数理解析研究所講究録 1401 巻 2004 年 82-108

Rotating Navier-Stokes Equations in $\mathbb{R}_+^3$ with Initial Data Nondecreasing at Infinity: The Ekman Boundary Layer Problem

Jürgen Saal

Darmstadt University of Technology

Abstract

This is a survey article of the results obtained in [8] by Y. Giga, K. Inui, A. Mahalov, S. Matsui, and me. There, existence and uniquness of local-in-time solutions for the Ekman boundary layer problem is proved.

1 Introduction

We study the initial value problem for the three-dimensional Navier-Stokes equations with Coriolis force in a half-space $\mathbf{R}_+^3$ and time interval $(0,T)$:

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \Omega e_3 \times \mathbf{U} + \nu curl^2 \mathbf{U} = -\nabla p,\ \nabla\cdot\mathbf{U} = 0, \tag{1.1}$$

$$\mathbf{U}(t,x)|_{x_3=0} = (U_1(t,x), U_2(t,x), U_3(t,x))|_{x_3=0} = (0,0,0), \tag{1.2}$$

$$\mathbf{U}(t,x)|_{t=0} \ = \ \mathbf{U}_0(x) \tag{1.3}$$

where $x=(x_1,x_2,x_3)$, $\mathbf{U}(t,x)=(U_1,U_2,U_3)$ is the velocity field and p is the pressure. In Eqs. (1.1) e_3 denotes the vertical unit vector and Ω is a constant Coriolis parameter (Ω is twice the frequency of rotation). Eqs. (1.1)-(1.3) are the 3D Navier-Stokes equations written in a rotating frame. The initial velocity field $\mathbf{U}_0(x)$ depends on three variables x_1, x_2 and x_3. We require the velocity field $\mathbf{U}(t,x)$ to satisfy Dirichlet (no slip) boundary conditions on the plane $\{x_3=0\}$.

Ekman spiral is the famous exact solution (time-independent) of the nonlinear problem (1.1)-(1.2). It describes rotating boundary layers in geophysical fluid dynamics (atmospheric and oceanic boundary layers). The boundary layer in the theory of rotating fluids known as the Ekman layer is between a geostrophic flow and a solid boundary at which the no slip condition applies. In the geostrophic flow region corresponding to large x_3 (far away from the solid boundary at $x_3 = 0$), there is a uniform flow with velocity U_∞ in the x_1 direction. Associated with U_∞, there is a pressure gradient in the x_2 direction. The Ekman spiral solution in $\mathbf{R}^3_+$ matches this uniform velocity for large x_3 with the no slip boundary condition at $x_3 = 0$. The corresponding velocity field $\mathbf{U}^E(x_3) : \mathbf{U}^E(x_3) = (U_1^E(x_3), U_2^E(x_3), 0)$ depends only on the vertical variable x_3:

$$U_1^E(x_3) = U_\infty \left(1 - e^{-\frac{x_3}{\delta}} \cos(\frac{x_3}{\delta})\right), \; U_2^E(x_3) = U_\infty e^{-\frac{x_3}{\delta}} \sin(\frac{x_3}{\delta}), \tag{1.4}$$

where δ is the rotating boundary layer (Ekman layer) thickness:

$$\delta = \left(\frac{2\nu}{\Omega}\right)^{1/2}. \tag{1.5}$$

The corresponding pressure field $p^E(x_2)$ depends only on x_2 and it is given by

$$p^E(x_2) = -\Omega U_\infty x_2. \tag{1.6}$$

Clearly, the nonlinear term in (1.1) is zero for $\mathbf{U} = \mathbf{U}^E(x_3)$ and, therefore, $(\mathbf{U}^E(x_3), p^E(x_2))$ which is called 'Ekman spiral' is an exact solution of the full nonlinear problem. Remarkable persistent (stability) of the Ekman spiral in atmospheric and oceanic rotating boundary layers has been noticed in geophysical literature. We note that the velocity field satisfies

$$\lim_{x_3 \to +\infty} \mathbf{U}^E(x_3) = (U_\infty, 0, 0), \tag{1.7}$$

and that the velocity field corresponding to the Ekman spiral solution is bounded as

$$|\mathbf{U}^E(x_3)| \leq 2\mathbf{U}_\infty. \tag{1.8}$$

Since the Ekman spiral has velocity field nondecreasing at infinity, it is essential in the mathematical theory of geophysical rotating boundary layers to study solvability of (1.1)-(1.3) for initial data in spaces of functions nondecreasing at infinity.

We write

$$\mathbf{U}(t, x_1, x_2, x_3) = \mathbf{U}^E(x_3) + \mathbf{V}(t, x_1, x_2, x_3), \; p(t, x_1, x_2, x_3) = p^E(x_2) + q(t, x_1, x_2, x_3). \tag{1.9}$$

Since the Ekman spiral is an exact solution of the full nonlinear problem, the vector field $\mathbf{V}(t, x_1, x_2, x_3)$ satisfies the following equations

$$\partial_t \mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + (\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{V} + V_3\frac{\partial \mathbf{U}^E}{\partial x_3} + \Omega e_3 \times \mathbf{V} + \nu curl^2\mathbf{V} = -\nabla q, \qquad (1.10)$$
$$\nabla\cdot\mathbf{V} = 0,$$

$$\mathbf{V}(t,x)|_{x_3=0} = (V_1(t,x), V_2(t,x), V_3(t,x))|_{x_3=0} = (0,0,0), \qquad (1.11)$$

$$\mathbf{V}(t,x)|_{t=0} \;=\; \mathbf{V}_0(x) \qquad (1.12)$$

Let $\mathbf{J}$ be the matrix such that $\mathbf{Ja} = e_3 \times \mathbf{a}$ for any vector field $\mathbf{a}$. Then

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (1.13)$$

Let $\mathbf{P}_+$ be the Helmholtz projection operator on $\mathbf{R}^3_+$. We define the Stokes operator $\mathbf{A}$ by

$$\mathbf{A}(\nu)v = \nu\mathbf{P}_+ curl^2 v = -\nu\mathbf{P}_+\Delta v \qquad (1.14)$$

on solenoidal vector fields v. The operator $\mathbf{P}_+$ can be represented by

$$\mathbf{P}_+ f = r\mathbf{P}Ef. \qquad (1.15)$$

Here, r is the restriction operator to the half space and $\mathbf{P}$ is the Helmholtz projection operator in the whole space, defined by

$$\mathbf{P} = \{P_{ij}\}_{i,j=1,2,3},\; P_{ij} = \delta_{ij} + R_i R_j; \qquad (1.16)$$

$R_j (j = 1,2,3)$ are the scalar Riesz operators $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$ with the symbols $\frac{i\xi_j}{|\xi|}$, e.g. [21]. Besides, the operator E is defined as follows:

For a function $h(x)$ on $\mathbb{R}^3_+$ we define an extended function $e^{\pm}h$ by

$$\left(e^{\pm}h\right)(x) = \begin{cases} h(x) & \text{if } x_3 > 0, \\ \pm h(x^*) & \text{if } x_3 < 0, \end{cases} \qquad (1.17)$$

where $x^* = (x_1, x_2, -x_3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^2 \times \mathbb{R}$.

For a vector field $f(x) = (f^1, f^2, f^3)$ on $\mathbb{R}^3_+$ we define an extended vector field Ef by

$$i\text{–th component of } (Ef)(x) = \begin{cases} (e^+ f^i)(x) & \text{for } 1 \le i \le 2, \\ (e^- f^3)(x) & \text{for } i = 3. \end{cases}$$

That is, $Ef = \operatorname{diag}[e^+, e^+, e^-]\,({}^T f)$, here diag represents a diagonal matrix, ${}^T f$ is the transposed vector field of f.

We transform (1.10)-(1.12) into an abstract operator differential equation for $\mathbf{V}$

$$\mathbf{V}_t + \mathbf{A}\mathbf{V} + \Omega\mathbf{S}\mathbf{V} + \mathbf{C}_E\mathbf{V} + \mathbf{P}_+(\mathbf{V}\cdot\nabla)\mathbf{V} = 0, \qquad \mathbf{V}|_{t=0} = \mathbf{V}_0, \tag{1.18}$$

where

$$\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+, \ \mathbf{C}_E\mathbf{V} = \mathbf{P}_+\left((\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{V} + V_3\frac{\partial \mathbf{U}^E}{\partial x_3}\right) \tag{1.19}$$

and we have used $\mathbf{P}_+\mathbf{J}\mathbf{V} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+\mathbf{V}$ on solenoidal vector fields. The main difference between the problem in a half-space $\mathbf{R}^3_+$ with the problem in $\mathbf{R}^3$ is that the Stokes operator $\mathbf{A}$ and the operator $\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+$ do not commute in $\mathbf{R}^3_+$ and there is an additional 'Ekman operator' $\mathbf{C}_E$ in Eqs. (1.18). We consider initial data $\mathbf{V}_0(x)$ for Eqs. (1.18) in spaces of solenoidal vector fields nondecreasing at infinity in x_1, x_2. The consideration of solutions not decaying at infinity in x_1, x_2 is essential in the development of rigorous mathematical theory of the Ekman rotating boundary layer problem. In view of (1.7) it is natural to consider vector fields $\mathbf{V}$ which belong to L_p, $1 < p < +\infty$ in x_3.

