative time.*

In what follows, we shall prove Proposition 2.2.

Let $n = 1, 2$, and let $v_a$ be a given asymptotic profile of the equation (2.1), namely an approximate solution for that equation as $t \to \infty$. We introduce the following function:

$$R = \mathcal{L}v_a - F_n(t, v_a), \tag{2.5}$$

where

$$\mathcal{L} = i\partial_t + \frac{1}{2}\Delta.$$

The function $R$ is difference between the left hand sides and the right hand ones in the equation (2.1) substituted $v = v_a$.

We can prove the following proposition (see Propositions 3.4 and 3.5 in [8]).

**Proposition 2.3.** *Assume that there exists a constant $\eta' > 0$ such that*

$$\|v_a(t)\|_{L^2} \leq \eta',$$

$$\|v_a(t)\|_{L^\infty} \leq \eta'(1+t)^{-1/2},$$

$$\left\|\int_t^\infty U(t-s)R(s)\, ds\right\|_{L_x^2} + \left\|\int_\tau^\infty U(\tau-s)R(s)\, ds\right\|_{L_\tau^4((t,\infty);Y_n)} \leq \eta'(1+t)^{-d},$$

*for $t \geq 0$, where $Y_1 = L_x^\infty$ and $Y_2 = L_x^4$, and assume that $\eta' > 0$ is sufficiently small. Then there exists a unique solution $v$ for the equation (2.1) satisfying*

$$\begin{aligned} v &\in C([0,\infty); L^2), \\ \sup_{t\geq 1} &\left(t^d \|v(t) - v_a(t)\|_{L^2}\right) < \infty, \\ \sup_{t\geq 1} &\left[t^d \left(\int_t^\infty \|v(s) - v_a(s)\|_{Y_n}^4 \, ds\right)^{1/4}\right] < \infty, \end{aligned}$$

*where $d$ is a constant satisfying $n/4 < d < 1$, $Y_1 = L_x^\infty$ and $Y_2 = L_x^4$.*

*A similar result holds for negative time.*

## 3. Remainder Estimates and Proof of Theorem 1.1

In this section, we prove Proposition 2.2 to obtain Theorem 1.1.

First we introduce the Strichartz estimate for the free Schrödinger equation obtained by Yajima [10]. We define the linear operator

$$(\Gamma h)(t) = \int_t^\infty U(t-s)h(s)\, ds,$$

where $h$ is a function of $(t, x)$.

**Lemma 3.1.** *Let $n$ denote the space dimension, and let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be pairs of positive numbers satisfying $2/q = n(1/2 - 1/r)$, $2 < q \leq \infty$, $2/\tilde{q} = n(1/2 - 1/\tilde{r})$ and $2 < \tilde{q} \leq \infty$. Then $\Gamma$ is a bounded operator from $L_t^{\tilde{q}'}((T_0,\infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$ into $L_t^q((T_0,\infty); L_x^r(\mathbb{R}^n))$ with norm uniformly bounded with respect to $T_0$, where $(\tilde{q}', \tilde{r}')$ is a pair of positive numbers satisfying $1/\tilde{q} + 1/\tilde{q}' = 1$ and $1/\tilde{r} + 1/\tilde{r}' = 1$. Furthermore, if $h \in L_t^{\tilde{q}'}((T_0,\infty); L_x^{\tilde{r}'}(\mathbb{R}^n))$, then $\Gamma h \in C_t([T_0,\infty); L_x^2(\mathbb{R}^n))$.*

Let

$$\begin{aligned} v_a(t,x) &= (U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi)(x) \\ &= \frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right) e^{i|x|^2/2t - iS(t,x/t)}, \end{aligned} \tag{3.1}$$

where $S$ is defined by (1.3). This modified free dynamics was introduced by Ozawa [7] for the ordinary nonlinear Schrödinger equation with a nonlinearity $\lambda|u|^2u$ in one space dimension. In order to prove

Proposition 2.2, we show that $v_a$ satisfies the assumptions in Proposition 2.3. It is sufficient to show only the estimates

$$\|v_a(t)\|_{L^2} \le \eta', \tag{3.2}$$

$$\|v_a(t)\|_{L^\infty} \le \eta' t^{-1/2}, \tag{3.3}$$

$$\begin{aligned} &\left\| \int_t^\infty U(t-s)R(s)\,ds \right\|_{L^2_x} \\ &+ \left\| \int_s^\infty U(s-\tau)R(\tau)\,d\tau \right\|_{L^4_s((t,\infty);Y_n)} \le \eta' t^{-d}, \end{aligned} \tag{3.4}$$

where $R$ is defined by (2.5). In fact, in order to avoid a singularity at $t=0$, multiplying a cut off function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t)=0$ if $t \le 1/2$ and $\theta(t)=1$ if $t \ge 3/4$ to $v_a$, we easily see from the estimates (3.2)–(3.4) that the resulting function satisfies the assumptions in Proposition 2.3.

