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An explicit Shimura canonical model for the quaternion algebra of discriminant 6

By

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Abstract

According to K. Takeuchi ([Tku1], [Tku2]), all the arithmetic triangle groups are listed up (1977). They are classified in 19 commensurable classes (Table 2.1 below). It corresponds a quaternion algebra for each class. (a) The first class is of non-compact type, and it induces the usual elliptic modular function. (b) Among 18 remained classes, in 16 cases we have triangle unit groups. (c) For the rest 2 cases (the class II and XII) it appears quadrangle unit groups.

Already we reported the result about the case (b) in the RIMS workshop of the previous year. There, we showed how to determine the Shimura canonical model modular function for them. As an application we exposed several defining equations of the Hilbert class fields of CM fields of higher degree.

In this survey article we explain how to obtain the exact Shimura canonical model for the class II quadrangle case (that is for the quaternion algebra $\left(\frac{-3,2}{\boldsymbol{Q}}\right)$) using the modular function coming from the Appell's hypergeometric differential equation. We state only the framework of the argument together with several back grounds. Detailed proofs will be published elesewhere.

This Shimura curve is studied by many mathematicians. See [Elk] for it.

§ 1. Recall the Picard modular function 1883 – 1988

In 1883 E. Picard ([Pcd]) made a study about a modular function for a special family of curves of genus 3 which induces an analogue of the elliptic λ-function in two variables. Later in 1988, the author ([Shg]) obtained an exact theta constants representation for it. We use it for our present study.

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Period map. We set the complex two dimensional hyper ball

$$\mathscr{D} = \{[\eta] = [\eta_0, \eta_1, \eta_2] \in \boldsymbol{P}^2 : {}^t\eta H \overline{\eta} < 0\},$$

here we put $H = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,1 \end{pmatrix}$. We set the Picard modular group

$$\Gamma = \{g \in \mathrm{GL}_3(\boldsymbol{Z}[\omega]) : {}^t\overline{g} H g = H\} \,\mathrm{with}\, \omega = e^{2\pi i/3}.$$

It acts on $\mathscr{D}$. For our study of the Shimura curve, we use the hyperball $\mathscr{D}$ together with the group Γ or its congruence subgroup. Our domain $\mathscr{D}$ is the period domain of the family of Picard curves:

$$\begin{aligned} C(\lambda) = C(\xi) : w^3 &= z(z-1)(z-\lambda_1)(z-\lambda_2) \\ \cong w^3 &= z(z-\xi_0)(z-\xi_1)(z-\xi_2). \end{aligned} \tag{1.1}$$

Set

$$\begin{aligned} \Lambda &= \{\lambda = (\lambda_1, \lambda_2) \in \boldsymbol{C}^2 : \lambda_1\lambda_2(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2) \neq 0\} \\ &= \{[\xi_0, \xi_1, \xi_2] \in \boldsymbol{P}^2(\boldsymbol{C}) : \xi_0\xi_1\xi_2(\xi_1 - \xi_0)(\xi_2 - \xi_0)(\xi_1 - \xi_2) \neq 0\} \end{aligned}$$

with $\lambda_1 = \dfrac{\xi_1}{\xi_0}, \lambda_2 = \dfrac{\xi_2}{\xi_0}$. For $\lambda \in \Lambda$, $C(\lambda)$ is a curve of genus three and is a three sheeted branched covering over the complex z plane. The Jacobian variety $Jac(C(\lambda))$ of $C(\lambda)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2,1)$. In fact, we have a basis system of holomorphic differentials given by

$$\varphi = \varphi_1 = \frac{dz}{w}, \quad \varphi_2 = \frac{dz}{w^2}, \quad \varphi_3 = \frac{zdz}{w^2}.$$

For the moment, we assume $0 < \lambda_1 < \lambda_2 < 1$. Under this condition we choose the symplectic basis $\{A_1, \ldots, B_3\}$ of $H_1(C(\lambda), \mathbb{Z})$ described in Figure 1.1, that is already used in [Shg]. Here we put cut lines starting from branch points in the lower half z-plane to get simply connected sheets. The real line (resp. dotted line, chained line) indicates an arc on the first sheet (resp. second sheet, third sheet).

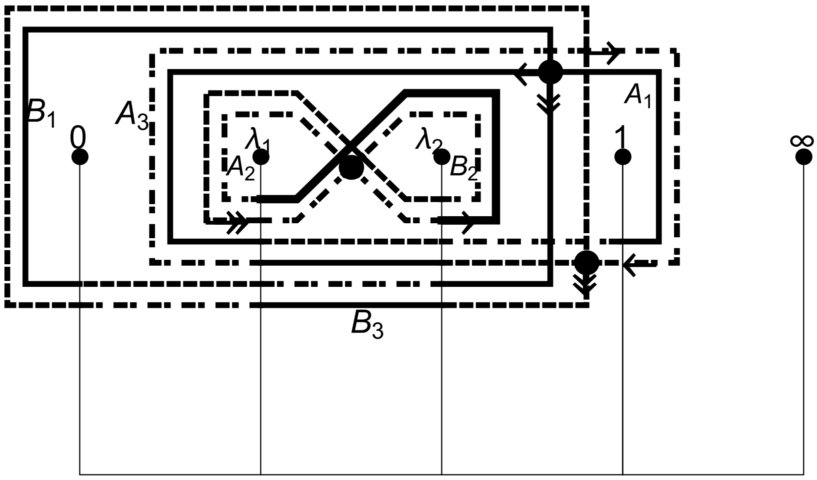

Figure 1.1. homology basis

Setting $\rho(z,w)=(z,\omega w)$, it holds

$$B_3=\rho(B_1),\quad A_3=-\rho^2(A_1),\quad B_2=-\rho^2(A_2).$$

We have the intersections $A_i\cdot B_j=\delta_{ij}$ $(i,j\in\{1,2,3\})$. Put

$$\eta_0=\int_{A_1}\varphi,\quad \eta_1=-\int_{B_3}\varphi,\quad \eta_2=\int_{A_2}\varphi. \tag{1.2}$$