The first step in the analysis of the nonlinear problem (1.10)-(1.12) is to show that the corresponding linear problem generates a semigroup in appropriate spaces. Note that the L_p, $1 < p < +\infty$ case is usually simpler than the L_∞ case due to the fact that Riesz operators are bounded operators in L_p but not in L_∞. We recall that for $\Omega = 0$ (non-rotating case) Green's function of the Stokes operator in $\mathbf{R}^3$ and $\mathbf{R}^3_+$ (half space) belong to $L_1(\mathbf{R}^3)$ implying that the corresponding operator generates a semigroup in $L_\infty(\mathbf{R}^3)$ and $L_\infty(\mathbf{R}^3_+)$. On the other hand, for $\Omega \neq 0$ Green's function of the (Stokes + Coriolis) problem even in $\mathbf{R}^3$ does not belong to $L_1(\mathbf{R}^3)$. Moreover, it behaves as $|x|^{-3}$ for large $|x|$ and the corresponding integral operator is not a bounded operator in $L_\infty(\mathbf{R}^3)$. One needs to restrict initial data on a subspace of $L_\infty(\mathbf{R}^3)$. Similar situation of unboundedness in $L_\infty(\mathbf{R}^2)$ (for horizontal x_1, x_2 planes) holds for the linear (Stokes+Coriolis) problem in a half space. One needs to restrict initial data on a subspace of $L_\infty(\mathbf{R}^2)$ where Riesz' operators and, consequently, the operator $\mathbf{P}_+\mathbf{J}\mathbf{P}_+$ are bounded. The natural space for this purpose for initial data $\mathbf{V}_0$ is the space $X = \dot{B}^0_{\infty,1}(\mathbf{R}^2; L^p(\mathbf{R}_+))$. Here $\dot{B}^0_{\infty,1}$ is the homogeneous Besov space. It contains almost periodic functions in x_1, x_2 and the Riesz operators are bounded in this space. We study local (in time) unique solvability of the rotating Ekman Navier-Stokes equations in $\mathbf{R}^3_+$ under the condition that the initial velocity $\mathbf{V}_0 \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbf{R}^2; L^p(\mathbf{R}_+))$. $2 < p < +\infty$. For the linear problem (Stokes + Coriolis) we employ the solution formula derived in [6] for the Stokes resolvent in terms of the resolvent of the Dirichlet Laplacian and certain remainder terms. Detailed information on the linear problem (Stokes + Coriolis + Ekman operators) is then used to construct (local-in-time) mild solutions to the full nonlinear rotating Navier-Stokes equations in $\mathbf{R}^3_+$. To derive the estimates for the linear part we will employ theory for E-valued Besov spaces, where E is a Banach space. The main ingredient will be an operator-valued version of Mikhlin's multiplier result. Among other things it will be the basis for an operator-valued H^∞-calculus for the Laplacian on E-valued Besov spaces, which serves as a useful tool in estimating the formulas for the Helmholtz projection and the resolvent of the Stokes operator. The generation result for the Stokes operator and a standard perturbation argument will then lead to the generation result for the full linear operator (Stokes+Coriolis+Ekman).

Note that we do not expect the solutions of the nonlinear equations to be an element of the space of initial data $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. This is essentially due to the fact that normal derivatives act merely on the L^p part of the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ (see Remark 3.8). To overcome this problem we applied the contraction mapping principle in the larger space $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. The unboundedness of the Helmholtz projection in that space is handled by using a splitting of the term $\mathbf{P}_+\partial_3$ in a term with pure normal derivative and terms containing only tangential derivatives and Riesz operators. This leads to the slightly technical Section 3.2.

In the subsequent sections we will be brief in details, in particular with the proofs of our results. For detailed versions of the proofs see [8].

In order to get a space of functions instead of equivalence classes we use an alternative definition of the homogeneous Besov spaces and denote them by $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ (see Definition 2.1). By $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ we denote the solenoidal part of $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ defined as the image of the Helmholtz projection $\mathbf{P}_+$ (observe that $\mathbf{P}_+$ is bounded on $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ according to Corollary 2.9). Our main result reads as

Theorem 1.1. *Let $2 < p < \infty$. For each $v_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ there exists $T_0 > 0$ and a unique (mild) solution v of (1.18) such that*

$$v \in \mathrm{BC}([0,T_0); \mathcal{BUC}_\sigma(\mathbb{R}^2; L^p(\mathbb{R}_+)))$$

and

$$t^{1/2}\nabla v \in \mathrm{BC}((0,T_0); \mathcal{BUC}_\sigma(\mathbb{R}^2; L^p(\mathbb{R}_+))).$$

2 Basic ingredients

In this section we define E-valued homogeneous Besov spaces and provide the required basics for the treatment of the linear and nonlinear problems in the subsequent sections.

Usually the homogeneous Besov space $\dot{B}^s_{r,q}(\mathbb{R}^n)$ is defined as a space of equivalence classes, see e.g. [22], where it is defined as a subspace of $\mathcal{Z}'(\mathbb{R}^n)$, the topological dual of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : D^\alpha \hat{f}(0) = 0,\ \alpha \in \mathbb{N}_0^n\},$$

(see also [3] for an equivalent definition). The space $\mathcal{Z}'(\mathbb{R}^n)$ can be identified with $\mathcal{S}'(\mathbb{R}^n)$ modulo all polynomials in $\mathbb{R}^n$, where $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This leads to the fact that elements of the equivalence classes in $\dot{B}^s_{r,q}(\mathbb{R}^n)$ have different derivatives in general. Therefore it is not appropriate to construct solutions of a concrete PDE in such a space. In such a situation it is desirable to have a space of functions, which motivates the alternative definition given below.

Recall that a Littlewood-Paley decomposition is given by a family of functions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\hat{\phi}_j(\xi) := \hat{\phi}_0(2^{-j}\xi)$ and $0 \neq \phi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\phi}_0 \subseteq \{1/2 \leq |\xi| \leq 2\}$. Moreover, for a Banach space E, we denote by $\mathcal{S}'(\mathbb{R}^n; E)$ the space of all E-valued linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$, i.e. $\mathcal{S}'(\mathbb{R}^n; E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n); E)$. Note that then

$$\mathcal{S}(\mathbb{R}^n; E) \hookrightarrow L^q(\mathbb{R}^n; E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n; E), \quad q \in [1, \infty].$$

Definition 2.1. Let E be a Banach space, $1 \leq r, q \leq \infty$, $s \in \mathbb{R}$, and $\{\phi_j\}_{j\in\mathbb{Z}}$ a Littlewood-Paley decomposition. If

$$\text{either} \quad s < n/r \qquad \text{or} \qquad s = n/r \ \text{ and } \ q = 1, \tag{2.1}$$

then the E-valued homogeneous Besov space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E)$ is defined by

$$\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E) := \{f \in \mathcal{S}'(\mathbb{R}^n; E) : \|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;E)} < \infty,\ f = \sum_{j\in\mathbb{Z}} \phi_j * f \text{ in } \mathcal{S}'(\mathbb{R}^n; E)\}, \tag{2.2}$$

where $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;E)} := \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\phi_j * f\|_{L^r(\mathbb{R}^n;E)})^q\right)^{1/q}$. On the other hand, if E is additionally the dual space of a Banach space F, $s \in \mathbb{R}$, $1 < r, q \leq \infty$, and

$$\text{either} \quad s > n/r \qquad \text{or} \qquad s = n/r \ \text{ and } \ q \neq 1, \tag{2.3}$$

we set

$$\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E) := (\dot{\mathcal{B}}^{-s}_{r',q'}(\mathbb{R}^n; F))'. \tag{2.4}$$

Remark 2.2. (1) Definition 2.1 relies on the fact that under condition (2.1) the series $\sum_{j\in\mathbb{Z}} \phi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n; E)$ for $f \in \mathcal{S}'(\mathbb{R}^n; E)$ with $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;E)} < \infty$. For $E = \mathbb{C}$ a proof of this fact can be found in [3], [13]. We omit the proof here, since the one given in [13] directly transfers to the E-valued case. Note that $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;E)} < \infty$ is not sufficient for the convergence of $\sum_{j\in\mathbb{Z}} \phi_j * f$ in $\mathcal{S}'(\mathbb{R}^n; E)$, if the parameters s, r, q satisfy the inverse condition (2.3). Therefore we used definition (2.4) in that case. Also note that the first ones who made use of definition (2.2) in the case $E = \mathbb{C}$ for the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n)$ related to the Navier-Stokes equations were O. Sawada and Y. Taniuchi in [18] and O. Sawada in [17].

(2) By standard arguments it can be easily shown that $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E)$ is a Banach space.

(3) Demanding f to have the representation $f = \sum_{j\in\mathbb{Z}} \phi_j * f$ ensures, that (E-valued) constants are not element of $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n; E)$. This yields the continuous imbedding

$$\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n; E) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n; E).$$

(Observe that $\|c\|_{\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n;E)} = 0$ for $c \in E$!)

(4) In this work we do not make use of $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E)$ for r, q, s satisfying (2.3) with $r = 1$ or $q = 1$. Therefore we skipped a proper definition of those spaces.

(5) In the scalar-valued case $E = \mathbb{C}$ for all values of the parameters s, r, q as in Definition 2.1 the space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n)$ is isomorphic to $\dot{B}^s_{r,q}(\mathbb{R}^n)$, see [3], [13].

The embedding in Remark 2.2 (3) is of crucial importance for estimating the nonlinear term in Section 3.2. But, since the Helmholtz projection $\mathbf{P}_+$ is expected to be unbounded in $\mathrm{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$, we also need to employ the larger space $\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^2; L^p(\mathbb{R}_+))$, which admits the boundedness of $\mathbf{P}_+$. For this purpose we define

$$\mathcal{BUC}(\mathbb{R}^n; E) := \{f \in \mathrm{BUC}(\mathbb{R}^n; E) : f = \sum_{j\in\mathbb{Z}} \phi_j * f \text{ in } \mathcal{S}'(\mathbb{R}^n; E)\}.$$

Since the series $\sum_{j\in\mathbb{Z}} \phi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n; E)$ for $f \in \mathrm{BUC}(\mathbb{R}^n; E)$ this space is well-defined and it is isomorphic to $\mathrm{BUC}(\mathbb{R}^n; E)$ modulo constants. An essential ingredient for the calculations in Section 3.2 will be

Lemma 2.3. *Let $n \in \mathbb{N}$, E be the dual of a Banach space F. Then*

$$\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n; E) \hookrightarrow \mathcal{BUC}(\mathbb{R}^n; E) \hookrightarrow \dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^n; E).$$

Proof. The first embedding is clear. The second one follows easily by the definition of the space $\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^n; E)$. ☐

In order to obtain densely defined generators the next Lemma will be useful.

