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On poly-Euler numbers of the second kind

By

Takao KOMATSU*

Abstract

For an integer k, define poly-Euler numbers of the second kind $\widehat{E}_n^{(k)}$ $(n = 0, 1, \dots)$ by

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4\sinh t} = \sum_{n=0}^\infty \widehat{E}_n^{(k)} \frac{t^n}{n!}\,.$$

When $k = 1$, $\widehat{E}_n = \widehat{E}_n^{(1)}$ are *Euler numbers of the second kind* or *complimentary Euler numbers* defined by

$$\frac{t}{\sinh t} = \sum_{n=0}^\infty \widehat{E}_n \frac{t^n}{n!}\,.$$

Euler numbers of the second kind were introduced as special cases of hypergeometric Euler numbers of the second kind in [7], so that they would supplement hypergeometric Euler numbers. In this paper, we give several properties of Euler numbers of the second kind. In particular, we determine their denominators. We also show several properties of poly-Euler numbers of the second kind, including duality formulae and congruence relations.

§ 1. Introduction

For an integer k, poly-Euler numbers $E_n^{(k)}$ $(n = 0, 1, \dots)$ are defined by

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4t\cosh t} = \sum_{n=0}^\infty E_n^{(k)} \frac{t^n}{n!} \tag{1.1}$$

([11, 12, 13]), where

$$\mathrm{Li}_k(z) = \sum_{n=1}^\infty \frac{z^n}{n^k} \quad (|z| < 1,\ k \in \mathbb{Z})$$

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*Department of Mathematical Sciences, School of Science, Zhejiang Sci-Tech University, Hangzhou 310018 China.
e-mail: `komatsu@zstu.edu.cn`

is the k-th polylogarithm function. When $k=1$, $E_n=E_n^{(1)}$ are the Euler numbers defined by

$$\frac{1}{\cosh t}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}\,. \tag{1.2}$$

Euler numbers have been extensively studied by many authors (see e.g. [8, 11, 12, 13, 14] and references therein), in particular, by means of Bernoulli numbers. In [7], for $N\ge 0$ *hypergeometric Euler numbers* $E_{N,n}$ ($n=0,1,2,\dots$) are defined by

$$\begin{aligned}\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}&=\frac{t^{2N}/(2N)!}{\cosh t-\sum_{n=0}^{N-1}t^{2n}/(2n)!}\\ &=\sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!}\,,\end{aligned} \tag{1.3}$$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$${}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}\,.$$

Here $(x)^{(n)}$ is the rising factorial, defined by $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ ($n\ge 1$) with $(x)^{(0)}=1$. Note that When $N=0$, $E_n=E_{0,n}$ are the Euler numbers defined in (1.2).

The sums of products of hypergeometric Euler numbers can be expressed as for $N\ge 1$ and $n\ge 0$,

$$\sum_{i=0}^n\binom{n}{i}E_{N,i}E_{N,n-i}=\sum_{k=0}^n\binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\,.$$

where $\widehat{E}_{N,n}$ are the *hypergeometric Euler numbers of the second kind* or *complementary hypergeometric Euler numbers* defined by

$$\begin{aligned}\frac{1}{{}_1F_2(1;N+1,(2N+3)/2;t^2/4)}&=\frac{t^{2N+1}/(2N+1)!}{\sinh t-\sum_{n=0}^{N-1}t^{2n+1}/(2n+1)!}\\ &=\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}\end{aligned} \tag{1.4}$$

([7, Theorem 4]). When $n=0$, $\widehat{E}_n=\widehat{E}_{0,n}$ are the *Euler numbers of the second kind* or *complementary Euler numbers* defined by

$$\frac{t}{\sinh t}=\sum_{n=0}^\infty \widehat{E}_n\frac{t^n}{n!}\,. \tag{1.5}$$

In [8], $\widehat{E}_n$ are called *weighted Bernoulli numbers*. But they mean different in different literatures. On the other hand, the sums of products of hypergeometric Euler numbers of the second kind can be also expressed as

$$\sum_{i=0}^{n}\binom{n}{i}\widehat{E}_{N,i}\widehat{E}_{N,n-i}=\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}$$

([7, Theorem 6]).