By the analytic continuation, they are multivalued analytic functions on the domain Λ. It holds

$$\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\int_{A_1}\varphi_1\\ -\omega^2\int_{B_1}\varphi_1\\ \int_{A_2}\varphi_1\end{pmatrix},\quad \begin{pmatrix}\int_{A_1}\varphi_i\\ -\omega\int_{B_1}\varphi_i\\ \int_{A_2}\varphi_i\end{pmatrix}=\begin{pmatrix}-\omega\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega\int_{B_2}\varphi_i\end{pmatrix}\ \ (i=2,3). \tag{1.3}$$

Set

$$\Omega_1=\Big(\int_{A_j}\varphi_i\Big),\quad \Omega_2=\Big(\int_{B_j}\varphi_i\Big),\quad (1\le i,j\le 3).$$

The normalized period matrix of $C(\xi)$ is given by $\Omega=\Omega_1^{-1}\Omega_2$. By the relations of periods (1.3) together with the symmetricity ${}^t\Omega=\Omega$, we can rewrite

$$\Omega=\Omega_1^{-1}\Omega_2=\begin{pmatrix}\frac{u^2+2\omega^2v}{1-\omega} & \omega^2u & \frac{\omega u^2-\omega^2v}{1-\omega}\\ \omega^2u & -\omega^2 & u\\ \frac{\omega u^2-\omega^2v}{1-\omega} & u & \frac{\omega^2u^2+2\omega^2v}{1-\omega}\end{pmatrix}, \tag{1.4}$$

here we put $u=\frac{\eta_2}{\eta_0}$, $v=\frac{\eta_1}{\eta_0}$. So we set $\Omega=\Omega(u,v)$. The Riemann period relation $\Im(\Omega)>0$ induces the defining inequality $2\Re(v)+|u|^2<0$ of $\mathscr{D}$. So, we define our period map $\Phi:\Lambda\to\mathscr{D}$ by

$$\Phi(\lambda_1,\lambda_2)=[\eta_0,\eta_1,\eta_2].$$

Setting the Picard modular group $\Gamma = \{g \in \mathrm{GL}_3(\boldsymbol{Z}[\omega]) : {}^t g H \overline{g} = H\}$, the element $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in \Gamma$ acts on $\mathscr{D}$ by

$$g(u,v) = \left(\frac{p_3 + q_3 v + r_3 u}{p_1 + q_1 v + r_1 u}, \frac{p_2 + q_2 v + r_2 u}{p_1 + q_1 v + r_1 u} \right). \tag{1.5}$$

Let us denote the congruence subgroup $\{g \in \Gamma : g \equiv I_3 \mod \sqrt{-3}\}$ by $\Gamma(\sqrt{-3})$. Set $\overline{\Gamma} = \Gamma/\langle -\omega^2 \rangle$ and set $\overline{\Gamma(\sqrt{-3})} = \Gamma(\sqrt{-3})/\langle \omega \rangle$. We have $\overline{\Gamma}/\overline{\Gamma(\sqrt{-3})} \cong S_4$, the symmetric group of degree 4.

Theta representation of the modular function. Using the Riemann theta constant on $\mathfrak{S}_3$ with a characteristic $(a,b) \in (\boldsymbol{Q}^3)^2$:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{n \in \boldsymbol{Z}^3} \exp[\pi i \, {}^t(n+a)\Omega(n+a) + 2\pi i \, {}^t(n+a)b],$$

we set

$$\vartheta_k(u,v) = \vartheta \begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix} (0, \Omega(u,v)). \tag{1.6}$$

Apparently it holds $\vartheta_k(u,v) = \vartheta_{k+3}(u,v)$, so k runs over $\{0,1,2\} = \boldsymbol{Z}/3\boldsymbol{Z}$.

The following properties are established.

Proposition 1.1.

(i) ([Shg] p.349) The period map Φ induces a biholomorphic isomorphism from ξ-space $\mathbb{P}^2(\mathbb{C})$ to the Satake compactification $\overline{\mathscr{D}/\Gamma(\sqrt{-3})}$ of $\mathscr{D}/\Gamma(\sqrt{-3})$. This compactification is obtained by atatching 4 boundary points corresponding to 4 points $[\xi_0, \xi_1, \xi_2] = [0,0,1], [0,1,0], [1,0,0], [1,1,1]$.

(ii) ([Shg] p.327) The map $\Theta : \mathscr{D} \longrightarrow \boldsymbol{P}^2$ defined by

$$\Theta([\eta_0, \eta_1, \eta_2]) = [\vartheta_0(u,v)^3, \vartheta_1(u,v)^3, \vartheta_2(u,v)^3] \tag{1.7}$$

gives the inverse of the period map Φ.