Lemma 2.4. *Let E be a Banach space, $1 \leq p < \infty$, and s, r, q be as in condition (2.1). Then*

(i) $\left\{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E),\ \alpha \in \mathbb{N}_0^n\right\} \stackrel{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; E).$

(ii) $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; \bigcap_{k\in\mathbb{N}_0} W^{k,p}(\mathbb{R})) \stackrel{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})).$

(iii) $\left\{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})),\ \alpha \in \mathbb{N}_0^n\right\} \stackrel{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n; L^p(\mathbb{R})).$

Proof. This follows by applying standard mollifier arguments. □

The following operator-valued Mikhlin type multiplier result is fundamental for the treatment of the linearized equations in Section 3.1. Its proof is based on results in [1] and is given in [8].

Theorem 2.5. *Let $N \in \mathbb{N}$. Let E, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ be as in Definition 2.1. Furthermore, let $m \in C^{N+1}(\mathbb{R}^n \setminus \{0\}, \mathcal{L}(E))$ such that*

$$\|m\|_{M(E)} := \max_{|\alpha|\leq N+1} \sup_{\xi\in\mathbb{R}^N\setminus\{0\}} |\xi|^{|\alpha|} \|D^\alpha m(\xi)\|_{\mathcal{L}(E)} < \infty. \tag{2.5}$$

Then $\mathcal{F}^{-1}m\mathcal{F}$ is a bounded operator on $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E)$ and we have

$$\|\mathcal{F}^{-1}m\mathcal{F}\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N;E))} \leq C\|m\|_{M(E)}, \tag{2.6}$$

where $C = C(n) > 0$ is independent of p,q,s and m.

We call $m : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(E)$, satisfying the assumptions of Theorem 2.5, an operator-valued multiplier on $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E)$. Easy examples of operator-valued multipliers are given by scalar-valued multipliers, i.e. functions $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ that satisfy the assumptions of Theorem 2.5 with $E = \mathbb{C}$. Indeed, by the identification $m = m \cdot I$, where I is the identity on E, it is easy to verify that m is also an operator-valued multiplier.

In the sequel we will also make use of the following type of an operator-valued H^∞-calculus.

Definition 2.6. Let $N \in \mathbb{N}$, $\phi \in (0, \pi)$. Let E, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ be as in (2.1). A sectorial operator A in $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E)$ is said to admit an operator-valued H^∞-calculus on $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E)$ if there exists a $C_\phi > 0$ such that

$$\|h(A)\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;E))} \leq C_\phi \|h\|_{L^\infty(\Sigma_\phi;\mathcal{L}(E))} \tag{2.7}$$

for all $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(E)) := \{h : \Sigma_\phi \to \mathcal{K}_A(E) : h \text{ bounded and holomorphic}\}$, where

$$\mathcal{K}_A(E) := \{T \in \mathcal{L}(E) : T(\lambda + A)^{-1} = (\lambda + A)^{-1}T,\ \lambda \in \rho(-A)\}. \tag{2.8}$$

The angle

$$\phi^\infty_{Op}(A) := \inf\{\phi \in (0,\pi) : \text{ there is a } C_\phi > 0 \text{ such that (2.7) holds}\}$$

is called the (operator-valued) H^∞-angle of A in $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E)$.

Remark 2.7. (a) It is clear that the definition above extends to arbitrary E-valued Banach spaces.
(b) Denote the class of all operators A admitting an operator-valued H^∞-calculus as above by $\mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E))$. Setting $E = \mathbb{C}$ we see that an operator $A \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E))$ in particular admits a scalar H^∞-calculus, i.e. $A \in \mathcal{H}^\infty(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N))$.
(c) By abstract results it follows that $A \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E))$ implies $A^s \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E))$ with $\phi^\infty_{A^s} \le \phi^\infty_{A^s}$ for $s \in [0,1]$.

Note that for

$$h \in H^\infty_0(\Sigma_\phi; \mathcal{K}_A(E)) \;:=\; \Big\{h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(E)) : \|h(z)\|_{\mathcal{L}(E)} \le C\frac{|z|^s}{(1+|z|)^{2s}}, z \in \Sigma_\phi, \text{ for some } C, s > 0\Big\}$$

the operator $h(A)$ is defined by

$$h(A) := \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - A)^{-1}\mathrm{d}\lambda,$$

where Γ is the path $\Gamma := \{re^{i\theta}; \infty > r \ge 0\} \cup \{re^{-i\theta}; 0 \le r < \infty\}$ for $\theta \in (0,\phi)$, passing from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. This representation explains the restriction of the values of the functions h to the subalgebra $\mathcal{K}_A(E)$. Otherwise there would be a second, possibly different, way to define $h(A)$, namely by the integral

$$h(A) := \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1}h(\lambda)\mathrm{d}\lambda.$$

This differers from the scalar-valued case, where these two definitions always coincide. Thus, in order to obtain a compatible definition for the operator-valued case it is reasonable to use this restriction.

By the additional decay in 0 and ∞ it is obvious that $h(A) \in \mathcal{L}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; E))$ for $h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(E))$. To define $h(A)$ for arbitrary $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(E))$ we take $z \mapsto g(z) := z/(1+z)^2 \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(E))$ and set

$$h(A) := (hg)(A)g(A)^{-1}$$

initially defined on $D(A) \cap R(A)$. Since the convergence lemma (see [4]) is still true for operator-valued holomorphic functions (see [10]), as in the scalar-valued case it suffices to prove (2.7) for all $h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(E))$ in order to obtain the validity of (2.7) for all $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(E))$. For a more comprehensive introduction to operator-valued H^∞-calculi we refer to [12] and [10], for the scalar-valued case see [4] and [5].

Examples of operators that admit an operator-valued H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E)$ are in order.

Proposition 2.8. *Let $N \in \mathbb{N}$ and E, $s \in \mathbb{R}$, $1 \le r, q \le \infty$ be as in (2.1). The Laplacian $-\Delta$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E)$ with domain $D(-\Delta) = \{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E), \alpha \in \mathbb{N}_0^N, |\alpha| \le 2\}$ admits an operator-valued H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E)$ with H^∞-angle $\phi^\infty_{-\Delta} = 0$.*

Proof. Note that the sectoriality of $-\Delta$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E)$ with spectral angle $\phi_{-\Delta} = 0$ is an immediate consequence of Theorem 2.5 and Lemma 2.4 (i). Indeed, it follows from the well-known fact that $\mathcal{F}\lambda(\lambda - \Delta)^{-1} = \lambda(\lambda + |\xi|^2)^{-1}$ satisfies the scalar Mikhlin conditions also for $|\alpha| \le N+1$ (instead of $|\alpha| \le [N/2]+1)$!) and for all $\lambda \in \Sigma_{\pi-\varphi_0}$, and arbitrary $\varphi_0 \in (0, \pi)$.

Now let $\phi \in (0, \pi)$ and $h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(E))$. Taking Fourier transform yields

$$\mathcal{F}h(-\Delta) = \frac{1}{2\pi i}\int_\Gamma h(\lambda)\mathcal{F}(\lambda - (-\Delta))^{-1} \mathrm{d}\lambda = h(|\cdot|^2).$$

By copying the proof for scalar-valued h verbatim, simply replacing absolute value $|\cdot|$ by the operator norm $\|\cdot\|_{\mathcal{L}(E)}$ it can be shown that $\xi \mapsto h(|\xi|^2)$ satisfies the Mikhlin condition of Theorem 2.5. This yields the assertion. □

By the preparations above we are in the situation to give an elegant proof of the boundedness of the Helmholtz projection on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$.

Corollary 2.9. *Let $n \in \mathbb{N}$, $1 < p < \infty$. Let $s \in \mathbb{R}$, $1 \le r, q \le \infty$ be as in Definition 2.1. The Helmholtz projection $\mathbf{P}_+$ is bounded on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$.*

Proof. We use the representation

$$\mathbf{P}_+ = r(I + RR^T)E$$

as given in (1.15) and (1.16). Obviously $r \in \mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})), \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)))$ and $E \in \mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)), \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))$. It remains to prove the boundedness of $R = (R_1, \dots, R_n)$ on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}))$. For $j = 1, \dots, n-1$ we write formally

$$R_j = \partial_j(-\Delta)^{-1/2} = R'_j h(-\Delta'),$$

where $R'_j := \partial_j(-\Delta')^{-1/2}$ is the tangential Riesz operator and

$$h : \Sigma_\phi \to \mathcal{K}_{-\Delta'}(L^p(\mathbb{R})), \quad h(z) := [z(z - \Delta_n)^{-1}]^{1/2}$$

for some $\phi \in (0, \pi)$ and $\Delta_n := \partial_n^2$. Theorem 2.5 easily yields $R'_j = \mathcal{F}^{-1}\left[\frac{i\xi_j}{|\xi'|} \cdot I\right]\mathcal{F} \in \mathcal{L}(\dot{\mathcal{B}}^0_{\infty,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))$, since $\frac{i\xi_j}{|\xi'|}$ satisfies the scalar Mikhlin conditions. Furthermore, from well-known resolvent estimates for the Laplacian $-\Delta_n$ on $L^p(\mathbb{R})$ we obtain

$$\|z(z - \Delta_n)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}))} \leq C_\phi, \quad z \in \Sigma_\phi.$$

This implies $h \in H^\infty(\Sigma_\phi, \mathcal{K}_{-\Delta'}(L^p(\mathbb{R})))$ and therefore $h(-\Delta') \in \mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))$ by Proposition 2.8, which proves the boundedness of R_j for $j = 1, \dots, n-1$.

