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Upper bound of the best constant of the Trudinger-Moser inequality and its application to the GagliardO-Nirenberg inequality

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We consider the best constant of the Trudinger-Moser inequality in $\mathbb{R}^n$. Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. It is well known that the Sobolev space $H_0^{1/p,p}(\Omega)$, $1 < p < \infty$, is continuously embedded into $L^q(\Omega)$ for all $q$ with $p \leq q < \infty$. However, we cannot take $q = \infty$ in such an embedding. For bounded domains $\Omega$, Trudinger [18] treated the case $p = n(\geq 2)$, i.e., $H_0^{1,n}(\Omega)$ and proved that there are two constants $\alpha$ and $C$ such that

\[
\| \exp(\alpha |u|^{n'}) \|_{L^1(\Omega)} \leq C|\Omega| \tag{0.1}
\]

holds for all $u \in H_0^{1,n}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} \leq 1$. Here and hereafter $p'$ represents the Hölder conjugate exponent of $p$, i.e., $p' = p/(p-1)$. Moser [9] gave the optimal constant for $\alpha$ in (0.1), which shows that one cannot take $\alpha$ greater than $1/(n^{n-2}\omega_n^{n-1})$, where $\omega_n$ is the volume of the unit $n$-ball, that is, $\omega_n := |B_1| = 2\pi^{n/2}/(n\Gamma(n/2))$ ($\Gamma$ : the gamma function). Adams [2] generalized Moser's result to the case $H_0^{m,n/m}(\Omega)$ for positive integers $m < n$ and obtained the sharp constant corresponding to (0.1).

When $\Omega = \mathbb{R}^n$, Ogawa [10] and Ogawa-Ozawa [11] treated the Hilbert space $H^{n/2,2}(\mathbb{R}^n)$ and then Ozawa [14] gave the following general embedding theorem in the Sobolev space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives which states that

\[
\| \Phi_p(\alpha |u|^{p'}) \|_{L^1(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}^p \tag{0.2}
\]
holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1$, where

$$\Phi_p(\xi) = \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geq p - 1\}.$$ 

The advantage of (0.2) gives the scale invariant form. Concerning the sharp constant for $\alpha$ in (0.2), Adachi-Tanaka [1] proved a similar result to Moser's in $H^{1,n}(\mathbb{R}^n)$.

Our purpose is to generalize Adachi-Tanaka's result to the space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives. We show an upper bound of the constant $\alpha$ in (0.2). Indeed, the following theorem holds:

**Theorem 0.1.** Let $2 \leq p < \infty$. Then, for every $\alpha \in (A_p, \infty)$, there exists a sequence $\{u_k\}_{k=1}^\infty \subset H^{n/p,p}(\mathbb{R}^n) \setminus \{0\}$ with $\|(-\Delta)^{n/(2p)}u_k\|_{L^p(\mathbb{R}^n)} \leq 1$ such that

$$\frac{\|\Phi_p(\alpha|u_k|^p')\|_{L^1(\mathbb{R}^n)}}{\|u_k\|_{L^p(\mathbb{R}^n)}^p} \to \infty \text{ as } k \to \infty,$$

where $A_p$ is defined by

$$A_p := \frac{1}{\omega_n} \left[ \frac{\pi^{n/2}2^{n/p}\Gamma(n/(2p'))}{\Gamma(n/(2p'))} \right]^{p'}.$$  

**Remark.** Let $\alpha_p$ be the best constant of (0.2), i.e.,

$$\alpha_p := \sup\{\alpha > 0 \mid \text{The inequality (0.2) holds with some constant } C.\}.$$ 

Then Theorem 0.1 implies that $\alpha_p \leq A_p$ for $2 \leq p < \infty$.