(iii) ([Shg] p.329) The group $\overline{\Gamma(\sqrt{-3})}$ is the projective monodromy group of the multivalued map $\Phi : \Lambda \to \mathscr{D}$. It has a system of explicit generator system $\{g_1, g_2, g_3, g_4, g_5\}$.

(iv) ([Shg] p.346) We have the automorphic property:

$$\vartheta_k(g(u,v))^3 = (p_1 + q_1 v + r_1 u)^3 \, \vartheta_k(u,v)^3 \tag{1.8}$$

for $g = \begin{pmatrix} p_1\, q_1\, r_1 \\ p_2\, q_2\, r_2 \\ p_3\, q_3\, r_3 \end{pmatrix} \in \Gamma(\sqrt{-3})$. The sytem $\{\vartheta_k(g(u,v))^3\}_{k=0,1,2}$ is a basis of the vector space of automorphic forms with the property (1.8).

(v) The system of periods $\{\eta_0, \eta_1, \eta_2\}$ is a basis of the space of solutions for the Appel hypergeometric differential equation $E_1(a,b,b',c)$ with $(a,b,b',c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$:

$$E_1(a,b,b',c) : \begin{cases} r\ (1-x)\ x + p\ (c - (1+a+b)\ x) - b\, q\, y + s\ (1-x)\ y - a\, b\, z = 0 \\ \\ -(b'\, p\, x) + s\, x\ (1-y) + t\ (1-y)\ y + q\ (c - (1+a+b')\ y) - a\, b'\, z = 0, \end{cases}$$

with $r = z_{xx}, s = z_{xy}, t = z_{yy}, p = z_x, q = z_y$. It has singularities along $\boldsymbol{P}^2 - \Lambda$. $\overline{\Gamma(\sqrt{-3})}$ is the projective monodromy group of $E_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$ also. Here we used (x,y) in stead of (λ_1, λ_2).

§ 2. Shimura's Complex Multiplication Theorem

Let F be a totally real number field. Set a quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right)$ over F. Let $\mathcal{O} = \mathcal{O}_{\boldsymbol{B}}$ be a fixed maximal order. We assume

$$(\mathrm{Cd}) : \begin{cases} a, b \in F : a < 0, b > 0 \\ \\ \text{all their conjugates other than } a, b \text{ are negative.} \end{cases}$$

By them we set a quaternion algebra $\boldsymbol{B} = \left(\frac{a,b}{F}\right) := F + F\alpha + F\beta + F\alpha\beta\ , \beta\alpha = -\alpha\beta, \alpha^2 = a, \beta^2 = b$. With

$$M_1 = \begin{pmatrix} 1\, 0 \\ 0\, 1 \end{pmatrix}, M_x = \begin{pmatrix} 0\, a \\ 1\, 0 \end{pmatrix}, M_y = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}, M_z = M_x M_y,$$

we have

$$\boldsymbol{B} \cong F\ M_1 + F\ M_x + F\ M_y + F\ M_z \subset M_2(\boldsymbol{R}).$$

Definition 2.1. The unit group of $\boldsymbol{B}$ is defined by

$$\Gamma^+(\boldsymbol{B}, \mathcal{O}) \quad (\textit{Takeuchi Notation}) = \Gamma(\mathcal{O}, 1) \quad (\textit{Shimura Notation})$$
$$:= \{\gamma \in \mathcal{O} : \det(\gamma) \in E_0\}.$$

Where E_0 is the group of totally positive units in F.

Note that, by (Cd), $\Gamma(\mathcal{O}, 1)$ is a discrete group acting on $\boldsymbol{H}$, and $\boldsymbol{H}/\Gamma(\mathcal{O}, 1)$ is a compact curve.

Shimura's Main theorem of CM(Existence of the canonical model according to [Smr]). Take above mentioned F and $\boldsymbol{B}$. Suppose a CM field M over F. Assume an embedding $f : M \hookrightarrow \boldsymbol{B}$ satisfying $f(\mathcal{O}_M) \subset \mathcal{O}_{\boldsymbol{B}}$, where $\mathcal{O}_M$ stands for the ring of integers of M. Then there is a pair (ψ, V) of a complex algebraic curve V and a modular map $\psi(z)$ on $\boldsymbol{H}$ with respect to $\Gamma(\mathcal{O}, 1)$ satisfying the following condition:

(1) $\psi(z)$ induces a biholomorphic correspondence $\boldsymbol{H}/\Gamma(\mathcal{O}, 1) \cong V$,

(2) V is defined over $C(F)$ (the Hilbert class field of F),

(3) for a regular fixed point (that is explained below) $z_0 \in \boldsymbol{H}$ of M, it holds $M(\psi(z_0)) \cdot C(F) = C(M)$, where $C(M)$ stands for the Hilbert class field of M.