Euler numbers of the second kind are complementary in view of determinants too. It is known that the Euler numbers are given by the determinant

$$E_{2n}=(-1)^n(2n)!\begin{vmatrix}\frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix} \tag{1.6}$$

(*Cf.* [3, p.52]). Euler numbers of the second kind ([6, Corollary 2.2]) can be expressed as

$$\widehat{E}_{2n}=(-1)^n(2n)!\begin{vmatrix}\frac{1}{3!} & 1 & & & \\ \frac{1}{5!} & \frac{1}{3!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & & \frac{1}{3!} & 1 \\ \frac{1}{(2n+1)!} & \frac{1}{(2n-1)!} & \cdots & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}\,. \tag{1.7}$$

Since Bernoulli numbers can be expressed as

$$B_n=(-1)^n n!\begin{vmatrix}\frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & & \frac{1}{2!} & 1 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!}\end{vmatrix}$$

(*Cf.* [3, p.53]), Euler numbers and those of second kind fill the gaps each other in Bernoulli numbers.

In [7, Proposition 1.1], it is shown that hypergeometric Euler numbers $E_{N,n}$ satisfy the relation:

$$\sum_{i=0}^{n/2}\frac{1}{(2N+n-2i)!(2i)!}E_{N,2i}=0\quad(n\ge 2\text{ is even})$$

with $E_{N,0}=1$.

From (1.4), we have

$$\begin{aligned}\frac{t^{2N+1}}{(2N+1)!} &= \left(\sum_{n=N}^\infty \frac{t^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}\right)\\ &= t^{2N+1}\left(\sum_{n=0}^\infty \frac{\frac{1+(-1)^n}{2}t^n}{(2N+n+1)!}\right)\left(\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}\right)\\ &= t^{2N+1}\sum_{n=0}^\infty\left(\sum_{i=0}^n \frac{\frac{1+(-1)^{n-i}}{2}}{(2N+n-i+1)!}\frac{\widehat{E}_{N,i}}{i!}\right)t^n\,.\end{aligned}$$

Therefore, the hypergeometric Euler numbers of the second kind satisfy the recurrence relation for even $n\ge 2$

$$\sum_{i=0}^{n/2}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!}=0$$

or for $n\ge 1$

$$\widehat{E}_{N,2n}=-(2n)!(2N+1)!\sum_{i=0}^{n-1}\frac{\widehat{E}_{N,2i}}{(2N+2n-2i+1)!(2i)!}\,. \tag{1.8}$$

It turns that $\widehat{E}_{N,2n}$ can be given by the determinant ([6, Theorem 2.1]).

Theorem 1.1. *For $N\ge 0$ and $n\ge 1$, we have*

$$\widehat{E}_{N,2n}=(-1)^n(2n)!\begin{vmatrix}\frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \frac{(2N+1)!}{(2N+5)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1\\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!}\end{vmatrix}\,.$$

When $N=0$, we obtain the determinant expression of Euler numbers of the second kind in (1.7).

Similarly to Theorem 1.1, we get the determinant expression of hypergeometric Euler numbers ([6, Theorem 2.3]).

Theorem 1.2. *For $N\ge 0$ and $n\ge 1$, we have*

$$E_{N,2n}=(-1)^n(2n)!\begin{vmatrix}\frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1\\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!}\end{vmatrix}\,.$$

When $N = 0$, we obtain the determinant expression of Euler numbers in (1.6).

In Section 2, we shall show several properties of Euler numbers of the second kind. In particular, we determine the denominator of $\widehat{E}_{2n}$. In Section 3, we introduce poly-Euler numbers of the second kind as one directed generalizations of the original Euler numbers of the second kind. We give some expressions of poly-Euler numbers of the second kind with both positive and negative indices. In Section 4, we show one type of duality formula for poly-Euler numbers of the second kind. In Section 5, we shall give several congruence relations of poly-Euler numbers of the second kind with negative indices.

§ 2. Euler numbers of the second kind

In this section, we shall show several properties of Euler numbers of the second kind. In particular, we determine the denominator of $\widehat{E}_{2n}$. We also give some identities involving Euler numbers of the second kind, as analogous results of those in Euler numbers.