In the case $j = n$ we directly write $R_n = g(-\Delta')$ with

$$g : \Sigma_\phi \to \mathcal{K}_{-\Delta'}(L^p(\mathbb{R})), \quad g(z) = \partial_n(z - \Delta_n)^{-1/2}.$$

Again by well-known estimates for the operator $-\Delta_n$ we deduce $g \in H^\infty(\Sigma_\phi; \mathcal{K}_{-\Delta'}(L^p(\mathbb{R})))$ implying $R_n \in \mathcal{L}(\dot{\mathcal{B}}^0_{\infty,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))$ and the proof is complete. □

As another consequence of Proposition 2.8 and Remark 2.7 (c) we obtain the following further example of an operator admitting an operator-valued H^∞-calculus. It will turn out to be the key-ingredient in the proof of the resolvent estimates of the Stokes operator in Theorem 3.1.

Corollary 2.10. *Let $N \in \mathbb{N}$ and E, $s \in \mathbb{R}$, $1 \leq r, q \leq \infty$ be as (2.1). The Poisson operator $|\nabla| := (-\Delta)^{1/2}$ admits an operator-valued H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; E)$ with H^∞-angle $\phi^\infty_{|\nabla|} = 0$.*

3 Local existence for the nonlinear problem with initial data in $\dot{B}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ for $1 < p < \infty$

3.1 Linear problem

In this section we consider the linear problem

$$\begin{aligned}\partial_t\mathbf{\Phi} - \nu\Delta\mathbf{\Phi} + \Omega\mathbf{e}_3 \times \mathbf{\Phi} + (\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{\Phi} + \Phi_3\frac{\partial\mathbf{U}^E}{\partial x_3} &= -\nabla\pi,\ \nabla\cdot\mathbf{\Phi} = 0, \\ \mathbf{\Phi}(t,x)|_{x_3=0} &= (0,0,0), \\ \mathbf{\Phi}(t,x)|_{t=0} &= \mathbf{\Phi}_0(x).\end{aligned} \tag{3.1}$$

in $\mathbb{R}^3_+ \times (0,\infty)$. After applying the Helmholtz projection $\mathbf{P}_+$, the above equation (3.1) can be written in operator form as follows

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi} + \Omega\mathbf{S}\mathbf{\Phi} + \mathbf{C}_E\mathbf{\Phi} = 0,\ \mathbf{\Phi}(t)|_{t=0} = \mathbf{\Phi}_0, \tag{3.2}$$

where $\mathbf{A}$ is the Stokes operator in a half-space, $\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+$ is the Coriolis operator in $\mathbf{R}^3_+$, and $\mathbf{C}_E$ is the Ekman operator. Most of the results below are stated in arbitrary dimension $n \geq 2$. Only if the Ekman operator comes into play we restrict dimension to the case $n = 3$. Since the results here are based on the results in Section 2 the proof works simultaneously in all homogeneous Besov spaces $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ as defined in Definition 2.1. Hence, for simplicity we put $X := \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ and $X_\sigma := \dot{\mathcal{B}}^s_{r,q,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) := \mathbf{P}_+\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ and start by stating the generation result for the Stokes operator

$$\begin{aligned}\mathbf{A} &:= \mathbf{A}_{\mathbb{R}^n_+} := -\nu\mathbf{P}_+\Delta, \\ D(\mathbf{A}) &:= D(\Delta_D) \cap X_\sigma \\ &:= \{u \in X : D^\alpha u \in X, \alpha \in \mathbb{N}^n_0, |\alpha| \leq 2, u|_{\partial\mathbb{R}^n_+} = 0\} \cap X_\sigma,\end{aligned}$$

where Δ_D denotes the Dirichlet-Laplacian in X and $\alpha \in \mathbb{N}^n_0$ is a multiindex. By a standard perturbation argument we will show afterwards that also

$$\begin{aligned}\mathbf{A}_E &:= \mathbf{A} + \mathbf{P}_+\mathbf{J}\mathbf{P}_+ + \mathbf{C}_E \\ D(\mathbf{A}_E) &= D(\mathbf{A})\end{aligned}$$

is the generator of a holomorphic semigroup on X_σ.

Theorem 3.1. *The Stokes operator* $\mathbf{A}$ *is the generator of a bounded holomorphic semigroup on* X_σ. *In particular, for each* φ_0 *there is a* C_{φ_0} *such that we have the resolvent estimates*

$$\sum_{k=0}^{2} |\lambda|^{k/2} \|\nabla^{2-k}(\lambda+\mathbf{A})^{-1}\|_{\mathcal{L}(X)} \le C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0}. \tag{3.3}$$

If s, r, q *satisfy condition (2.1) the semigroup is strongly continuous.*

The proof of this result requires some preparations. First let us recall a suitable representation for the solution of the Stokes resolvent problem

$$(SRP)_{u_0,\lambda} \left\{ \begin{array}{rcll} (\lambda-\Delta)u+\nabla p & = & u_0 & \text{in } \mathbb{R}^n_+, \\ \nabla\cdot u & = & 0 & \text{in } \mathbb{R}^n_+, \\ u & = & 0 & \text{in } \mathbb{R}^{n-1}. \end{array} \right.$$

In [6] (see also[16]) it was shown that $u = (\lambda+\mathbf{A})^{-1}u_0$ can be represented as

$$\begin{array}{rcl} u' & = & (\lambda-\Delta_D)^{-1}u_0' - R'v, \\ u^n & = & (\lambda-\Delta_D)^{-1}u_0^n + v, \end{array}$$

where R' denotes the tangential Riesz operator and the Fourier transform of the remainder v is given by

$$\hat{v}(\xi', x_n) = \frac{e^{-\omega(|\xi'|)x_n} - e^{-|\xi'|x_n}}{\omega(|\xi'|)-|\xi'|} \int_0^\infty e^{-\omega(|\xi'|)s}\hat{u}_0^n(\xi', s)ds, \quad (\xi, x_n) \in \mathbb{R}^n_+,$$

where $\omega(|\xi'|) = \sqrt{\lambda+|\xi'|^2}$. Furthermore, the Fourier transform of the related pressure p is given as

$$\hat{p}(\xi', x_n) = \frac{i\xi'}{|\xi'|}\cdot\frac{\omega(|\xi'|)+|\xi'|}{\omega(|\xi'|)} e^{-|\xi'|x_n} \int_0^\infty e^{-\omega(|\xi'|)s}\hat{u}_0'(\xi', x_n)ds, \quad (\xi, x_n) \in \mathbb{R}^n_+. \tag{3.4}$$

In order to estimate these formulae we follow the arguments in [16], i.e. we will prove

$$\|\nabla p\|_X \le C\|f\|_X. \tag{3.5}$$

Then, by plugging over ∇p to the right hand side of $(SRP)_{f,\lambda}$ it can be regarded as a resolvent problem for the Dirichlet-Laplacian with data $u_0 - \nabla p$. The estimates for the solution of this problem, which are proved first, in combination with (3.5) then yields the assertion. The essential ingredient for estimating the formulae for u and p in [16] is the H^∞-calculus for the tangential Poisson operator $|\nabla'| := (-\Delta')^{1/2} = \mathcal{F}^{-1}[|\xi'|]\mathcal{F}$ on $L^q(\mathbb{R}^{n-1})$. The corresponding ingredient in the situation considered here will be the stronger property of an operator-valued H^∞-calculus for $|\nabla'|$ on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; L^p(\mathbb{R}_+))$ as provided in Corollary 2.10. This is due to the fact that here we have to deal with E-valued spaces in contrast to [16].

As a further application of Proposition 2.8 we start with the desired resolvent estimates for the Dirichlet-Laplacian Δ_D. The proof follows by similar methods as in the proof of Corollary 2.9. Therefore we omit the details here.

Proposition 3.2. *Let $\varphi_0 \in (0, \pi)$. There is a $C_{\varphi_0} > 0$ such that the Dirichlet-Laplacian Δ_D with domain $D(\Delta_D) = \{u \in X : D^\alpha u \in X, \alpha \in \mathbb{N}_0^n, |\alpha| \le 2, u|_{\partial\mathbb{R}^n_+} = 0\}$ admits the resolvent estimates*

$$\sum_{k=0}^{2} |\lambda|^{k/2} \|\nabla^{2-k}(\lambda - \Delta_D)^{-1}\|_{\mathcal{L}(X)} \le C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0}.$$

With the above preparations in hand we can turn to the proof of the generation result for the Stokes operator.