$$
\begin{array}{rll}
\text{Hilbert Class F.:} & C(M) = M(\psi(z_0)) \cdot C(F) & \\
 & \Big| & \\
\text{CM field/F:} & M = F(\alpha) & (\alpha \in \mathcal{O}_M),\ f : M \hookrightarrow \boldsymbol{B} \text{ with (Cd)} \\
 & \Big| & \\
\text{Tot. real F.:} & F & \\
 & \Big| & \\
 & \boldsymbol{Q} & (\psi, V)\text{: } \exists! \text{ the Shimura canonical model} \\
 & & V\text{: Shimura curve, } \psi \text{ :modular function for } \Gamma(\mathcal{O}, 1)
\end{array}
$$

Diagram 2.1: Scheme of the Shimura complex multiplication theorem

[Regular fixed point $z_0 \in \boldsymbol{H}$ of M]. Recall the embedding $f : M \hookrightarrow \boldsymbol{B}$. We can put $M = F(\alpha), \alpha \in \mathcal{O}_M$. The linear transformation $g = f(\alpha)$ has unique fixed point in $\boldsymbol{H}$ (note that $(\boldsymbol{tr}(\alpha))^2 - 4\boldsymbol{det}(\alpha) < 0$). That gives our regular fixed point.

Definition 2.2. The above pair (ψ, V) is called **a canonical model** for $\boldsymbol{H}/\Gamma(\mathcal{O}, 1)$.

Remark. The canonical model is unique up to $\mathrm{Aut}_{C(F)}(V)$. (see [Smr] Theorem 3.3).

Class	F	Disc.	$\Gamma^+(\boldsymbol{B},\mathcal{O})$	(a,b)
I	$\boldsymbol{Q}$	(1)	$(2,3,\infty)$	$(-3,4)$
II	$\boldsymbol{Q}$	$(2)(3)$	$(0;2,2,3,3)$	$(-3,2)$
III	$\boldsymbol{Q}(\sqrt{2})$	$\mathfrak{p}_2$	$(3,3,4)$	$(-3,4\cos^2(\frac{\pi}{8})(-3+4\cos^2(\frac{\pi}{8})))$
IV	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_2$	$(2,3,12)$	$(-3,1+\sqrt{3})$
V	$\boldsymbol{Q}(\sqrt{3})$	$\mathfrak{p}_3$	$(2,4,12)$	$(-4,6+4\sqrt{3})$
VI	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_2$	$(2,5,5)$	$(-4,1+\sqrt{5})$
VII	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_3$	$(3,5,5)$	$(-\frac{1}{2}(5+\sqrt{5}),3(2+\sqrt{5}))$
VIII	$\boldsymbol{Q}(\sqrt{5})$	$\mathfrak{p}_5$	$(3,3,5)$	$(-3,\sqrt{5})$
IX	$\boldsymbol{Q}(\sqrt{6})$	$\mathfrak{p}_2$	$(3,4,6)$	$(-4,6(2+\sqrt{6}))$
X	$\boldsymbol{Q}(\cos(\frac{\pi}{7}))$	(1)	$(2,3,7)$	$(-3,4\cos^2(\frac{\pi}{7})(-3+4\cos^2(\frac{\pi}{7})))$
XI	$\boldsymbol{Q}(\cos(\frac{\pi}{9}))$	(1)	$(2,3,9)$	$(-3,4\cos^2(\frac{\pi}{9})(-3+4\cos^2(\frac{\pi}{9})))$
XII	$\boldsymbol{Q}(\cos(\frac{\pi}{9}))$	$\mathfrak{p}_2\mathfrak{p}_3$	$(0;2,2,9,9)$	$(-4,8\cos^2(\frac{\pi}{18})(-2+4\cos^2(\frac{\pi}{18})))$
XIII	$\boldsymbol{Q}(\cos(\frac{\pi}{8}))$	$\mathfrak{p}_2$	$(3,3,8)$	$(-3,4\cos^2(\frac{\pi}{16})(-3+4\cos^2(\frac{\pi}{16})))$
XIV	$\boldsymbol{Q}(\cos(\frac{\pi}{10}))$	$\mathfrak{p}_2$	$(2,5,20)$	$\begin{pmatrix} -\frac{1}{2}(5+\sqrt{5}), \\ (3+\sqrt{5})\cos^2(\frac{\pi}{20})(-5+\sqrt{5}+8\cos^2(\frac{\pi}{20})) \end{pmatrix}$
XV	$\boldsymbol{Q}(\cos(\frac{\pi}{12}))$	$\mathfrak{p}_2$	$(2,3,24)$	$\begin{pmatrix} -3, \\ 4\cos^2(\frac{\pi}{24})(-3+4\cos^2(\frac{\pi}{24})) \end{pmatrix}$
XVI	$\boldsymbol{Q}(\cos(\frac{\pi}{15}))$	$\mathfrak{p}_3$	$(2,5,30)$	$\begin{pmatrix} -\frac{1}{2}(5+\sqrt{5}), \\ (3+\sqrt{5})\cos^2(\frac{\pi}{30})(-5+\sqrt{5}+8\cos^2(\frac{\pi}{30})) \end{pmatrix}$
XVII	$\boldsymbol{Q}(\cos(\frac{\pi}{15}))$	$\mathfrak{p}_5$	$(2,3,30)$	$(-3,4\cos^2(\frac{\pi}{30})(-3+4\cos^2(\frac{\pi}{30})))$
XVIII	$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$\mathfrak{p}_2$	$(4,5,5)$	$\begin{pmatrix} -\frac{1}{2}(5+\sqrt{5}), \\ \frac{1}{8}(3+\sqrt{5})\cos^2(\frac{\pi}{8})(-5+\sqrt{5}+\cos^2(\frac{\pi}{8})) \end{pmatrix}$
XIX	$\boldsymbol{Q}(\cos(\frac{\pi}{11}))$	(1)	$(2,3,11)$	$(-3,4\cos^2(\frac{\pi}{11})(-3+4\cos^2(\frac{\pi}{11})))$

