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<th>On a removable isolated singularity theorem for the stationary Navier-Stokes equations (Harmonic Analysis and Nonlinear Partial Differential Equations)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2004), 1401: 152-160</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/26059">http://hdl.handle.net/2433/26059</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
On a removable isolated singularity theorem for the stationary Navier-Stokes equations

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1 Introduction

The purpose of this note is to provide a removable isolated singularity theorem for smooth solutions of the Navier-Stokes equations

\[-\Delta u + \text{div}(u \otimes u) + \nabla p = f \quad \text{and} \quad \text{div} \, u = 0 \quad \text{(NS)},\]

where $\Omega$ is a nonempty open subset of $\mathbb{R}^n$ with $n \geq 3$. Here $u = (u^1, u^2, \ldots, u^n)$ and $p$ denote the unknown velocity and pressure fields of a stationary viscous incompressible fluid driven by an external force $f$. We also denote by $\text{div}(u \otimes u)$ the vector field whose $j$-th component is $\text{div}(uu^j) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u^i u^j)$.

Our main result reads

**Theorem 1** Let $(u, p)$ be a $C^\infty$-solution of the Navier-Stokes equations (NS) in $B_R \setminus \{0\}$. Suppose that

\[ f \in C^\infty(B_R) \]

and

\[ u \in L^n(B_R) \quad \text{or} \quad |u(x)| = o(|x|^{-1}) \quad \text{(1)} \]

as $x \to 0$. Then $(u, p)$ can be defined at 0 so that it is a $C^\infty$-solution of (NS) in $B_R$.

Theorem 1 improves the previous results by Dyer and Edmunds [2], Shapiro [9, 10] and by Choe and Kim [1]. Moreover, for the three-dimensional case ($n=3$), Theorem 1 is best possible due to singular solutions constructed by Tian

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*supported by Japan Society for the Promotion of Science under JSPS Postdoctoral Fellowship For Foreign Researchers.
and Xin [12]. For any real number $c$ with $|c| > 1$, let us define $u = (u^1, u^2, u^3)$ and $p$ by

$$u^1(x) = 2 \frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \quad u^2(x) = 2 \frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2},$$
$$u^3(x) = 2 \frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2} \quad \text{and} \quad p(x) = 4 \frac{cx_1 - |x|}{|x|(c|x| - x_1)^2}.$$

Then a straightforward calculation shows that $(u, p)$ is a $C^\infty$-solution of (NS) in $B_1 \setminus \{0\}$ with $f = 0$, $|u(x)| = O(|x|^{-1})$ as $x \to 0$ but the singularity at 0 is irremovable.

Our proof of Theorem 1 is based on Shapiro’s removable singularity result and our new regularity result for distribution solutions of (NS). In [10], Shapiro proved

**Theorem 2 (Shapiro [10])** Suppose that

1. $u \in L^\beta_{loc}(B_R)$ for some $\beta > 2$, $p \in L^1_{loc}(B_R \setminus \{0\})$, $f \in L^1_{loc}(B_R)$,

2. $(u, p)$ is a distribution solution of (NS) in $B_R \setminus \{0\}$

3. and $\left(r^{-n} \int_{B_r} |u|^\beta \, dx\right)^{1/\beta} = o(r^{-(n-1)/2})$ as $r \to 0$.

Then $p \in L^1_{loc}(B_R)$ and $(u, p)$ is a distribution solution of (NS) in $B_R$.

To state our regularity result, let us introduce the definition of the weak $L^n(\Omega)$-norm:

$$||u||_{L^n(\Omega)} = \sup_{\sigma > 0} \sigma |\{x \in \Omega : |u(x)| > \sigma\}|^{1/n}.$$

Then since

$$||u||_{L^n(\Omega)} \leq ||u||_{L^n(B_r)} \quad \text{and} \quad |||x|^{-1}||_{L^n(\mathbb{R}^n)} = C(n) < \infty,$$

we easily show that if $u$ satisfies the condition (1), then

$$||u||_{L^n(B_r)} \to 0 \quad \text{as} \quad r \to 0.$$

Therefore, in view of Theorem 2, Theorem 1 is an immediate consequence of the following regularity result.