Proof. (of Theorem 3.1)
First we show that $\mathbf{A}$ is densely defined if s, r, q satisfy condition (2.1). Parallel to Proposition 3.2 it can be proved that Δ with its domain

$$D(\Delta) = \left\{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})),\ \alpha \in \mathbb{N}_0^n, |\alpha| \le 2\right\}$$

is the generator of a bounded holomorphic semigroup on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}))$. Thanks to Lemma 2.4 (iii), $D(\Delta)$ is dense in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}))$, hence this semigroup is strongly continuous, which implies

$$\lambda(\lambda - \Delta)^{-1} f \to f, \quad \text{in } \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})) \text{ if } \lambda \to \infty.$$

In view of

$$(\lambda - \Delta_D)^{-1} f = r(\lambda - \Delta)^{-1} e^- f,$$

where r is the restriction on $\mathbb{R}^n_+$, e^- the extension by odd reflection as defined in (1.17), we obtain also for the Dirichlet Laplacian

$$\lambda(\lambda - \Delta_D)^{-1} f \to r e^- f = f \quad \text{in } \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) \text{ if } \lambda \to \infty.$$

By the representation

$$\lambda(\lambda + \mathbf{A}_{\mathbb{R}^n_+})^{-1} f = \lambda(\lambda - \Delta_D)^{-1}(f - \nabla p(\lambda))$$

it therefore remains to show

$$\nabla p(\lambda) \to 0 \quad \text{in } X \text{ if } \lambda \to \infty$$

to obtain $D(\mathbf{A})$ to be dense in X_σ, provided s, r, q fulfill (2.1). This will be done in Lemma 3.4 below.

To prove the resolvent estimates (3.3) we regard $(SRP)_{u_0,\lambda}$ as the problem

$$\begin{cases} (\lambda - \Delta)u &= u_0 - \nabla p \quad \text{in} \quad \mathbb{R}^n_+, \\ u &= 0 \quad\quad\quad\;\, \text{in} \quad \mathbb{R}^{n-1}, \end{cases}$$

Proposition 3.2 yields

$$\sum_{k=0}^{2} |\lambda|^{k/2} \|\nabla^{2-k} u\|_X \le C_{\varphi_0} \|u_0 - \nabla p\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0}.$$

So, if we can show

$$\|\nabla p\|_X \le C_{\varphi_0} \|u_0\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

the proof of Theorem 3.1 is complete. But this is an immediate consequence of the next lemma for $\delta = 0$. □

For later purposes we state the estimate for the pressure in a more general form. To this end define the operator $S(\lambda)$ by

$$S(\lambda)u_0 := \nabla p, \quad u_0 \in X, \tag{3.6}$$

where p is given by formula (3.4).

Lemma 3.3. *Let $\varphi_0 \in (0, \pi)$, $1 < p < \infty$, and $\delta \in [0, 1/p']$. Then there is a constant $C = C(\delta, \varphi_0)$ such that*

$$\| |\nabla'|^{-\delta} S(\lambda) \|_{\mathcal{L}(X)} \le \frac{C}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0}. \tag{3.7}$$

Proof. Fix $\varphi_0 \in (0, \pi)$ and $\delta \in [0, 1/p']$. Let $\phi \in (0, \varphi_0/4)$ and define for $f \in L^p(\mathbb{R}_+)$,

$$(h_\lambda(z)f)(x_n) := \left(1 + \frac{z}{\omega(z)}\right) z^{1-\delta} \mathrm{e}^{-zx_n} \int_0^\infty \mathrm{e}^{-\omega(z)s} f(s) \mathrm{d}s, \quad z \in \Sigma_\phi, \ x_n > 0.$$

Then, by representation (3.4) we see that $|\nabla'|^{-\delta}(S(\lambda)u_0)^n$ can be written as

$$|\nabla'|^{-\delta}(S(\lambda)u_0)^n = -R' \cdot h_\lambda(|\nabla'|)u_0', \quad u_0 \in X.$$

We already know that $R' \in \mathcal{L}(X)$. Therefore, in view of Corollary 2.10, it remains to show that $h_\lambda \in H^\infty(\Sigma_\phi; \mathcal{L}(L^p(\mathbb{R}_+)))$ with the upper bound given in (3.7). But for $f \in L^p(\mathbb{R}_+)$ we have

$$\begin{aligned}
\|h_\lambda(z)f\|_{L^p(\mathbb{R}_+)} &\leq \left|\left(1+\frac{z}{\omega(z)}\right)z^{1-\delta}\right| \|e^{-zx_n}\|_{L^p(\mathbb{R}_+)} \int_0^\infty |e^{-\omega(z)s}f(s)|ds \\
&\leq C\left|\left(1+\frac{z}{\omega(z)}\right)z^{1-\delta}\right| |z|^{-1/p}\|e^{-\omega(z)s}\|_{L^{p'}(\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}_+)} \\
&\leq C\left(1+\left|\frac{z}{\omega(z)}\right|\right)\left|\frac{z}{\omega(z)}\right|^{\frac{1}{p'}-\delta}\frac{1}{|\omega(z)|^\delta}\|f\|_{L^p(\mathbb{R}_+)}.
\end{aligned}$$

Our choice $\phi \in (0, \varphi_0/4)$ (which is possible in view of $\phi^\infty_{|\nabla'|} = 0$) implies the existence of a $c_1 = c_1(\varphi_0) > 0$ such that $\mathrm{Re}\,\omega(z) \geq c_1(\sqrt{|\lambda|} + |z|)$ for $\lambda \in \Sigma_{\pi-\varphi_0}$, $z \in \Sigma_\phi$. Then, it easily follows

$$\left|\frac{z}{\omega(z)}\right| \leq C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi,$$

and

$$\frac{1}{|\omega(z)|^\delta} \leq \frac{C_{\varphi_0}}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi.$$

Hence, since $\delta \in [0, 1/p']$, i.e. $\frac{1}{p'} - \delta > 0$,

$$\|h_\lambda(z)\|_{\mathcal{L}(L^p(\mathbb{R}_+))} \leq \frac{C_{\varphi_0}}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi.$$

Employing Corollary 2.10 we finally may conclude

$$\begin{aligned}
\||\nabla'|^{-\delta}(S(\lambda)u_0)^n\|_X &= \|R' \cdot h_\lambda(|\nabla'|)u_0'\|_X \leq C\sum_{j=1}^{n-1}\|h_\lambda(|\nabla'|)u_0^j\|_X \\
&\leq C_{\varphi_0}\|h_\lambda\|_{L^\infty(\Sigma_\phi;\mathcal{L}(L^p(\mathbb{R}_+)))}\|u_0\|_X \\
&\leq C_{\varphi_0}\|u_0\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ u_0 \in X.
\end{aligned}$$

By the equality

$$i\xi'\hat{p} = \frac{i\xi'}{|\xi'|}|\xi|\hat{p} = -\frac{i\xi'}{|\xi'|}\partial_n\hat{p},$$

we have

$$|\nabla'|^{-\delta}(S(\lambda)u_0)' = -R'|\nabla'|^{-\delta}(S(\lambda)u_0)^n.$$

Again in view of $R' \in \mathcal{L}(X)$, we see that the corresponding estimate for $|\nabla'|^{-\delta}(S(\lambda)u_0)'$ is reduced to the just proved estimate for $|\nabla'|^{-\delta}(S(\lambda)u_0)^n$. Hence, the proof is complete. □

Lemma 3.4. *Let r, q, s, p be as in Lemma 2.4. Then*

$$S(\lambda)f \to 0 \quad \textit{in } X \textit{ if } \lambda \to \infty$$

for $f \in X_\sigma$.

Proof. The proof of Lemma 3.3 for $\delta = 0$ shows that we can obtain an estimate as

$$\|S(\lambda)f\|_X \leq C\|(-\Delta')^{1/2p'}(\lambda - \Delta')^{-1/2p'}f\|_X.$$

The operator on the right hand side can be written as

$$(-\Delta')^\alpha(\lambda - \Delta')^{-\alpha} = (\lambda^\alpha + (-\Delta')^\alpha)(\lambda - \Delta')^{-\alpha}(-\Delta')^\alpha(\lambda^\alpha + (-\Delta')^\alpha)^{-1}$$

with $\alpha = 1/2p'$. Now, in view of Proposition 2.8, $(\lambda^\alpha + (-\Delta')^\alpha)(\lambda - \Delta')^{-\alpha}$ is bounded on X_σ even with an upper bound independent of λ. Moreover, since the sectoriality of $-\Delta'$ in X_σ implies also $(-\Delta')^\alpha$ to be sectorial in X_σ (with $\phi_{(-\Delta')^\alpha} = 0$), we have

$$(-\Delta')^\alpha(\sigma + (-\Delta')^\alpha)^{-1}f \to 0 \quad \text{if} \quad \sigma \to \infty$$

for $f \in X_\sigma$. Consequently

$$\|S(\lambda)f\|_X \leq C\|(-\Delta')^{1/2p'}(\lambda^{1/2p'} + (-\Delta')^{1/2p'})^{-1}f\|_X \to 0 \quad \text{if} \quad \lambda \to \infty$$

for $f \in X_\sigma$. □

The boundedness of the operator $\mathbf{P}_+$ on X and of the Ekman spiral solution $\mathbf{U}^E$ now allows us to employ a standard perturbation argument for proving the generation result for the full linear operator $\mathbf{A}_E$. More precisely we have

Theorem 3.5. *Let $\varphi_0 \in (0, \pi/2]$. There are constants $K_1 = K_1(\varphi_0) > 0$, $K_2 = K_2(\varphi_0) \geq 1$ such that for $\omega_0 = \omega_0(\varphi_0) := 2K_2 \max\{1, [K_1(\Omega + \|\mathbf{U}^E\|_{1,\infty})]^2\}$ we have*

$$\Sigma_{\pi-\varphi_0} \subseteq \rho(-(\mathbf{A}_E + \omega_0))$$

and

$$\sum_{k=0}^{2} |\lambda|^{k/2}\|\nabla^{2-k}(\lambda + \mathbf{A}_E + \omega_0)^{-1}\|_{\mathcal{L}(X)} \leq C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

for some $C_{\varphi_0} > 0$. Hence, $\mathbf{A}_E$ is the generator of a holomorphic C_0-semigroup with growth bound $\omega_{\mathbf{A}_E} \leq \omega_0(\pi/2)$.