Table 2.1: Table of 19 commensurable classes according to K. Takeuchi

Note that for the quaternion algebra $\boldsymbol{B}=\left(\frac{-3,2}{\boldsymbol{Q}}\right)$, its unit group is given by the quadrangle group $\square(0;2,2,3,3)$. Moreover, according to A. Kurihara [Krh] the corresponding Shimura curve is given by the conic $KC: X^2+Y^2+3Z^2=0$ defined over $\boldsymbol{Q}$.

§ 3. The result of Petkova-Shiga and shifting to the canonical model

Here we summarize the results in [P-S].

§ 3.1. Another Shimura curve

Set $D_{\boldsymbol{c}}=\{(u,v)\in\mathscr{D}:v=-1\}$, $\Gamma_c=\{g\in\Gamma:g(D_c)=D_c\}/\{g\in\Gamma:g_{|D_c}=\mathrm{id}_{D_c}\}$. According to Petkova and Holzapfel ([Pet], [Hol]) we know that $D_{\boldsymbol{c}}/\Gamma_c$ is the Shimura curve for

$$\boldsymbol{B}=\left(\frac{-3,2}{\boldsymbol{Q}}\right)$$

with $\mathrm{Disc}(\boldsymbol{B})=6$ in the following sense. (We have $\mathrm{Disc}(\boldsymbol{B})=\prod_{(-3,2)_p=-1}p=2\cdot3=6$.)

Theorem 3.1. *Set a complex line* $L_{\boldsymbol{c}} = \{\lambda_1 + \lambda_2 = 1\}$ *in* (λ_1, λ_2) *space* $\boldsymbol{P}^2$. *Then we have*

$$\Theta(D_{\boldsymbol{c}}) = L_{\boldsymbol{c}}.$$

Hence, the Shimura curve is realized as a hyperplane section in the projective plane of the Picard modular forms generated by the homogeneous coordinates $\langle \vartheta_0^3, \vartheta_1^3, \vartheta_2^3 \rangle$.

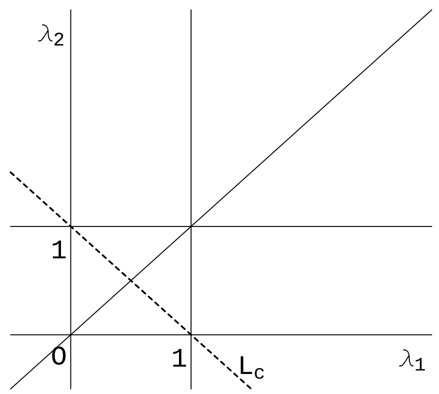

Figure 3.1: Figure of the Shimura curve $L_{\boldsymbol{c}}$

Theorem 3.2. *For a Picard curve* $C(\lambda) : w^3 = z(z-1)(z-\lambda_1)(z-\lambda_2)$ *with* $\lambda_1 + \lambda_2 = 1$, *we have a decomposition*

$$Jac(C(\lambda)) \cong \mathscr{E}_0 \times A'(\lambda) \quad (\textit{up to isogeny }),$$

where $\mathscr{E}_0 = \boldsymbol{C}/(\omega \boldsymbol{Z} + \boldsymbol{Z})$ *and* $A'(\lambda)$ *is a 2-dimentional abelian variety. And generically we have*

$$\mathrm{End}_0(A'(\lambda)) = \boldsymbol{B}.$$

Let

$$F_1(\lambda_1, \lambda_2) = F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; \lambda_1, \lambda_2) = 1 + \sum_{m+n>0} \frac{(\frac{1}{3}, m+n)(\frac{1}{3}, m)(\frac{1}{3}, n)}{(1, m+n)m!n!} \lambda_1^m \lambda_2^n$$

be the Appell hypergeometric series that is a solution of $E_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$.

Theorem 3.3. *Let* $f(s) = F_1(\frac{1}{2}(1+s), \frac{1}{2}(1-s))$ *be the restriction of* F_1 *on* $L_{\boldsymbol{c}}$. *Then* $f(s)$ *is an even function of* s. *So we put* $f(s) = g(t)$ *with* $t = s^2$. *In this situation* $g(t)$ *satisfies the Gauss hypergeometric differential equation:*

$$g''(t) + \frac{9t-5}{6t(t-1)} g'(t) + \frac{1}{18t(t-1)} g(t) = 0. \tag{3.1}$$

Remark. We can show (3.1) is the Gauss hypergeometric differential equation $E(\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$, and its monodromy group is the triangle group $\Delta(3,6,6)$ which is commensurable with $\square(0; 2, 2, 3, 3)$.

§ 3.2. Period map and the monodromy matrices.