From the definitions (1.2) and (1.5),

$$E_{2n+1} = \widehat{E}_{2n+1} = 0 \quad (n \ge 0)\,.$$

We also know that

$$\frac{1}{\cos t} = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{t^{2n}}{(2n)!} \tag{2.1}$$

and

$$\frac{t}{\sin t} = \sum_{n=0}^{\infty} (-1)^n \widehat{E}_{2n} \frac{t^{2n}}{(2n)!} \tag{2.2}$$

Euler numbers E_{2n} are integers, but Euler numbers of the second kind $\widehat{E}_{2n}$ are rational numbers. We can know the denominator of $\widehat{E}_{2n}$ completely.

Theorem 2.1. *For an integer $n \ge 1$, the denominator of Euler numbers of the second kind $\widehat{E}_{2n}$ is given by*

$$\prod_{(p-1)|2n} p\,,$$

where p runs over all odd primes with $(p-1)|2n$. In particular,

$$\left(\prod_{(p-1)|2n} p\right) \widehat{E}_{2n}$$

is an integer, where p runs over all odd primes with $(p-1)|2n$.

Proof. Notice that for $n \ge 1$, we have

$$\widehat{E}_n = 2^n \mathcal{B}_n\left(\frac{1}{2}\right) = (2-2^n)\mathcal{B}_n\,,$$

where $\mathcal{B}_n(x)$ is the Bernoulli polynomial, defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^\infty \mathcal{B}_n(x)\frac{t^n}{n!}\,.$$

When $x=0$, $\mathcal{B}_n = \mathcal{B}_n(0)$ is the classical Bernoulli number with $\mathcal{B}_1 = -1/2$. By Von Staud-Clausen theorem, for $n \ge 1$

$$\mathcal{B}_{2n} + \sum_{(p-1)|2n}\frac{1}{p}$$

is an integer, where the sum extends over all primes p with $(p-1)|2n$. By Fermat's Little Theorem, if $(p-1)|2n$, then $m^{2n} \equiv 1 \pmod p$ for $m = 1, 2, \dots, p-1$. Thus, $2^{2n} \equiv 1 \not\equiv 2 \pmod p$ for any odd prime p. Therefore, the denominator of Euler numbers of the second kind is given by

$$\prod_{(p-1)|2n} p$$

where the product extends over all odd primes p with $(p-1)|2n$. □

Example 2.2. The odd primes p satisfying $(p-1)|24$ are $3, 5, 7, 13$, and

$$\widehat{E}_{24} = \frac{1982765468311237}{1365} = \frac{47\cdot 103\cdot 178481\cdot 2294797}{3\cdot 5\cdot 7\cdot 13}\,.$$

The odd prime p satisfying $(p-1)|26$ is 3, and

$$\widehat{E}_{26} = -\frac{286994504449393}{3} = -\frac{13\cdot 31\cdot 601\cdot 1801\cdot 657931}{3}\,.$$

Remark. For any integer $n \ge 0$,

$$(2n+1)(2n-1)\cdots 3\widehat{E}_{2n} = \frac{(2n+1)!}{2^n n!}\widehat{E}_{2n}$$

is an integer.

It is known that Euler numbers satisfy the recurrence relation

$$\sum_{j=0}^n \binom{2n}{2j} E_{2j} = 0 \quad (n \ge 1)$$

with $E_0 = 1$. Similarly, Euler numbers of the second kind satisfy the following recurrence relation.

Theorem 2.3. *For $n \ge 1$,*

$$\sum_{j=0}^{n} \binom{2n+1}{2j} \widehat{E}_{2j} = 0$$

and $\widehat{E}_0 = 1$.

Proof. From the definition (1.5), we have

$$\begin{aligned} t &= \frac{t}{\sinh t} \sinh t \\ &= \left(\sum_{j=0}^{\infty} \widehat{E}_{2j} \frac{t^{2j}}{(2j)!} \right) \left(\sum_{l=0}^{\infty} \frac{t^{2l+1}}{(2l+1)!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{2n+1}{2j} \widehat{E}_{2j} \frac{t^{2n+1}}{(2n+1)!} \quad (n = j + l). \end{aligned}$$

Comparing the coefficients on both sides, we get the result. □

For a positive integer n and a nonnegative integer k, Euler numbers satisfy the relation

$$\sum_{j=0}^{n} \binom{2n}{2j} (2k+1)^{2n-2j} E_{2j} = 2 \sum_{l=1}^{k} (-1)^{k-l} (2l)^{2n}$$

(e.g. [9]). Euler numbers of the second kind satisfy the following relation.