First we consider the gauge invariant nonlinearity $G_n(u)$.

**Lemma 3.2.** *There exists a constant $C>0$ such that for $t \ge 1$,*

$$\|v_a(t)\|_{L^2} = \|\phi\|_{L^2},$$

$$\|v_a(t)\|_{L^\infty} \le C\|\phi\|_{L^1} t^{-n/2},$$

$$\|\mathcal{L}v_a(t) - G_n(v_a(t))\|_{L^2} \le C(\|\phi\|_{H^{0,2}} + \|\phi\|^3_{H^{0,2}})\frac{(\log t)^2}{t^2}.$$

Since we can prove this lemma in the same way as Lemma 2.2 in [8], we omit the proof.

We next consider the non-gauge invariant and non-autonomous nonlinearity $N_n(t,u)$. In order to obtain the estimate (3.4), we need the following lemma, which is shown in Lemma 3.3 in [9].

**Lemma 3.3.** *Assume that $\|\phi\|_{H^2 \cap H^{0,2}} \le 1$. Then, there exists a constant $C>0$ such that for $t \ge 1$,*

$$\begin{aligned} &\left\| \int_t^\infty U(t-s)N_n(s, v_a(s))\,ds \right\|_{L^2_x} \\ &+ \left\| \int_s^\infty U(s-\tau)N_n(\tau, v_a(\tau))\,d\tau \right\|_{L^4_s((t,\infty);Y_n)} \le C\|\phi\|_{H^2\cap H^{0,2}} t^{-d}, \end{aligned}$$

*where $0<d<1$.*

*Proof.* As mentioned above, this lemma was shown in Lemma 3.3 in [9]. For convenience of readers, we describe the proof of this lemma. It is sufficient to prove for a single power nonlinearity of the form

$$N_n(t,v) = \lambda v^l \bar{v}^m e^{-i(\alpha-1)(tE\cdot x - t^3|E|^2/3)},$$

where $\lambda \in \mathbb{C}$,

$$(l,m) = (3,0) \text{ or } (0,3), \text{ when } n = 1,$$
$$(l,m) = (2,0), \ (1,1) \text{ or } (0,2) \text{ when } n = 2,$$
$$\alpha = l - m.$$

Note that $l + m = 1 + 2/n$ and $\alpha \neq \pm 1$. Then

$$\begin{aligned}
&N_n(t, v_a) \\
=&\frac{1}{t^{1+n/2}} P\left(\frac{x}{t}\right) e^{i\alpha\theta_1(t,x)} e^{i(\alpha-1)(\theta_2(t,x)+\theta_3(t))} \\
=&\frac{1}{i(\alpha-1)|E|^2} \frac{1}{t^{3+n/2}} P\left(\frac{x}{t}\right) e^{i\alpha\theta_1(t,x)} e^{i(\alpha-1)\theta_2(t,x)} \partial_t (e^{i(\alpha-1)\theta_3(t)}),
\end{aligned} \tag{3.5}$$

where

$$P(x) = i^{-\alpha n/2} \widehat{\phi}(x)^l \overline{\widehat{\phi}(x)}^m,$$
$$\theta_1(t,x) = \frac{|x|^2}{2t} - S\left(t, \frac{x}{t}\right), \quad \theta_2(t,x) = -tE \cdot x, \quad \theta_3(t) = \frac{t^3|E|^2}{3}.$$

We calculate the integrand $U(-s)N_n(s, v_a(s))$:

$$\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)\theta_2(s,x)} \partial_s (e^{i(\alpha-1)\theta_3(s)})\right\} \\
=&\partial_s \left[U(-s)\left\{\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right] \\
&+ \frac{i}{2} U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)}\right) e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\} \\
&+ iU(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)}\right) \cdot \nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\} \\
&+ \frac{i}{2} U(-s)\left\{\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} \Delta\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\} \\
&- U(-s)\left\{\partial_s\left(\frac{1}{s^{3+n/2}} P\left(\frac{x}{s}\right) e^{i\alpha\theta_1(s,x)} e^{i(\alpha-1)\theta_2(s,x)}\right) e^{i(\alpha-1)\theta_3(s)}\right\}.
\end{aligned}$$

Noting the relation

$$\Delta\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right) = i(\alpha-1) e^{i(\alpha-1)\theta_2(s,x)} \partial_s (e^{i(\alpha-1)\theta_3(s)}),$$

we have

$$
\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
=&\partial_s\left[U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right]\\
&+\frac{i}{2}U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\\
&+iU(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}\\
&-\frac{\alpha-1}{2}U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
&-U(-s)\left\{\partial_s\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\right)e^{i(\alpha-1)\theta_3(s)}\right\}.
\end{aligned}
$$