By restricting Φ on this hyperplane L_c, we have a Schwarz map $\mathcal{F} : L_c \to \mathcal{D}$ with

$$\mathcal{D} = \{|u| < \sqrt{2}\} \subset \boldsymbol{C}.$$

Remark. Because L_c is parametrized by the variable $s = 2\lambda_1 - 1$, the inverse of $\mathcal{F}$ gives a single valued function $s(u)$ on $\mathcal{D}$. By putting $t(u) = s(u)^2$, we obtain the inverse of the Schwarz map of the Gauss hypergeometric differential equation $E(\frac{1}{6}, \frac{1}{3}, \frac{5}{6})$.

Set $\tilde{\mathcal{F}}(s) = \eta_2/\eta_0(\frac{1+s}{2}, \frac{1-s}{2})$, and set $\mathcal{G}(t) = \tilde{\mathcal{F}}(s)$.

We can define the branch of $\mathcal{G}(t)$ on the real interval $[0, 1]$ to stay on the real interval $[0, 1]$. Then, we make an analytic continuation of $\mathcal{G}(t)$ on the lower half plane $\boldsymbol{H}_-$. By observing the Riemann scheme, we know that $\mathcal{G}(\boldsymbol{H}_-)$ becomes to be a hyperbolic triangle with angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{6}$ corresponding to the singularities $t = 0, 1, \infty$ in this order. So, we denote the triangle $\mathcal{G}(\boldsymbol{H}_-)$ by $\nabla(3, 6, 6)$. It is described by the triangle $\nabla(P_0, P_1', P_m)$ in Fig. 3.2. We denote the Schwarz reflection of $\nabla(3, 6, 6)$ with respect to the edge P_0P_m by $\nabla'(3, 6, 6)$.

Fundamental regions. Let $\Delta(3, 6, 6)$ be the triangle group generated by even times of the Schwarz reflection procedures of $\nabla(3, 6, 6)$. Let us denote $\diamond_0 = \nabla(3, 6, 6) \cup \nabla'(3, 6, 6)$. It becomes to be a fundamental region of $\Delta(3, 6, 6)$. The function $t(u)$ is given as the inverse of $\mathcal{G}(t)$. It is a modular function with respect to $\Delta(3, 6, 6)$.

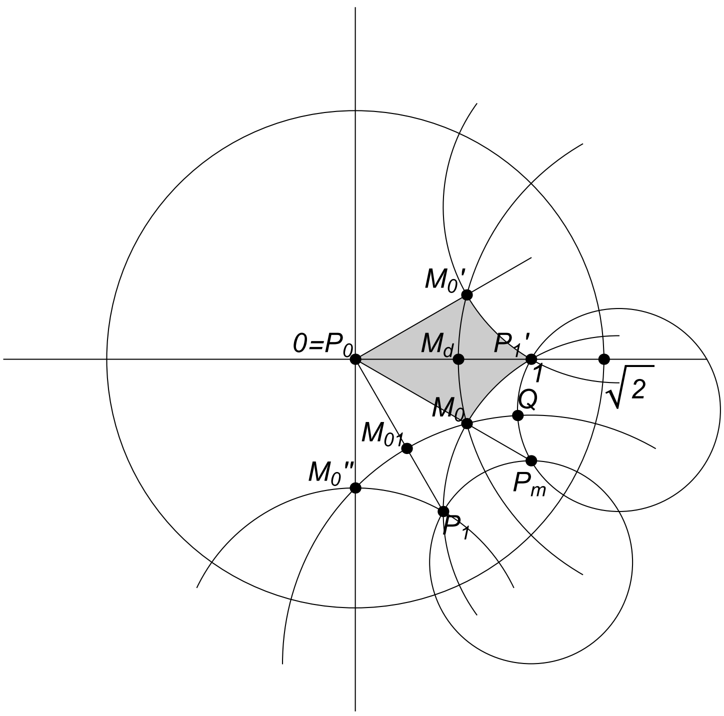

Fig. 3.2. Fundamental region $\nabla(P_0, P_1', P_m)$ of the Schwarz map of $E(\frac{1}{6}, \frac{1}{3}, \frac{5}{6})$ on $\mathcal{D}$

Remark. According to Fact 2.1, it holds

$$\lambda_1(u,v) = \left(\frac{\vartheta_1(u,v)}{\vartheta_0(u,v)}\right)^3, \quad \lambda_2(u,v) = \left(\frac{\vartheta_2(u,v)}{\vartheta_0(u,v)}\right)^3. \tag{3.2}$$

By its restriction on the disc $\mathcal{D}$, observing Remark 3.2 we obtain an explicit description :

$$t(u) = (2\lambda(u)-1)^2, \quad \text{with } \lambda(u) = \lambda_1(u,-1). \tag{3.3}$$

Commensurable arithmetic groups. Besides $\nabla(3,6,6)$, we set a triangle $\nabla(2,6,6) = \nabla(P_0, P_1', M_0)$ and a quadrangle

$$\diamond(0;2,2,3,3) = \diamond(P_0, M_0', P_1', M_0). \tag{3.4}$$

They generate corresponding triangle group $\Delta(2,6,6)$ and quadrangle group $\square(0;2,2,3,3)$, respectively. By observing Fig.3.2 we know the relation of these commensurable groups:

$$\square(0;2,2,3,3) \subset \Delta(2,6,6) \supset \Delta(3,6,6)$$

with

$$[\Delta(2,6,6):\Delta(3,6,6)] = [\Delta(2,6,6):\square(0;2,2,3,3)] = 2.$$

Set

$$\tilde{s}(u) = \left(\frac{t(u)-1}{t(u)+1}\right)^2. \tag{3.5}$$

It gives the double cover $\mathcal{D}/\Delta(3,6,6) \to \mathcal{D}/\Delta(2,6,6)$ ramifying at P_1' and M_0, namely $t^{-1}(1)$ and $t^{-1}(-1)$. Then it becomes a modular function for $\nabla(2,6,6)$, because of the fact $\nabla(P_1', P_0, P_m) = \nabla(P_1', P_0, M_0) \cup \nabla(P_1', M_0, P_m)$. Note that we have $t(M_d) = 3-2\sqrt{2}, t(Q) = 3+2\sqrt{2}, \tilde{s}(M_d) = \tilde{s}(Q) = \frac{1}{2}$.