Theorem 2.4. *For a positive integer n and a nonnegative integer k,*

$$\sum_{j=0}^{n} \binom{2n+1}{2j} (2k+1)^{2n-2j+1} \widehat{E}_{2j} = 2(2n+1) \sum_{l=1}^{k} (2l)^{2n}.$$

Proof. Put

$$A(t) = \sum_{k=0}^{\infty} t^k \cos kx \quad \text{and} \quad B(t) = \sum_{k=0}^{\infty} t^k \sin kx.$$

For $|t| < 1$, we have

$$\begin{aligned} A(t) + \sqrt{-1} B(t) &= \sum_{k=0}^{\infty} t^k (\cos x + \sqrt{-1} \sin x)^k \\ &= \frac{1}{1 - t\cos x - \sqrt{-1} t \sin x} = \frac{1 - t\cos x + \sqrt{-1} t \sin x}{1 - 2t\cos x + t^2}. \end{aligned}$$

Hence, we get

$$A(t)=\frac{1-t\cos x}{1-2t\cos x+t^2}\quad\text{and}\quad B(t)=\frac{t\sin x}{1-2t\cos x+t^2}\quad(|t|<1)\,,$$

yielding

$$\begin{aligned}\sum_{k=0}^\infty t^k\cos(2k+1)x&=\frac{A(\sqrt{t})-A(-\sqrt{t})}{2\sqrt{t}}=\frac{(1-t)\cos x}{(1+t)^2-4t\cos^2 x}\,,\\ \sum_{k=0}^\infty t^k\cos 2kx&=\frac{A(\sqrt{t})+A(-\sqrt{t})}{2}=\frac{1-2t\cos^2 x+t}{(1+t)^2-4t\cos^2 x}\,,\\ \sum_{k=0}^\infty t^k\sin(2k+1)x&=\frac{B(\sqrt{t})-B(-\sqrt{t})}{2\sqrt{t}}=\frac{(1+t)\sin x}{(1+t)^2-4t\cos^2 x}\,,\\ \sum_{k=0}^\infty t^k\sin 2kx&=\frac{B(\sqrt{t})+B(-\sqrt{t})}{2}=\frac{t\sin 2x}{(1+t)^2-4t\cos^2 x}\,.\end{aligned}$$

Thus, for $|t|<1$, we have

$$\begin{aligned}\sum_{k=0}^\infty t^k\sum_{j=0}^{2k}\cos(2k-2j)x&=\sum_{k=0}^\infty t^k\left(2\sum_{l=0}^\infty t^l\cos 2lx-1\right)\\ &=2\left(\sum_{k=0}^\infty t^k\right)\left(\sum_{l=0}^\infty t^l\cos 2lx\right)-\sum_{k=0}^\infty t^k\\ &=\frac{2}{1-t}\frac{1-2t\cos^2 x+t}{(1+t)^2-4t\cos^2 x}-\frac{1}{1-t}\\ &=\frac{1+t}{(1+t)^2-4t\cos^2 x}=\frac{1}{\sin x}\sum_{k=0}^\infty t^k\sin(2k+1)x\,.\end{aligned}$$

Therefore, we obtain

$$\sum_{j=0}^{2k}\cos(2k-2j)x=\frac{\sin(2k+1)x}{\sin x}\,. \tag{2.3}$$

The right-hand side of (2.3) is equal to

$$\begin{aligned}&\frac{x}{\sin x}\frac{\sin(2k+1)x}{x}\\ &=\left(\sum_{j=0}^\infty(-1)^j\widehat{E}_{2j}\frac{x^{2j}}{(2j)!}\right)\left(\sum_{m=0}^\infty(-1)^m(2k+1)^{2m+1}\frac{x^{2m}}{(2m+1)!}\right)\\ &=\sum_{n=0}^\infty(-1)^n\sum_{j=0}^n\binom{2n+1}{2j}(2k+1)^{2n-2j+1}\widehat{E}_{2j}\frac{x^{2n}}{(2n+1)!}\quad(j+m=n)\,.\end{aligned}$$