Next, if we give a similar type estimate to (0.2) by taking another normalization such as $\|(I - \Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leq 1$, then we can cover all $1 < p < \infty$. Moreover, when $p = 2$, it turns out that our constant $A_2$ of (0.3) is optimal. To state our second result, let us recall the rearrangement $f^*$ of the measurable function $f$ on $\mathbb{R}^n$. For detail, see Section 2 (Stein-Weiss [16]). We denote by $f^{**}$ the average function of $f^*$, i.e.,

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau)d\tau \quad \text{for } t > 0.$$ 

Our theorem now reads:
Theorem 0.2. Let $1 < p < \infty$ and $A_p$ be as in (0.3).

(i) For every $\alpha \in (A_p, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/(2p)} u_k\|_{L^p(\mathbb{R}^n)} \leq 1$ such that

$$\|\Phi_p(\alpha |u_k|^p')\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$ 

(ii) We define $A_p^*$ by

$$A_p^* = A_p / B_p^{1/(p-1)},$$

where

$$B_p := (p - 1)^p \sup \left\{ \int_0^\infty (f^{**}(t) - f^*(t))^p dt \mid \|f\|_{L^p(\mathbb{R}^n)} \leq 1 \right\}.$$ 

Then for every $\alpha \in (0, A_p^*)$, there exists a positive constant $C$ depending only on $p$ and $\alpha$ such that

$$\|\Phi_p(\alpha |u|^p)\|_{L^1(\mathbb{R}^n)} \leq C$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \leq 1$.

Remark. Later, we shall show that

$$1 \leq B_p \leq p^p - (p - 1)^p \quad \text{for } 1 < p < \infty.$$ 

In particular, for $2 \leq p < \infty$, there holds

$$B_p = (p - 1)^{p-1}.$$ 

(0.5)

In any case, we obtain $A_p^* \leq A_p$ for $1 < p < \infty$.

Since it follows from (0.5) that $B_2 = 1$, we see that $A_2 = A_2^* = (2\pi)^n/\omega_n$ is the best constant of (0.4). Hence, the following corollary holds:

Corollary 0.1. (i) For every $\alpha \in ((2\pi)^n/\omega_n, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/2,2}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/4} u_k\|_{L^2(\mathbb{R}^n)} \leq 1$ such that

$$\|\Phi_2(\alpha |u_k|^2)\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$ 

(ii) For every $\alpha \in (0, (2\pi)^n/\omega_n)$, there exists a positive constant $C$ depending only on $\alpha$ such that

$$\|\Phi_2(\alpha |u|^2)\|_{L^1(\mathbb{R}^n)} \leq C$$

holds for all $u \in H^{n/2,2}(\mathbb{R}^n)$ with $\|(I - \Delta)^{n/4} u\|_{L^2(\mathbb{R}^n)} \leq 1$. 

(0.6)
It seems to be an interesting question whether or not (0.6) does hold for 
\( \alpha = (2\pi)^n/\omega_n \).

Next, we consider the Gagliardo-Nirenberg interpolation inequality which is closely related to the Trudinger-Moser inequality. Ozawa [14] proved that for \( 1 < p < \infty \) there is a constant \( M \) depending only on \( p \) such that

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq M q^{1/p'} \|u\|_{L^p(\mathbb{R}^n)}^{p/q} \|(-\Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{0.7}
\]

holds for all \( u \in H^{n/p,p}(\mathbb{R}^n) \) and for all \( q \in [1, \infty) \). Ozawa [13],[14] also showed the fact that (0.2) and (0.7) are equivalent and he gave the relation between \( \alpha \) in (0.2) and \( M \) in (0.7). Combining his formula with our result, we obtain an estimate of \( M \) from below. Indeed, there holds the following theorem:

**Theorem 0.3.** Let \( 2 \leq p < \infty \). We define \( M_p \) and \( m_p \) as follows.

\[
M_p := \inf\{M > 0 \mid \text{The inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty) \},
\]

\[
m_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty) \}.\]

Then there holds

\[
M_p \geq m_p \geq \frac{1}{(p'eA_p^{*})^{1/p'}}.
\]