Since $\alpha \neq -1$, we have

$$
\begin{aligned}
&U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\partial_s(e^{i(\alpha-1)\theta_3(s)})\right\}\\
=&\frac{2}{\alpha+1}\partial_s\left[U(-s)\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\right]\\
&+\frac{i}{\alpha+1}U(-s)\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\}\\
&+\frac{2i}{\alpha+1}U(-s)\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\}\\
&-\frac{2}{\alpha+1}U(-s)\left\{\partial_s\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\right)e^{i(\alpha-1)\theta_3(s)}\right\}.
\end{aligned}
$$

By the identity (3.5), the above identity is equivalent to

$$
\begin{aligned}
&U(-s)N_n(s,v_a(s))\\
=&\frac{1}{i(\alpha-1)|E|^2}\left(\partial_s(U(-s)I_1(s))+\sum_{j=2}^{4}U(-s)I_j(s)\right),
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
I_1(s)&=\frac{2}{\alpha+1}\left\{\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\},\\
I_2(s)&=\frac{i}{\alpha+1}\left\{\Delta\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right\},\\
I_3(s)&=\frac{2i}{\alpha+1}\left\{\nabla\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}\right)\cdot\nabla\left(e^{i(\alpha-1)(\theta_2(s,x)+\theta_3(s))}\right)\right\},
\end{aligned}
$$

$$I_4(s) = -\frac{2}{\alpha+1}\left\{\partial_s\left(\frac{1}{s^{3+n/2}}P\left(\frac{x}{s}\right)e^{i\alpha\theta_1(s,x)}e^{i(\alpha-1)\theta_2(s,x)}\right)e^{i(\alpha-1)\theta_3(s)}\right\}.$$

Integrating the identity (3.6) over the interval $(t,\infty)$ and applying $U(t)$ to the resulting equality, we have

$$\begin{aligned}&\int_t^\infty U(t-s)N_n(s,v_a(s))\,ds\\ =&\frac{3}{i(\alpha-1)|E|^2}\left(-I_1(t)+\sum_{j=2}^{4}\int_t^\infty U(t-s)I_j(s)\,ds\right).\end{aligned}\tag{3.7}$$

By the definitions of $I_1$, $I_2$, $I_3$ and $I_4$, we have

$$\begin{aligned}\|I_1(t)\|_{L^2} &\le Ct^{-3}\|\hat{\phi}\|_{L^2}\|\hat{\phi}\|_{L^\infty}^{2/n},\\ \|I_1(t)\|_{L^\infty} &\le Ct^{-7/2}\|\hat{\phi}\|_{L^\infty}^3, \quad \text{when } n=1,\\ \|I_1(t)\|_{L^4} &\le Ct^{-4}\|\hat{\phi}\|_{L^8}^2, \quad \text{when } n=2,\\ \|I_2(s)\|_{L^2} &\le Cs^{-3}(\log s)^2\|\phi\|_{H^2\cap H^{0,2}},\\ \|I_3(s)\|_{L^2} &\le Cs^{-2}(\log s)\|\phi\|_{H^2\cap H^{0,2}},\\ \|I_4(s)\|_{L^2} &\le Cs^{-2}(\log s)\|\phi\|_{H^2\cap H^{0,2}}.\end{aligned}$$

We have used Hölder's inequality, the Sobolev embedding and the assumption $\|\phi\|_{H^2\cap H^{0,2}} \le 1$. We note that the $L^2$-norms of $I_2$, $I_3$ and $I_4$ are integrable over the interval $(t,\infty)$. Applying the above inequalities and Lemma 3.1 to the identity (3.7), we obtain this lemma. □

*Proof of Theorem 1.1.* Assume all the assumptions in Theorem 1.1. Let $v_a$ be the function defined by (3.1). According to Proposition 2.3, as mentioned before, it is sufficient to show the estimates (3.2)–(3.4). The estimates (3.2) and (3.3) immediately follow from the definition of $v_a$. We prove the estimate (3.4). Since

$$R = \mathcal{L}v_a - G_n(v_a) - N_n(t,v_a),$$

by Lemmas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}&\left\|\int_t^\infty U(t-s)R(s)\,ds\right\|_{L_x^2} + \left\|\int_s^\infty U(s-\tau)R(\tau)\,ds\right\|_{L_s^4((t,\infty);Y_n)}\\ \le& C\int_t^\infty \|\mathcal{L}v_a(s) - G_n(v_a(s))\|_{L^2}\,ds\\ &+\left\|\int_t^\infty U(t-s)N_n(s,v_a(s))\,ds\right\|_{L_x^2} + \left\|\int_s^\infty U(s-\tau)N_n(\tau,v_a(\tau))\,d\tau\right\|_{L_s^4((t,\infty);Y_n)}\\ \le& C\|\phi\|_{H^2\cap H^{0,2}}t^{-d},\end{aligned}$$

where $n/4 < d < 1$ appearing in the assumption of Theorem 1.1. Taking $\eta' = C\|\phi\|_{H^2\cap H^{0,2}}$, we see that the condition (3.4) is satisfied. According to Proposition 2.3, this completes the proof of Theorem 1.1.

□

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