**Theorem 3** For each integer $m \geq 0$, let $q$ be a real number such that

$q \in (1, \infty)$ if $m = 0$ \quad and \quad $q \in (1, \infty) \cap [n/4, \infty)$ if $m \geq 1.$
Then there exists a small constant $\epsilon = \epsilon(n, q) > 0$ with the following property. If $(u, p) \in L^2_{loc}(\Omega) \times L^1_{loc}(\Omega)$ is a distribution solution of (NS) in $\Omega$ with $f \in W^{m,q}_{loc}(\Omega)$ and if $u$ satisfies

$$||u||_{L^\infty(\Omega)} \leq \epsilon,$$

then

$$u \in W^{m+2,q}_{loc}(\Omega) \quad \text{and} \quad p \in W^{m+1,q}_{loc}(\Omega).$$

As an easy corollary of Theorem 3, we also obtain the following interior regularity theorem for the Navier-Stokes equations (NS).

**Corollary 4** Let $(u, p) \in L^n_{loc}(\Omega) \times L^1_{loc}(\Omega)$ be a distribution solution of (NS) in $\Omega$. Suppose that

$$f \in W^{m,q}_{loc}(\Omega)$$

for some integer $m$ and real number $q$ such that

$$m = 0 \quad \text{and} \quad q \in (1, \infty) \quad \text{or} \quad m \geq 1 \quad \text{and} \quad q \in (1, \infty) \cap [n/4, \infty).$$

Then

$$u \in W^{m+2,q}_{loc}(\Omega) \quad \text{and} \quad p \in W^{m+1,q}_{loc}(\Omega).$$

Corollary 4 improves an interior regularity result in a book [3] by Galdi as well as Shapiro’s one in [9]. It was shown in [3, Section VIII.5] that if $(u, p) \in L^n_{loc}(\Omega) \cap W^{1,2}_{loc}(\Omega) \times L^2_{loc}(\Omega)$ is a weak solution of (NS) in $\Omega$ and if $f \in W^{m,q}_{loc}(\Omega)$ for some $(m, q)$ such that $q \in [2n/(n + 2), \infty)$ if $m = 0$ and $q \in [n/2, \infty)$ if $m \geq 1$, then $u \in W^{m+2,q}_{loc}(\Omega)$ and $p \in W^{m+1,q}_{loc}(\Omega)$.

Theorem 3 and its proof are inspired by our recent works [5, 7] on the interior regularity of weak solutions with small $L^\infty(0, T; L^3_{w}(\Omega))$-norm of the non-stationary Navier-Stokes equations in three dimensions. The remaining part of the note is devoted to giving a sketch of the proof of Theorem 3. For a more complete proof, please refer to our original paper [6].

2 A sketchy proof of Theorem 3

Let us first consider the following boundary value problem for the perturbed Stokes equations

$$\begin{align*}
-\Delta v + \text{div}(u \otimes v) + \nabla p &= f \quad \text{in} \quad B \\
\text{div} v &= g \quad \text{in} \quad B \\
v &= 0 \quad \text{on} \quad \partial B,
\end{align*}$$

(2)

where $u$ is a known divergence-free vector field in $L^n_{w}(B)$ and $B = B_1, B_2$ or $B_3$.

The following lemma is of basic importance to derive estimates for the convective term in (2).
Lemma 5 If \( v \in L^q_{\omega}(B) \) and \( w \in W^{1,q}(B) \) with \( 1 < q < n \), then

\[
  v \cdot w \in L^q(B) \quad \text{and} \quad ||v \cdot w||_{L^q(B)} \leq C||v||_{L^q_{\omega}(B)}||w||_{W^{1,q}(B)}.
\]

Here and after \( C \) denotes a positive constant depending only on \( n \) and \( q \).