Next, we consider a double cover $\mathcal{D}/\square(2,2,3,3) \to \mathcal{D}/\Delta(2,6,6)$ caused by the folding of $\nabla(P_0, P_1', M_0)$ and $\nabla(P_0, P_1', M_0')$ along the side P_0P_1' , it is the real segment $[0,1]$. It is realized by the map

$$\tilde{s} = -\frac{(\tau-1)^2}{4\tau}. \tag{3.6}$$

We have the following table of the values at possible angular points of the modular functions, the angular points are designated by the brackets indexed with corresponding angular parameters. In the table it appears several equality symbols. They mean the coincidences of the images of the corresponding angular points by the indicated functions:

function	P_1'	P_0	P_m	M_0	M_0'	M_d	Q
$t(u)$	$\langle 1\rangle_3$	$\langle 0\rangle_6$	$\langle\infty\rangle_6$	-1	$=-1$	$3-2\sqrt{2}$	$3+2\sqrt{2}$
$\tilde{s}(u)$	$\langle 0\rangle_6$	$\langle 1\rangle_6$	$=\langle 1\rangle_6$	$\langle\infty\rangle_2$	$=\langle\infty\rangle_2$	$\frac{1}{2}$	$=\frac{1}{2}$
$\tau(u)$	$\langle 1\rangle_3$	$\langle -1\rangle_3$	$=\langle -1\rangle_3$	$\langle\infty\rangle_2$	$\langle 0\rangle_2$	$\sqrt{-1}$	$-\sqrt{-1}$

.

Table 3.1: Angular points and the values of modular functions

By considering the order of verteces on the quadrangle $\diamond(P_1', Q, M_0, M_d)$, we may assume $\tau(M_d) = \sqrt{-1}, \tau(Q) = -\sqrt{-1}$.

Remark. By construction, $\tau(u)$ is a modular function for the quadrangle group $\square(2,2,3,3)$. The quadrangle $\diamond(P_0, M_0', P_1', M_0)$ is mapped to the lower half complex plane by $\tau(u)$ (see Fig. 3.2). So, a fundamental region of $\tau(u)$ is given by the pentagon $\mathrm{Pent}(P_0, M_0', P_1', P_1, M_0'')$ in Fig.3.2.

Proposition 3.4. *The inverse of the Schwarz map of the following differential equation gives* $\tau(u)$ *:*

$$144(1-\tau^2)\tau^2 f_{\tau\tau} - 48\tau(5\tau^2-1)f_\tau - 7(\tau^2-1)f = 0.$$

We define the cross ratio of 4 points on $\boldsymbol{P}_1(\boldsymbol{C})$ by $(z_1, z_2, z_3, z_4) = (z_1 - z_3)(z_2 - z_4)/((z_1 - z_4)(z_2 - z_3))$. From Table 4.1 we have:

Proposition 3.5.

$$(3.7)\qquad \begin{cases} (\tau(P_0), \tau(P_1), \tau(M_0'), \tau(M_0)) = -1, \\ (\tau(P_0), \tau(P_1), \tau(M_d), \tau(M_{01})) = -1, \\ (\tau(M_0'), \tau(M_0), \tau(M_d), \tau(M_{01})) = -1. \end{cases}$$

§ 4. The Shimura canonical model for $\boldsymbol{B} = \left(\frac{-3,2}{\boldsymbol{Q}}\right)$

§ 4.1. General tactics.

According to Shimura's complex multiplication theorem, there exists a pair (ψ, V) that is the canonical model for $\boldsymbol{H}/\Gamma(\mathcal{O}, 1)$, where $\Gamma(\mathcal{O}, 1)$ is the unit group $\square(0; 2,2,3,3)$ for $\boldsymbol{B} = \left(\frac{-3,2}{\boldsymbol{Q}}\right)$. According to Kurihara, V is given by the affine equation KC : $x^2 + y^2 + 3 = 0$ as an algebraic variety over $\boldsymbol{Q}$. We are looking for an exact expression of ψ.

Already, we have the modular function $\tau(u)$ defined on $\mathcal{D}$. That is also a modular function for $\square(0; 2,2,3,3)$, where $\square(0; 2,2,3,3)$ is the quadrangle group on $\mathcal{D}$. We are requested to get the relation between these two modular functions.

We have the following Diagram 4.1.

We can obtain the exact modular map ψ by knowing the isomorphism q.