Proof. Set $\mathbf{B} := \Omega \mathbf{P}_+ \mathbf{J} \mathbf{P}_+ + \mathbf{C}_E$. For $\omega_0 > 0$ the resolvent of $\mathbf{A}_E + \omega_0 = \mathbf{A} + \mathbf{B} + \omega_0$ can be written as

$$(\lambda + (\omega_0 + \mathbf{A} + \mathbf{B}))^{-1} = (\lambda + \omega_0 + \mathbf{A})^{-1}[I + \mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}]^{-1}. \tag{3.8}$$

Next we estimate $\|\mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)}$. Since $\mathbf{U}^E$ depends only on x_n we obtain

$$\begin{aligned} \|\mathbf{C}_E(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} &\leq C\Big(\|\mathbf{U}^E\|_\infty \|\nabla(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} \\ &\qquad + \|\partial_n \mathbf{U}^E\|_\infty \|(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)}\Big) \\ &\leq \frac{C_{\varphi_0}}{\sqrt{|\lambda + \omega_0|}} \|\mathbf{U}^E\|_{1,\infty}, \quad |\lambda + \omega_0| \geq 1, \end{aligned}$$

where we applied (3.3). This implies by the boundedness of $\mathbf{P}_+$ on X

$$\begin{aligned} \|\mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} &\leq C_{\varphi_0}\left(\Omega \|(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} + \frac{1}{\sqrt{|\lambda + \omega_0|}} \|\mathbf{U}^E\|_{1,\infty} \right) \\ &\leq \frac{K_1}{\sqrt{|\lambda + \omega_0|}} (\Omega + \|\mathbf{U}^E\|_{1,\infty}), \quad |\lambda + \omega_0| \geq 1, \end{aligned} \tag{3.9}$$

where $K_1 = K_1(\varphi_0)$ depends on upper bounds for $\|\mathbf{P}_+\|_{\mathcal{L}(X)}$ and $\|\lambda(\lambda + \mathbf{A})^{-1}\|_{\mathcal{L}(X)}$ only. Note that there is a constant $K_2 = K_2(\varphi_0) \geq 1$ such that $|\lambda + \omega_0| \geq K_2^{-1}\omega_0$ for all $\lambda \in \Sigma_{\pi - \varphi_0}$ and $\omega_0 > 0$. Now we set $\omega_0 := 2K_2 \max\{1, [K_1(\Omega + \|\mathbf{U}^E\|_{1,\infty})]^2\}$. Then we may employ the Neumann series obtaining

$$\begin{aligned} & \|\nabla^k(\lambda + (\omega_0 + \mathbf{A} + \mathbf{B}))^{-1}\|_{\mathcal{L}(X)} \\ \leq\ & \|\nabla^k(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} \|[I + \mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}]^{-1}\|_{\mathcal{L}(X)} \\ \leq\ & \frac{C}{|\lambda + \omega_0|^{(2-k)/2}} \sum_{j=0}^{\infty} \left[\left(\frac{\omega_0}{2K_2|\lambda + \omega_0|} \right)^{1/2} \right]^j \\ \leq\ & \frac{C}{|\lambda + \omega_0|^{(2-k)/2}} \frac{1}{1 - (1/2)^{1/2}} \\ \leq\ & \frac{C}{|\lambda + \omega_0|^{(2-k)/2}}, \quad \lambda \in \Sigma_{\pi - \varphi_0}, \ k \in \{0, 1, 2\}, \end{aligned}$$

where we applied again estimate (3.3) for the Stokes operator $\mathbf{A}$. □

3.2 Nonlinear problem - local existence

We start with some Lemmata stating useful estimates we will need in order to estimate the nonlinear term. Exemplary we will give the proof of some of the statements. The first one is already proved in [7].

Lemma 3.6. *Let $n \in \mathbb{N}$ and $\alpha > 0$. Then there exists $C_\alpha = C(\alpha) > 0$ such that*
(1) $||(-\Delta)^\alpha G_t(x)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \leq Ct^{-\alpha}$ *for* $t > 0$,
where G_t denotes the heat kernel, and
(2) $||(-\Delta)^\alpha e^{t\Delta} f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \leq Ct^{-\alpha}||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $t > 0$.

Lemma 3.7. *Assume $1 < q \leq p < \infty$, $\delta \in (0, 1/2]$.*
(1) *It holds*

$$\begin{aligned} &||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \\ \leq\ & C(\delta)t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\left(||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + ||\nabla' f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\right) \end{aligned}$$

for $j = 1, ..., n-1$ and any $t > 0$.
(2) $||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{W^{1,\infty}(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))}$ *for any $t > 0$.*
(3) $||\partial_j e^{t\Delta_D} f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$
for $j = 1, ..., n-1$ and any $t > 0$.
(4) $||\partial_n e^{t\Delta_D} f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$ *for any $t > 0$.*

Remark 3.8. (a) Due to the fact that ∂_n acts on the third component (L^p part) there is no regularizing effect in that case, i.e. we cannot expect to generalize the estimates (2) and (4) to

$$\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \to \dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$$

as (1) and (3).
(b) The properties of the Dirichlet Laplacian Δ_D we use in the proof of Lemma 3.7 are known also for the Neumann Laplacian Δ_N. Hence all assertions of Lemma 3.7 are valid for Δ_N as well.

Proof. (1) Since ∂_j for $1 \leq j \leq n-1$, $-\Delta'$, and $e^{t\Delta'}$ act only on the tangential direction we have

$$\begin{aligned} ||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} &= \sum_{k=-\infty}^{\infty} ||\,|\phi_k * e^{t\Delta_D}\partial_j f|_{L^p(\mathbb{R}_+)}||_{L^\infty(\mathbb{R}^{n-1})} \\ &= \sum_{k=-\infty}^{\infty} ||\,|(-\Delta')^\delta e^{t\Delta'}\phi_k * e^{t\Delta_{D,n}}\partial_j(-\Delta')^{-\delta} f|_p||_\infty. \end{aligned}$$

Here, $\phi_k = \phi_k(x_1, x_2)$ for all $k \in \mathbb{Z}$. Multiplying $1 = \sum_{l\in\mathbb{Z}} \phi_l *$, it follows from $e^{t\Delta_{D,n}}(\phi_l * f) = \phi_l * e^{t\Delta_{D,n}} f$ that

$$\begin{aligned}
\|e^{t\Delta_D}\partial_j f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} &= \sum_{k,l\in\mathbb{Z},|k-l|\le 2} \| \, |e^{t\Delta'}\phi_k * \phi_l * e^{t\Delta_{D,n}}\partial_j f|_p \|_\infty \\
&= \sum_{k,l\in\mathbb{Z},|k-l|\le 2} \| \, |(-\Delta')^{\delta} e^{t\Delta'}\phi_k * e^{t\Delta_{D,n}}(\phi_l * \partial_j(-\Delta')^{-\delta} f)|_p \|_\infty.
\end{aligned}$$

Then vector-valued Young's inequality yields (see page 13 in [1])

$$\begin{aligned}
&\|e^{t\Delta_D}\partial_j f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \le \\
\le &\sum_{k,l\in\mathbb{Z},|k-l|\le 2} \|(-\Delta')^{\delta} e^{t\Delta'}\phi_k\|_{L^1(\mathbb{R}^{n-1};\mathbb{R})} \|e^{t\Delta_{D,n}}(\phi_l * \partial_j(-\Delta')^{-\delta} f)\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|e^{t\Delta_D}\partial_j f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \le \\
\le &\sup_{l\in\mathbb{Z}} \|e^{t\Delta_{D,n}}(\phi_l * \partial_j(-\Delta')^{-\delta} f)\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z},|k-l|\le 2} \|(-\Delta')^{-\delta} e^{t\Delta'}\phi_k\|_{L^1(\mathbb{R}^{n-1};\mathbb{R})} \\
= &\; 5 \sup_{l\in\mathbb{Z}} \|e^{t\Delta_{D,n}}(\phi_l * \partial_j(-\Delta')^{-\delta} f)\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \sum_{k\in\mathbb{Z}} \|(-\Delta')^{-\delta} e^{t\Delta'}\phi_k\|_{L^1(\mathbb{R}^{n-1};\mathbb{R})}.
\end{aligned}$$

The $L^p - L^q$-estimate of the operator $e^{t\Delta_{D,n}}$ yields

$$\begin{aligned}
\|e^{t\Delta_{D,n}}(\phi_l * f)\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} &= \| \, |e^{t\Delta_{D,n}}(\phi_l * f)|_p \|_\infty \le C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \| \, |\phi_l * f|_q \|_\infty \\
&= C t^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|\phi_l * f\|_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}.
\end{aligned}$$

On the other hand, it follows from Lemma 3.6 (1) that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \|(-\Delta')^{-\delta} e^{t\Delta'}\phi_k\|_{L^1(\mathbb{R}^{n-1};\mathbb{R})} &= \sum_{k\in\mathbb{Z}} \|(-\Delta')^{-\delta} G_t(x') * \phi_k\|_{L^1(\mathbb{R}^{n-1};\mathbb{R})} \\
&= \|(-\Delta')^{-\delta} G_t(x')\|_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})} \le C t^{-\delta}.
\end{aligned}$$