The left-hand side of (2.3) is equal to

$$\sum_{j=0}^{2k}\sum_{n=0}^{\infty}(-1)^n(2k-2j)^{2n}\frac{x^{2n}}{(2n)!}\,.$$

Comparing the coefficients on both sides, we have

$$\begin{aligned}\sum_{j=0}^{n}\binom{2n+1}{2j}(2k+1)^{2n-2j+1}\frac{\widehat{E}_{2j}}{(2n+1)!}&=\sum_{l=0}^{2k}\frac{(2k-2l)^{2n}}{(2n)!}\\&=2\sum_{l=1}^{k}\frac{(2l)^{2n}}{(2n)!}\,.\end{aligned}$$

Therefore, we get the desired result. □

§ 3. Poly-Euler numbers of the second kind

In this section we introduce poly-Euler numbers of the second kind as one directed generalizations of the original Euler numbers of the second kind. A different direction of generalizations is in (1.4) as hypergeometric Euler numbers of the second kind. Similar poly numbers are poly-Bernoulli numbers ([4]) and poly-Cauchy numbers ([5]). We shall give some expressions of poly-Euler numbers of the second kind with both positive and negative indices.

For an integer k, define *poly-Euler numbers of the second kind* $\widehat{E}_n^{(k)}$ $(n=0,1,\dots)$ by

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4\sinh t}=\sum_{n=0}^{\infty}\widehat{E}_n^{(k)}\frac{t^n}{n!}\,. \tag{3.1}$$

When $k=1$, $\widehat{E}_n=\widehat{E}_n^{(1)}$ are *Euler numbers of the second kind* or complimentary Euler numbers defined in (1.5). Several values of poly-Euler numbers of the second kind can be seen in Table 1.

Poly-Euler numbers of the second kind can be expressed explicitly in terms of poly-Bernoulli numbers $B_n^{(k)}$ ([4]) defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}\,.$$

When $k=1$, $B_n=B_n^{(1)}$ is the Bernoulli number with $B_1=1/2$. Notice that poly-Bernoulli numbers can be expressed explicitly ([4, Theorem 1]) in terms of the Stirling numbers of the second kind $\left\{{m \atop j}\right\}$:

$$B_m^{(k)}=\sum_{j=0}^{m}\frac{(-1)^{m-j}j!}{(j+1)^k}\left\{{m \atop j}\right\}\,.$$

Table 1. The numbers $\widehat{E}_n^{(k)}$ for $1 \le n \le 7$ and $1 \le k \le 5$

k	1	2	3	4	5
$\widehat{E}_1^{(k)}$	0	-1	$-\frac{3}{2}$	$-\frac{7}{4}$	$-\frac{15}{8}$
$\widehat{E}_2^{(k)}$	$-\frac{1}{3}$	$\frac{5}{9}$	$\frac{59}{27}$	$\frac{275}{81}$	$\frac{1004}{243}$
$\widehat{E}_3^{(k)}$	0	1	$-\frac{11}{6}$	$-\frac{211}{36}$	$-\frac{985}{108}$
$\widehat{E}_4^{(k)}$	$\frac{7}{15}$	$-\frac{679}{225}$	$-\frac{12737}{3375}$	$\frac{245789}{50625}$	$\frac{12383617}{759375}$
$\widehat{E}_5^{(k)}$	0	$-\frac{7}{3}$	$\frac{527}{30}$	$\frac{47171}{2700}$	$-\frac{85361}{9000}$
$\widehat{E}_6^{(k)}$	$-\frac{31}{21}$	$\frac{60001}{2205}$	$\frac{483221}{231525}$	$-\frac{1961354909}{24310125}$	$-\frac{205924986214}{2552563125}$
$\widehat{E}_7^{(k)}$	0	$\frac{31}{3}$	$-\frac{45853}{210}$	$-\frac{1250393}{132300}$	$\frac{763114237}{2315250}$

Here, the Stirling numbers of the second kind are defined by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n ,$$

yielding from

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\cdots(x-k+1) .$$

Lemma 3.1. *For integers n and k with $n \ge 0$, we have*

$$\widehat{E}_n^{(k)} = \frac{1}{2} \sum_{m=0}^{n} \binom{n}{m} 4^m \big((-1)^{n-m} + (-3)^{n-m} \big) B_m^{(k)} .$$

When the index is negative, we had a more explicit formula without Bernoulli numbers [6].