Proof. Note that \( L^q(B) = L^{q,q}(B) \) and \( L^n_{\omega}(B) = L^{n,\infty}(B) \). Hence it follows from Hölder and Sobolev inequalities in Lorenz spaces (see Proposition 2.1 and Proposition 2.2 in [8]) that

\[
  ||v \cdot w||_{L^q(B)} = ||v \cdot w||_{L^{q,q}(B)} \leq C||v||_{L^n,\infty(B)}||w||_{L^{n,q,q}(B)}^\frac{n}{n-q}
  = C||v||_{L^n_{\omega}(B)}||w||_{W^{1,q}(B)}.
\]

\[\square\]

In view of Lemma 5, we have

\[
  \int_B |u \otimes v : \nabla \Phi| \, dx \leq C||v||_{L^q(B)}||u||_{W^{2,q'}(B)}||\nabla \Phi||_{L^{q'}(B)}
  \leq C||v||_{L^q(B)}||u||_{L^n_{\omega}(B)}||\Phi||_{W^{2,q}(B)}
\]

whenever

\[
v \in L^q(B), \quad \Phi \in W^{2,q'}(B) \quad \text{and} \quad 1 < q' = \frac{q}{q-1} < n.
\]

Hence if \( \frac{n}{n-1} < q < \infty \), then weak solutions in \( L^q(B) \) to the problem (2) can be defined as follows.

Definition 6 A vector field \( v \in L^q(B) \) with \( \frac{n}{n-1} < q < \infty \) is called a \( q \)-weak solution or simply a weak solution to the problem (2), provided that

\[
  -\int_B \{v \cdot \Delta \Phi + u \otimes v : \nabla \Phi\} \, dx = <f, \Phi> \quad (4)
\]

and

\[
  -\int_B v \cdot \nabla \varphi \, dx = <g, \varphi> \quad (5)
\]

for all \( \Phi \in C^\infty(\overline{B}) \) and \( \varphi \in C^\infty(\overline{B}) \) such that \( \text{div} \Phi = 0 \) in \( B \) and \( \Phi = 0 \) on \( \partial B \). Here \( f \) and \( g \) are sufficiently regular distributions so that the right hand sides of (4) and (5) are well-defined.

The uniqueness of \( q \)-weak solutions to the problem (2) can be proved under the assumption that \( ||u||_{L^n_{\omega}(B)} \) is sufficiently small.
Lemma 7 For each \( q \in (\frac{n}{n-1}, \infty) \), there exists a small positive number \( \varepsilon_1 = \varepsilon_1(n, q) \) such that if \( u \) satisfies

\[
||u||_{L^n(B)} \leq \varepsilon_1,
\]

then \( q \)-weak solutions to the problem (2) are unique.

Proof. We prove the lemma by an elementary duality argument. Let \( v \) be a weak solution to (2) with \( f = 0 \) and \( g = 0 \) so that

\[
\int_B \{v \cdot \Delta \Phi + u \otimes v : \nabla \Phi\} \, dx = 0 \quad \text{and} \quad \int_B v \cdot \nabla \varphi \, dx = 0
\]

for all \( \Phi \in C^\infty(\overline{B}) \) and \( \varphi \in C^\infty(\overline{B}) \) such that \( \text{div} \Phi = 0 \) in \( B \) and \( \Phi = 0 \) on \( \partial B \).

Let \( w \in C^\infty(\overline{B}) \) be fixed. Then in view of a classical theory (see [3] for instance), the Stokes problem

\[-\Delta \Phi + \nabla \varphi = w, \quad \text{div} \Phi = 0 \quad \text{in} \quad B \quad \text{and} \quad \Phi = 0 \quad \text{on} \quad \partial B\]

has a unique solution \((\Phi, \varphi)\) such that

\[\Phi \in C^\infty(\overline{B}), \quad \varphi \in C^\infty(\overline{B}) \quad \text{and} \quad ||\Phi||_{W^{2,q'}(B)} \leq C ||w||_{L^{q'}(B)}.\]

Hence by virtue of (6) and (3), we have

\[
\int_B v \cdot w \, dx = \int_B v \cdot (-\Delta \Phi + \nabla \varphi) \, dx \leq C ||v||_{L^n(B)} ||u||_{L^n(B)} ||\Phi||_{W^{2,q'}(B)} \leq C_1 ||v||_{L^n(B)} ||u||_{L^n(B)} ||w||_{L^{q'}(B)}.
\]