Thus we conclude

$$\begin{aligned}
\|e^{t\Delta_D}\partial_j f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} &\le C t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \sup_{l\in\mathbb{Z}} \|\phi_l * \partial_j(-\Delta')^{-\delta} f\|_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\
&= C t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|\partial_j(-\Delta')^{-1/2}(-\Delta')^{\frac{1}{2}-\delta} f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\
&\le C t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|(-\Delta')^{\frac{1}{2}-\delta} f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}, \qquad (3.10)
\end{aligned}$$

where we used $R'_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$ for the tangential Riesz operator $R' = \partial_j(-\Delta')^{-1/2}$ in the last inequality. By Proposition 2.8 the operator $(-\Delta')^{\frac{1}{2}-\delta}(1-\Delta')^{-1/2}$ is bounded on $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))$. Moreover, by general results for fractional powers of sectorial operators we know that the norms $\|(1-\Delta')^{1/2}\cdot\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$ and $\|\cdot\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + \|(-\Delta')^{1/2}\cdot\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$ are equivalent. This implies

$$\begin{aligned}\|(-\Delta')^{\frac{1}{2}-\delta}f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} &= \|(-\Delta')^{\frac{1}{2}-\delta}(1-\Delta')^{-1/2}(1-\Delta')^{-1/2}f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\\ &\leq C\left(\|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + \|(-\Delta')^{1/2}f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\right).\end{aligned}$$

Combining this with (3.10) it remains to show

$$\|(-\Delta')^{1/2}f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \leq C\|\nabla' f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}.$$

But this estimate follows easily from the representation $(-\Delta')^{1/2} = \sum_{j=1}^{n-1} R'_j\partial_j$ by applying once again $R'_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$. □

By using the lemma above we can estimate terms of the form $\partial_j e^{t\Delta_D}\mathbf{P}_+ f$ for $1 \leq j \leq n$. The main problem occuring here is to handle the term with normal derivative ∂_n. The idea is to split $\mathbf{P}_+\partial_n$ into a normal derivative term without Riesz operators and terms including only tangential derivatives and Riesz operators.

Lemma 3.9. *Let $1 < q \leq p < \infty$ and $\delta \in (0, 1/2]$. Then for $1 \leq j \leq n$ we have*

$$\begin{aligned}&(1) \quad \|e^{t\Delta_D}\mathbf{P}_+\partial_j f\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq C_\delta t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{W^{1,\infty}(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))}, \quad t>0,\\ &(2) \quad \|\partial_j e^{t\Delta_D}\mathbf{P}_+ f\|_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}, \quad t>0.\end{aligned}$$

Proof. In the case that $1 \leq j \leq n-1$ we have $e^{t\Delta_D}\mathbf{P}_+\partial_j f = e^{t\Delta_D}\partial_j\mathbf{P}_+ f$. On the other hand, by Corollary 2.9 the operator $\mathbf{P}_+$ is bounded on $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$. Hence, by Lemma 3.7 (1) we have

$$\begin{aligned}&\|e^{t\Delta_D}\mathbf{P}_+\partial_j f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq\\ &\leq Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + \|\nabla' f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\right), \quad t>0, \quad (3.11)\end{aligned}$$

yielding (1) for $1 \leq j \leq n-1$ in virtue of Lemma 2.3. For $j = n$ we use the following splitting of $\mathbf{P}_+\partial_n f$:

$$\begin{aligned}
\mathbf{P}_+\partial_n f &= r\mathbf{P}E\partial_n f = r\mathbf{P}\partial_n\widetilde{E}f && (3.12)\\
&= r\partial_n((\widetilde{E}f)',0) + \sum_{j=1}^{n-1} r\partial_j R(R_n(\widetilde{E}f)^j) + \\
& \quad r(\nabla',0)R_n^2(\widetilde{E}f)^n - \sum_{j=1}^{n-1} r\partial_j R_j R_n(\widetilde{E}f)^n e_n \\
&= \partial_n \mathbf{Q}_0 f + \sum_{j=1}^{n-1}\partial_j \mathbf{Q}_j f \\
&=: I + II,
\end{aligned}$$

where the operators $\mathbf{Q}_0, \mathbf{Q}_1, \ldots, \mathbf{Q}_{n-1}$ are defined by

$$\mathbf{Q}_0 g = r((\widetilde{E}g)',0) = (g',0), \tag{3.13}$$

$$\mathbf{Q}_j g = rR(R_n(\widetilde{E}g)^j) + rR_n^2(\widetilde{E}g)^n e_j - rR_jR_n(\widetilde{E}g)^n e_n, \quad j = 1,\ldots,n-1. \tag{3.14}$$

Here we denote by e_j the unit vector whose j-th component is 1 and $Rh = (R_1h, \ldots, R_nh)$ for scalar function h. To derive (3.12) we used the facts that

$$\partial_n R_j = \partial_n\partial_j(-\Delta)^{-1/2} = \partial_j\partial_n(-\Delta)^{-1/2} = \partial_j R_n \quad \text{for } 1 \le j \le n, \quad \text{and} \quad R_n^2 = -1 - \sum_{j=1}^{n-1} R_j^2.$$

By the boundedness of r, E and in view of $R \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R})))$ this implies that $\mathbf{Q}_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$, $j = 1, \ldots, n-1$. Applying Lemma 3.7 (2) to I and Lemma 3.7 (1) to II then yields

$$\begin{aligned}
&||e^{t\Delta_D}\mathbf{P}_+\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \le \\
\le\ & ||e^{t\Delta_D}\partial_n\mathbf{Q}_0 f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} + \sum_{j=1}^{n-1}||e^{t\Delta_D}\partial_j\mathbf{Q}_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\
\le\ & Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\Bigg[||\mathbf{Q}_0 f||_{W^{1,\infty}(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))} \\
& + \sum_{j=1}^{n-1}\Big(||\mathbf{Q}_j f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + ||\mathbf{Q}_j\nabla' f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\Big)\Bigg] \\
\le\ & Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{W^{1,\infty}(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))}.
\end{aligned}$$

□

Lemma 3.10. *Let $\varphi_0 \in (0,\pi)$, $2 < p < \infty$. There is a $C = C_{\varphi_0} > 0$ such that*

$$(1) \qquad \|(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{B}^0_{\infty,1}(L^p)} \le \frac{C}{|\lambda|^{1-\frac{1}{2p}}} \|f\|_{\dot{B}^0_{\infty,\infty}(L^{p/2})},$$

$$(2) \qquad \|\nabla(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{B}^0_{\infty,1}(L^p)} \le \frac{C}{|\lambda|^{\frac{1}{2}-\frac{1}{2p}}} \|f\|_{\dot{B}^0_{\infty,\infty}(L^{p/2})},$$

$$(3) \qquad \|B(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{B}^0_{\infty,1}(L^p)} \le \frac{C}{|\lambda|^{\frac{1}{2}-\frac{1}{2p}}} \|f\|_{\dot{B}^0_{\infty,\infty}(L^{p/2})},$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| \ge 1$, $f \in \dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})$.

The next proposition contains the crucial estimates that allow us to construct solutions in the space $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. Note again that due to the fact mentioned in Remark 3.8 (a) we are not able to carry out the iteration in the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$.

Proposition 3.11. *Let $2 < p < \infty$, $\varphi_0 \in (0,\pi/2)$, $\delta \in (0,1/4)$, and $\omega_0 = \omega_0(\varphi_0)$ as in Theorem 3.5. There exist $C = C(\varphi_0,\delta) > 0$ and $\omega_1 \ge \omega_0$ such that*

$$\| \nabla^\ell e^{-t(\mathbf{A}_E+\omega_1)} \mathbf{P}_+ \partial_j f\|_{L^\infty(\mathbb{R}^2;L^p(\mathbb{R}_+))} \le C t^{-\frac{\ell}{2}-\frac{1}{2p}-\delta(1-\ell)} \|f\|_{W^{1,\infty}(\mathbb{R}^2;W^{1,p/2}(\mathbb{R}_+))} \tag{3.15}$$

for $t > 0$, $\ell = 0,1$, $j = 1,2,3$, $f \in \mathcal{BUC}^1(\mathbb{R}^2; W^{1,p/2}(\mathbb{R}_+)) := \{u \in \mathcal{BUC}(\mathbb{R}^2; L^{p/2}(\mathbb{R}_+)) : \nabla u \in \mathcal{BUC}(\mathbb{R}^2; L^{p/2}(\mathbb{R}_+))\}$.

Proof. For simplicity we omit the $\mathbb{R}$ notation in the spaces, i.e. we write $W^{1,\infty}(L^p) = W^{1,\infty}(\mathbb{R}^2; L^p(\mathbb{R}_+))$, $L^\infty(L^p) = L^\infty(\mathbb{R}^2; L^p(\mathbb{R}_+))$ and so on in the sequel. We will prove the corresponding estimates for the resolvent of $\mathbf{A}_E$, i.e.

$$\|\nabla^\ell(\lambda + \omega_1 + \mathbf{A}_E)^{-1} \mathbf{P}_+ \partial_j f\|_{L^\infty(L^p)} \le \frac{C}{|\lambda|^{1-\frac{\ell}{2}-\frac{1}{2p}-\delta(1-\ell)}} \|f\|_{W^{1,\infty}(W^{1,p/2})} \tag{3.16}$$

for $j = 1,2,3$, $\ell = 0,1$, $\lambda \in \Sigma_{\pi-\varphi_0}$, and $f \in W^{1,\infty}(W^{1,p/2})$. Then (3.15) easily follows by the representation

$$\mathrm{e}^{-t(\mathbf{A}_E+\omega_1)} = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-t\lambda} (\lambda + \omega_1 + \mathbf{A}_E)^{-1} \mathrm{d}\lambda.$$

Now fix $\varphi_0 \in (0,\pi/2)$ and set $\mu := \lambda + \omega_1$. Observe that the resolvent of the Stokes operator $\mathbf{A}$ can be written as

$$(\mu + \mathbf{A})^{-1} = (\mu - \Delta_D)^{-1}(I - S(\mu)), \tag{3.17}$$

where $S(\mu)$ is defined as in (3.6). Thus, according to (3.8) the resolvent of $\mathbf{A}_E$ is represented as

$$(\mu+\mathbf{A}_E)^{-1}=(\mu-\Delta_D)^{-1}(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}.$$