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\tau"] \arrow[dr] & & S = \boldsymbol{P}^1 \arrow[dd, "q"] \\
& \mathcal{D}/\square \arrow[ur, "\tilde{\tau}"] & \\
\boldsymbol{H} \arrow[uu, Rightarrow, "h"] \arrow[rr, "\psi"] \arrow[dr] & & KC \\
& \boldsymbol{H}/\square \arrow[uu, Rightarrow, "\tilde{h}"] \arrow[ur, "\tilde{\psi}"] &
\end{tikzcd}$$

Diagram 4.1FThe canonical model modular function ψ

§ 4.2. Conclusion.

Let $KC : X^2 + Y^2 + 3Z^2 = 0$ be the Kurihara conic defined over $\boldsymbol{Q}$ described in the homogeneous coordinate $[X, Y, Z]$. Let $\square[2, 2, 3, 3]$ be the quadrangle group acting on the disc $\mathcal{D}$ which is generated by the quadrangle $\diamond(P_0, M_d', P_1', M_d)$. The modular function $\tau(u)$ maps $\diamond(P_0, M_d', P_1', M_d)$ to the lower half complex plane. We have a theta representation of $\tau(u)$ via (11) and (12), (14), (15).

Theorem 4.1. *Looking at Fig. 3.2 and the definition of elliptic points in section 3.4, It holds*

$$\begin{cases} q \circ \tau(P_0) = q(-1) = [\sqrt{-3}, 0, 1], \\ q \circ \tau(P_1') = q(1) = [-\sqrt{-3}, 0, 1](= q \circ \tau(P_1)), \\ \\ q \circ \tau(M_0) = q(\infty) = [\sqrt{-1}, 1, 0], \\ q \circ \tau(M_0') = q(0) = [-\sqrt{-1}, 1, 0], \\ \\ q \circ \tau(M_d) = q(\sqrt{-1}) = [0, 1, \frac{\sqrt{-3}}{3}], \\ q \circ \tau(M_{01}) = q(-\sqrt{-1}) = [0, 1, -\frac{\sqrt{-3}}{3}]. \end{cases} \tag{4.1}$$

This is the equality up to $\mathrm{Aut}_{\boldsymbol{Q}}(KC)$.

Set a $\boldsymbol{Q}(\sqrt{-3})$ isomorphism $\pi : KC \to \Sigma = \boldsymbol{P}^1$ by

$$\begin{cases} \pi([a, 1, c]) = a - \sqrt{-3}c \\ \pi([\sqrt{-3}, 0, 1]) = 0 \\ \pi([-\sqrt{-3}, 0, 1]) = \infty. \end{cases}$$

Theorem 4.2. *The canonical model modular function is given by*

$$\pi \circ q \circ \tau = \frac{2\lambda(u) - 1}{2(\lambda^2(u) - \lambda(u))}.$$

§ 4.3. Procedures to obtain the Main result .

1) The 6 elliptic points on $\mathcal{D}$ are regular fixed points in the sense of Shimura. So, the coordinates of ψ values are restricted. ... Due to Shimura's complex multiplication theorem.

2) Suppose an element $\rho \in \mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})$ acts on KC. For an elliptic point $\zeta \in \boldsymbol{H}$, it holds $\psi(\zeta)^\rho = \psi(\exists\zeta')$, for some elliptic point ζ'.... Due to the Shimura riciprocity law.

3) Because KC does have no real point, the 6 elliptic points are pairwise complex conjugates on KC.

4) By observing the values of the modular function $\tau(u)$, we can calculate the cross ratios of them, $\cdots$ Proposition 3.2.

5) The group $\mathrm{Aut}_{\boldsymbol{Q}}(KC)$ acts on the set of $\boldsymbol{Q}(\sqrt{-3})$-rational points on KC in a transitive way. Same for $\boldsymbol{Q}(\sqrt{-1})$-rational points and $\boldsymbol{Q}(\sqrt{2}, \sqrt{-3})$-rational points.

6) Computer calculation of the coordinates of the values of $\psi = q \circ \tau \circ h$ on these elliptic points brings the required result.

§ 5. Numerical Examples

(i) $\boldsymbol{Q}(\sqrt{-3})$-point $u_1 = \frac{\sqrt{2}-\sqrt{6}}{1+\sqrt{3}} \in \mathcal{D}$ is a regular fixed point for $\boldsymbol{Q}(\sqrt{-3})$ that corresponds to the linear transformation $z \mapsto (3(-1+z))/(3+z)$ acting on $\boldsymbol{H}$. By making approximate calculation up to 60 digits, we know that it holds $\pi \circ q \circ \tau \circ (u_1) = \frac{13}{55} \in \Sigma(= \pi(KC))$.

(ii) $\boldsymbol{Q}(\sqrt{-19})$-point $u_3 = \frac{1}{3}(-2\sqrt{2}+i\sqrt{19}) \in \mathcal{D}$ is a regular fixed point for $\boldsymbol{Q}(\sqrt{-19})$ that corresponds to the linear transformation $z \mapsto \frac{-9+z-2\sqrt{2}z}{1+2\sqrt{2}+3z}$ acting on $\boldsymbol{H}$. By making approximate calculation up to 70 digits, we know that it holds $\pi^{-1} \circ q \circ \tau \circ (u_3) = [-\frac{9}{56} - \frac{42\sqrt{-19}}{211} : 1 : \frac{53}{379} + \frac{133\sqrt{-19}}{400}] \in KC$.

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