Since \( w \in C^\infty(\overline{B}) \) is arbitrary and \( C^\infty(\overline{B}) \) is dense in \( L^{q'}(B) \), it follows that

\[
||v||_{L^n(B)} \leq C_1 ||u||_{L^n(B)} ||w||_{L^{q'}(B)}.
\]

Therefore, taking \( \varepsilon_1 = 1/2C_1 \), we conclude that if \( ||u||_{L^n(B)} \leq \varepsilon_1 \), then \( ||v||_{L^n(B)} = 0 \). This completes the proof of Lemma 7. \( \square \)

We can also prove the existence of weak solutions in \( W^{1,q}(B) \) and \( W^{2,q}(B) \).

Lemma 8 For each \( q \in (1,n) \), there exists a small positive constant \( \varepsilon_2 = \varepsilon_2(n, q) \) such that if \( u \) satisfies

\[
||u||_{L^n(B)} \leq \varepsilon_2,
\]

then for every

\[f \in W^{-1,q}(B) \quad \text{and} \quad g \in L^q(B) \quad \text{with} \quad \int_B g \, dx = 0,
\]

there exists a unique weak solution \( v \) in \( W^{1,q}_0(B) \) to the problem (2).
Remark 9 This solution $v$ is actually a $nq/(n - q)$-weak solution in the sense of Definition 6 since $W^{1,q}_0(B) \subset L^{nq/(n-q)}(B)$ and $\frac{n}{n-1} < \frac{nq}{n-q} < \infty$.

Proof. By virtue of Lemma 5, we have

$$||u \otimes v||_{L^q(B)} \leq C||u||_{L^q_0(B)}||v||_{W^{1,q}(B)}$$

for all $v \in W^{1,q}_0(B)$.

Hence it follows from the classical theory of the Stokes equations (see [3]) that for each $v \in W^{1,q}_0(B)$, there exists a unique weak solution $\bar{v} = Lv \in W^{1,q}_0(B)$ to the problem

$$\begin{cases}
-\Delta \bar{v} + \nabla \overline{p} = f - \text{div}(u \otimes v) & \text{in } B \\
\text{div} \bar{v} = g & \text{in } B \\
\bar{v} = 0 & \text{on } \partial B
\end{cases}$$

which satisfies the estimate

$$||\bar{v}||_{W^{1,q}(B)} \leq C \left(||f||_{W^{-1,q}(B)} + ||g||_{L^q(B)} + ||u \otimes v||_{L^q(B)} \right).$$

Moreover, the operator $L$ on $W^{1,q}_0(B)$ satisfies

$$||Lv_1 - Lv_2||_{W^{1,q}(B)} \leq C||u \otimes (v_1 - v_2)||_{L^q(B)}$$

$$\leq C_2||u||_{L^q_0(B)}||v_1 - v_2||_{W^{1,q}(B)}$$

for all $v_1, v_2 \in W^{1,q}_0(B)$. Therefore, taking $\varepsilon_2 = 1/(2C_2)$, we conclude that if $||u||_{L^q_0(B)} \leq \varepsilon_2$, then $L$ is a contraction on $W^{1,q}_0(B)$ and so have a unique fixed point. This proves Lemma 8. □

Lemma 10 For each $q \in (1, n)$, there exists a small positive constant $\varepsilon_3 = \varepsilon_3(n, q)$ such that if $u$ satisfies

$$||u||_{L^q_0(B)} \leq \varepsilon_3,$$

then for every $f \in L^q(B)$ and $g \in W^{1,q}(B)$ with $\int_B g \, dx = 0$, there exists a unique weak solution $v$ in $W^{1,q}_0(B) \cap W^{2,q}(B)$ to the problem (2).

Proof. Similar to the proof of Lemma 8. □

Now Theorem 3 can be deduced from the following result by a standard scaling argument and induction on $m$. 
Proposition 11 Assume that $\Omega = B_3$ and $q \in (1, n)$. Then there exists a small positive constant $\varepsilon = \varepsilon(n, q)$ with the following property.