Let us first consider the easier case of tangential derivatives. Since ∂_j commutes with $\mathbf{P}_+$ and all parts of $\mathbf{A}_E$ for $j=1,2$, in this case we have

$$\begin{aligned}
&\|(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\\
=\ & C\|(\mu-\Delta_D)^{-1}\partial_j(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+f\|_{\dot{B}^0_{\infty,1}(L^p)}\\
\le\ & \frac{C}{|\mu|^{1-\frac{1}{2p}-\delta}}\Big(\|(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+f\|_{\dot{B}^0_{\infty,\infty}(L^{p/2})}\\
&+\|(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+\nabla' f\|_{\dot{B}^0_{\infty,\infty}(L^{p/2})}\Big)\\
\le\ & \frac{C}{|\mu|^{1-\frac{1}{2p}-\delta}}\|f\|_{W^{1,\infty}(L^{p/2})},
\end{aligned}\tag{3.18}$$

where we applied Lemma 3.7 (1) as well as the boundedness of $S(\mu)$, $[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}$, and $\mathbf{P}_+$ in the space $\dot{B}^0_{\infty,\infty}(L^{p/2})$ given by Lemma 3.3, by $\omega_1\ge\omega_0$ and our choice of ω_0 (see Theorem 3.5), and Corollary 2.9, respectively. Applying Lemma 3.7 (3) instead of Lemma 3.7 (1) we can obtain in an analogous way

$$\|\partial_i(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\le\frac{C_{\varphi_0}}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}}\|\partial_j f\|_{L^\infty(L^{p/2})}\tag{3.19}$$

for $i=1,2$ and $j=1,2,3$.

The case of normal derivatives is more involved. Here we employ Neumann series and use the representation of the form

$$(\mu+\mathbf{A}_E)^{-1}=\sum_{k=0}^{\infty}(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k.$$

In order to estimate this expression we need

Lemma 3.12. *There are constants $K=K(\varphi_0)>0$ and $\omega_1\ge\omega_0$ such that*

$$\|(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)}\le\frac{K}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\frac{1}{\sqrt{2}}\right)^k\|f\|_{W^{1,\infty}(W^{1,p/2})}\tag{3.20}$$

for all $f \in \mathcal{BUC}^1(W^{1,p/2})$, $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$, $k = 0, 1, 2, \ldots$, *and*

$$\|\partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{P}_+ \partial_j f\|_{L^\infty(L^p)} \le \frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|f\|_{W^{1,\infty}(W^{1,p/2})} \qquad (3.21)$$

for all $f \in \mathcal{BUC}^1(W^{1,p/2})$, $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$, $j = 1, 2, 3$, $k = 0, 1, 2, \ldots$.

Proof. The assertion follows by induction over k. The proof is very technical, but for the single steps we basically use the same methods as in the proofs of the previous Lemmata, i.e. employing Besov spaces in the case of terms including tangential derivatives or $S(\lambda)$, or using the splitting of $\mathbf{P}_+\partial_3$. □

We complete the proof of Proposition 3.11. From (3.20) we immediately conclude

$$\begin{aligned} \|(\mu + \mathbf{A}_E)^{-1}\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)} &\le \sum_{k=0}^{\infty} \frac{C_{\varphi_0}}{|\mu|^{1-\frac{1}{2p}-\delta}} \left(\frac{1}{\sqrt{2}}\right)^k \|f\|_{W^{1,\infty}(W^{1,p/2})} \\ &\le \frac{C_{\varphi_0}}{|\mu|^{1-\frac{1}{2p}-\delta}} \|f\|_{W^{1,\infty}(W^{1,p/2})} \end{aligned}$$

On the other hand (3.21) implies

$$\begin{aligned} \|\partial_3(\mu + \mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)} &\le \sum_{k=0}^{\infty} \frac{C_{\varphi_0}}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|f\|_{W^{1,\infty}(W^{1,p/2})} \\ &\le \frac{C_{\varphi_0}}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \|f\|_{W^{1,\infty}(W^{1,p/2})} \end{aligned}$$

for $j = 1, 2, 3$. Combining these two inequalities with (3.18) and (3.19), estimate (3.16) and thus the assertion of Proposition 3.11 follows. □

Theorem 1.1 is now essentially a consequence of Proposition 3.11 and the contraction mapping principle applied on $v = Fv$ with

$$Fv(t) := e^{-t\mathbf{A}_E} v_0 - \int_0^t e^{-(t-s)\mathbf{A}_E} \mathbf{P}_+ \mathrm{div}(v(s) \otimes v(s)) \mathrm{d}s$$

in the space

$$X_{T,K} = \{v \in \mathrm{BC}((0,T); \mathcal{BUC}_\sigma(\mathbb{R}^2; L^p(\mathbb{R}_+)); \|v\|_{X_T} < K\}$$

with norm

$$\|v\|_{X_T} := \sup_{0 \le s \le T} \|v\|_{L^\infty(L^p)}(s) + \sup_{0 \le s \le T} s^{1/2} \|\nabla v\|_{L^\infty(L^p)}(s).$$

We omit the proof here and refer to [8] for the details. We just remark that there remains one difficuly concerning the continuity of the solution, i.e. the fact that we can construct the solutions in $\mathrm{BC}((0,T);\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+)))$ instead of $L^\infty((0,T);\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+))$ only. Note that $\mathrm{e}^{-t\mathbf{A}_E}$ is not even expected to be bounded on $\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+)))$. Essentially the continuity is a consequence of estimate (3.16) and representation

$$\mathrm{e}^{-t(\omega_1+\mathbf{A}_E)}\mathbf{P}_+\partial_j = \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{-t\lambda}(\lambda-(\omega_1+\mathbf{A}_E))^{-1}\mathbf{P}_+\partial_j \mathrm{d}\lambda, \quad j=1,2,3,\ t>0,$$

which is valid even on $\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+)))$, since $\{\mathrm{e}^{-t\mathbf{A}_E}\}_{t\geq 0}$ is the generator of a holomorphic semigroup on the larger space $\dot{\mathcal{B}}^0_{\infty,\infty,\sigma}(\mathbb{R}^2;L^p(\mathbb{R}_+))$.

References

[1] H. Amann (1997), Operator-Valued Fourier Multipliers, Vector-Valued Besov Spaces, and Applications, *Math. Nachr.*, **186**, p. 5-56.

[2] H. Amann (2003), Vector-Valued Distributions and Fourier Multipliers, http://www.math.unizh.ch/amann/books.html.

[3] J. Bergh and J. Löfström, (1976), *Interpolation Spaces. An Introduction*, Springer-Verlag.

[4] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, (1996), Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc.*, **60**, p. 51-89.

[5] R. Denk, M. Hieber, J. Prüss, (2001), R-Boundedness, Fourier multipliers and problems of elliptic and parabolic type, *Mem. Amer. Math. Soc.*, **166**, p. viii-114.

[6] W. Desch, M. Hieber, J. Prüss, (2001), L^p-Theory of the Stokes equation in a half-space, *J. Evol. Equ.*, **1**, p. 115-142.

[7] Y. Giga, K. Inui, A. Mahalov and S. Matsui, Navier-Stokes Equations in a Rotating Frame in $\mathbb{R}^3$ with Initial Data Nondecreasing at Infinity, submitted for publication in *Hokkaido Math. J.*

[8] Y. Giga, K. Inui, A. Mahalov, S. Matsui, and J. Saal, Rotating Navier-Stokes equations in $\mathbb{R}^3_+$ with initial data nondecreasing at infinity: The Ekman boundary layer problem, in preparation.

[9] H.P. Greenspan, (1968), *The Theory of Rotating Fluids*, Cambridge University Press.

[10] Robert Haller, (2003), *Poissonkern Abschätzungen und beschränkter H^∞-Kalkül*, PhD Thesis, TU Darmstadt, Wissenschaftlicher Verlag, Berlin.

[11] J.R. Holton, (1992), *An Introduction to Dynamic Meteorology*, Academic Press.

[12] N. Kalton and L. Weis, (2001), The H^∞-calculus and sums of closed operators, *Math. Ann.*, **321**, p. 319–345.

[13] H. Kozono and Masao Yamazaki, (1994), Semilinear heat equations and Navier-Stokes equation with distributions in new function spaces as initial data, *Commun. Partial Differ. Equations*, **19**, No. 5-6, p. 959–1014.

[14] J. Pedlosky, (1987), *Geophysical Fluid Dynamics*, 2nd edition, Springer–Verlag.

[15] J. Saal, (2003), *Robin Boundary Conditions and Bounded H^∞-Calculus for the Stokes Operator*, PhD Thesis, TU Darmstadt, Logos Verlag, Berlin.

[16] J. Saal, (2004), Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space, J. Mat. Fluid Mech., to appear.

[17] O. Sawada, (2004), The Navier-Stokes flow with linearly growing initial velocity in the whole space, Funkcial. Ekvac., to appear.

[18] O. Sawada and Y. Taniuchi (2003), On the Boussinesq flow with nondecaying initial data, Bol. Soc. Paran. Mat., to appear.

[19] Y. Shimizu (1999), L^∞-Estimate of first-order space derivatives of Stokes flow in a half space, *Funkcial. Ekvac.*, **42**, p. 291-309.

[20] V. A. Solonnikov (2003), On nonstationary Stokes problem and navier-Stokes problem in a half-space with initial data nondecreasing at infinity, *Journal of Mathematical Sciences*, **114**, No. 5, p. 1726–1740.

[21] E.M. Stein (1993), *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press.

[22] H. Triebel, (1992), *Theory of Function Spaces*, Monographs in Mathematics, **78**, Basel-Boston-Stuttgart: Birkhäuser Verlag.