Lemma 3.2. *For nonnegative integers n and k, we have*

$$\widehat{E}_n^{(-k)} = \frac{(-1)^k}{2} \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \big((4l+3)^n + (4l+1)^n \big) .$$

Lemma 3.2 can be stated as follows too.

Lemma 3.3. *For nonnegative integers n and k, we have*

$$\widehat{E}_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} (4l+2)^{n-2m} .$$

Table 2. The numbers $\widehat{E}_n^{(-k)}$ for $1 \le n \le 7$ and $0 \ge -k \ge -4$

$-k$	0	-1	-2	-3	-4
$\widehat{E}_1^{(-k)}$	2	6	14	30	62
$\widehat{E}_2^{(-k)}$	5	37	165	613	2085
$\widehat{E}_3^{(-k)}$	14	234	1826	10770	55154
$\widehat{E}_4^{(-k)}$	41	1513	19689	175465	1287657
$\widehat{E}_5^{(-k)}$	122	9966	210134	2741670	27930182
$\widehat{E}_6^{(-k)}$	365	66637	2236365	41809933	578341965
$\widehat{E}_7^{(-k)}$	1094	450834	23819306	628464090	11615023034

Several exact values can be seen in Table 2. As special cases, we have the following.

Lemma 3.4. *For nonnegative integers n and k, we have*

$$\widehat{E}_0^{(-k)} = 1, \quad \widehat{E}_1^{(-k)} = 2^{k+2} - 2, \quad \widehat{E}_2^{(-k)} = 32 \cdot 3^k - 2^{k+5} + 5,$$
$$\widehat{E}_3^{(-k)} = 384 \cdot 4^k - 576 \cdot 3^k + 220 \cdot 2^k - 14,$$
$$\widehat{E}_4^{(-k)} = 6144 \cdot 5^k - 12288 \cdot 4^k + 7616 \cdot 3^k - 1472 \cdot 2^k + 41,$$
$$\widehat{E}_5^{(-k)} = 122880 \cdot 6^k - 307200 \cdot 5^k + 264960 \cdot 4^k - 90240 \cdot 3^k + 9844 \cdot 2^k - 122,$$
$$\widehat{E}_n^{(0)} = \frac{3^n + 1}{2}, \quad \widehat{E}_n^{(-1)} = \frac{7^n + 5^n}{2}, \quad \widehat{E}_n^{(-2)} = \frac{2(11^n + 9^n) - (7^n + 5^n)}{2},$$
$$\widehat{E}_n^{(-3)} = \frac{6(15^n + 13^n) - 6(11^n + 9^n) + (7^n + 5^n)}{2},$$
$$\widehat{E}_n^{(-4)} = \frac{24(19^n + 17^n) - 36(15^n + 13^n) + 14(11^n + 9^n) - (7^n + 5^n)}{2},$$
$$\widehat{E}_n^{(-5)} = \frac{120(23^n + 21^n) - 240(19^n + 17^n) + 150(15^n + 13^n) - 30(11^n + 9^n) + (7^n + 5^n)}{2}.$$

§ 4. Duality formulae for poly-Euler numbers of the second kind

It is known that the duality formula $B_n^{(-k)} = B_k^{(-n)}$ $(n, k \ge 0)$ holds for poly-Bernoulli numbers ([4]). In this section, we shall show a different type of duality formula for poly-Euler numbers of the second kind.