If $u$ satisfies $\|u\|_{L^q_\omega(B_3)} \leq \varepsilon$ and if $(v, p) \in L^n_\omega(B_3) \times L^1(B_3)$ is a distribution solution of

$$
\begin{align*}
&\Delta v + \text{div}(u \otimes v) + \nabla p = f & \text{in } \Omega \\
&\text{div} v = 0 & \text{in } \Omega
\end{align*}
$$

with $f \in L^q(B_3)$, then

$$v \in W^{2,q}(B_1) \quad \text{and} \quad p \in W^{1,q}(B_1).$$

Proof. It is easy to show that $L^n_\omega(B_3) \subset L^{n-\delta}(B_3)$ for any $\delta > 0$. This fact together with Sobolev inequality yields

$$\nabla v - u \otimes v \in \text{div}(\nabla v - u \otimes v) \in W^{-1,n-\delta}(B_3)$$

for any $\delta > 0$ and so $\nabla p = f + \text{div}(\nabla v - u \otimes v) \in W^{-2,q}(B_3)$ because $1 < q < n$.

Hence it follows that $p \in W^{-1,q}(B_3)$.

Let us choose a cut-off function $\varphi \in C_c^\infty(B_3)$ such that $\varphi = 1$ in $B_2$ and $\varphi = 0$ in $B_3 \setminus B_{5/2}$. Then it is easy to show that $\overline{v} = \varphi v \in L^2(B_3) \cap L^q(B_3)$ is a 2-weak solution (in the sense of Definition 6) to the following problem

$$
\begin{align*}
&-\Delta \overline{v} + \text{div}(u \otimes \overline{v}) + \nabla \overline{p} = \overline{f} & \text{in } B_3 \\
&\text{div} \overline{v} = g & \text{in } B_3 \\
&\overline{v} = 0 & \text{on } \partial B_3,
\end{align*}
$$

where

$$\overline{p} = \varphi p \in W^{-1,q}(B_3), \quad g = \nabla \varphi \cdot v \in L^q(B_3)$$

and

$$\overline{f} = \varphi f + \nabla \varphi \cdot (u \otimes v - 2\nabla v + pI) - (\Delta \varphi)v \in W^{-1,q}(B_3).$$

We now assume that $u$ satisfies

$$\|u\|_{L^q_\omega(B_3)} \leq \varepsilon_2(n, q). \quad (9)$$

Then by virtue of Lemma 8, there exists a unique solution $w \in W^{1,q}_0(B_3)$ to the problem (8). Note that

$$w \in L^{\frac{nq}{n-q}}(B_3) \quad \text{and} \quad \frac{n}{n-1} < \frac{nq}{n-q} < \infty.$$ 

Hence by virtue of Lemma 7, we deduce that

$$\overline{v} = w \in W^{1,q}(B_3) \quad \text{and so} \quad v \in W^{1,q}(B_2),$$

and so
provided that
\[ ||u||_{L^q(B_3)} \leq \epsilon_1(n, q_1), \quad \text{where} \quad q_1 = \min \left( 2, \frac{mq}{n - q} \right). \quad (10) \]

Moreover, it follows from Lemma 5 that
\[
\begin{align*}
\nabla p & = f + \text{div}(\nabla v - u \otimes v) \in W^{-1,q}(B_2), \\
p & \in L^q(B_2), \quad \overline{f} \in L^q(B_2) \quad \text{and} \quad g \in W^{1,q}(B_2).
\end{align*}
\]

On the other hand, we observe that if we choose \( \varphi \in C^\infty_c(B_3) \) so that \( \varphi = 1 \) in \( B_1 \) and \( \varphi = 0 \) in \( B_3 \setminus B_{3/2} \), then \( \overline{v} = \varphi v \in W^{1,q}(B_2) \) is a \( q_1 \)-weak solution to the problem (8) with \( B_3 \) replaced by \( B_2 \).

Therefore, assuming in addition to (9) and (10) that
\[ ||u||_{L^q(B_3)} \leq \epsilon_3(n, q). \]

we conclude from Lemma 10 and Lemma 7 that
\[ \overline{v} \in W^{2,q}(B_2) \quad \text{and so} \quad v \in W^{2,q}(B_1), \]

which implies then that \( p \in W^{1,q}(B_1) \). This completes the proof of Proposition 11. \( \square \)

References