Since Ozawa [13],[14] gave the relation between the constants \( \alpha \) in (0.2) and \( M \) in (0.7), we obtain a lower bound of the best constant for the Sobolev inequality in the critical exponent:

**Theorem 0.4.** Let \( 1 < p < \infty \).

(i) For every \( M > (p'eA_p^{*})^{-1/p'} \), there exists \( q_0 \in [p, \infty) \) depending only on \( p \) and \( M \) such that

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq M q^{1/p'} \|I - \Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \tag{0.8}
\]

holds for all \( u \in H^{n/p,p}(\mathbb{R}^n) \) and for all \( q \in [q_0, \infty) \).

(ii) We define \( \overline{M}_p \) and \( \overline{m}_p \) as follows.

\[
\overline{M}_p := \inf\{M > 0 \mid \text{The inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty) \},
\]

\[
\overline{m}_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty) \}.
\]
Then there holds

\[ \overline{M}_{p} \geqq \overline{m}_{p} \geqq \frac{1}{(p'eA_{p})^{1/p'}}. \]

Since we have obtained \( A_{2} = A_{2}^{*} \) for \( p = 2 \), we see that

\[ \frac{1}{\sqrt{2eA_{2}}} = \frac{1}{\sqrt{2eA_{2}^{*}}} = \sqrt{\frac{\omega_{n}}{2^{n+1}e\pi^{n}}} \]

Hence, the above theorem gives the best constant for (0.8). Indeed, we have the following corollary:

**Corollary 0.2.** (i) For every \( M > \sqrt{\omega_{n}/(2^{n+1}e\pi^{n})} \), there exists \( q_{0} \in [2, \infty) \) such that

\[ \|u\|_{L^{q}(\mathbb{R}^{n})} \leqq Mq^{1/2}\|(I - \Delta)^{n/4}u\|_{L^{2}(\mathbb{R}^{n})} \]

holds for all \( u \in H^{n/2,2}(\mathbb{R}^{n}) \) and for all \( q \in [q_{0}, \infty) \).

(ii) For every \( 0 < M < \sqrt{\omega_{n}/(2^{n+1}e\pi^{n})} \) and \( q \in [2, \infty) \), there exist \( q_{0} \in [q, \infty) \) and \( u_{0} \in H^{n/2,2}(\mathbb{R}^{n}) \) such that

\[ \|u_{0}\|_{L^{q_{0}}(\mathbb{R}^{n})} > Mq_{0}^{1/2}\|(I - \Delta)^{n/4}u_{0}\|_{L^{2}(\mathbb{R}^{n})} \]

holds.

To prove our theorems, by means of the Riesz and the Bessel potentials, we first reduce the Trudinger-Moser inequality to some equivalent form of the fractional integral. The technique of symmetric decreasing rearrangement plays an important role for the estimate of fractional integrals in \( \mathbb{R}^{n} \). To this end, we make use of O'Neil's result [12] on the rearrangement of the convolution of functions. Such a procedure is similar to Adams [2]. First, we shall show that for every \( \alpha \in (0, A_{p}^{*}) \), there exists a positive constant \( C \) depending only on \( p \) and \( \alpha \) such that (0.4) holds for all \( u \in H^{n/p,p}(\mathbb{R}^{n}) \) with \( \|(I - \Delta)^{n/(2p)}u\|_{L^{p}(\mathbb{R}^{n})} \leqq 1 \). On the other hand, we shall show that the constant \( C \) holding (0.2) and (0.4) in \( \mathbb{R}^{n} \) can be also available for the corresponding inequality in bounded domains. Since Adams [2] gave the sharp constant \( \alpha \) in the corresponding inequality to (0.1), we obtain an upper bound \( A_{p} \) as in (0.3). For general \( p \), we have \( A_{p}^{*} \leqq A_{p} \). In particular, for \( p = 2 \), there holds \( A_{2}^{*} = A_{2} \), which provides us the best constant of (0.4).
References