Theorem 4.1. *For nonnegative integers n and k, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{2 - E_{n-m}}{4^n} \widehat{E}_m^{(-k)} = \sum_{m=0}^{k} \binom{k}{m} \frac{2 - E_{k-m}}{4^k} \widehat{E}_m^{(-n)}.$$

This theorem is proven by using the expression of poly-Bernoulli numbers in terms of poly-Euler numbers of the second kind. In [6, Theorem 3], poly-Euler numbers of the second kind are expressed in terms of poly-Bernoulli numbers:

$$\widehat{E}_n^{(k)}=\frac{1}{2}\sum_{m=0}^n\binom{n}{m}4^m\big((-1)^{n-m}+(-3)^{n-m}\big)B_m^{(k)}\,.$$

Proposition 4.2. *For integers n and k with $n\ge 0$, we have*

$$B_n^{(k)}=\sum_{m=0}^n\binom{n}{m}\frac{2-E_{n-m}}{4^n}\widehat{E}_m^{(k)}\,.$$

Proof. Since

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{1-e^{-4t}}=\frac{2}{e^{-t}+e^{-3t}}\frac{\mathrm{Li}_k(1-e^{-4t})}{2(e^t-e^{-t})}\,, \tag{4.1}$$

we have

$$\begin{aligned}\sum_{n=0}^\infty B_n^{(k)}\frac{(4t)^n}{n!}&=\left(2e^t-\frac{1}{\cosh t}\right)\left(\sum_{n=0}^\infty\widehat{E}_n^{(k)}\frac{t^n}{n!}\right)\\&=\left(2\sum_{l=0}^\infty\frac{t^l}{l!}\right)\left(\sum_{m=0}^\infty\widehat{E}_m^{(k)}\frac{t^m}{m!}\right)-\left(\sum_{l=0}^\infty E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^\infty\widehat{E}_m^{(k)}\frac{t^m}{m!}\right)\\&=2\sum_{n=0}^\infty\sum_{m=0}^n\binom{n}{m}\widehat{E}_m^{(k)}\frac{t^n}{n!}-\sum_{n=0}^\infty\sum_{m=0}^n\binom{n}{m}E_{n-m}\widehat{E}_m^{(k)}\frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients on both sides, we get

$$4^nB_n^{(k)}=\sum_{m=0}^n\binom{n}{m}(2-E_{n-m})\widehat{E}_m^{(k)}\,.$$

□

Proof of Theorem 4.1. From Proposition 4.2 and the duality formula $B_n^{(-k)}=B_k^{(-n)}$, we get the desired result. □

We can also describe the positivity of poly-Euler numbers of the second kind with negative index.

Theorem 4.3. *For nonnegative integers n and k, we have*

$$\begin{aligned}\widehat{E}_n^{(-k)} &= \sum_{j=0}^{\min(n,k)}(j!)^2\sum_{m=0}^{n}\sum_{\mu=0}^{k}\binom{n}{m}\binom{k}{\mu}\left\{{n-m \atop j}\right\}\left\{{\mu \atop j}\right\}4^{n-m}\widehat{E}_m^{(0)}\\ &= \sum_{j=0}^{\min(n,k)}(j!)^2\sum_{m=0}^{n}\sum_{\mu=0}^{k}\binom{n}{m}\binom{k}{\mu}\left\{{n-m \atop j}\right\}\left\{{\mu \atop j}\right\}\frac{4^{n-m}(3^m+1)}{2}\,.\end{aligned}$$

Proof. From (4.1), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\mathrm{Li}_{-k}(1-e^{-4x})}{4\sinh x}\frac{(4y)^k}{k!} &= \frac{e^{-x}+e^{-3x}}{2}\sum_{k=0}^{\infty}\frac{\mathrm{Li}_{-k}(1-e^{-4x})}{1-e^{-4x}}\frac{(4y)^k}{k!}\\ &= e^{-4x}\frac{\mathrm{Li}_0(1-e^{-4x})}{4\sinh x}\frac{e^{4(x+y)}}{e^{4x}+e^{4y}-e^{4(x+y)}}\\ &= \sum_{m=0}^{\infty}\widehat{E}_m^{(0)}\frac{x^m}{m!}e^{4y}\sum_{j=0}^{\infty}(j!)^2\frac{(e^{4x}-1)^j}{j!}\frac{(e^{4y}-1)^j}{j!}\\ &= \sum_{j=0}^{\infty}(j!)^2\left(\sum_{m=0}^{\infty}\widehat{E}_m^{(0)}\frac{x^m}{m!}\right)\left(\sum_{\nu=j}^{\infty}\left\{{\nu \atop j}\right\}\frac{(4x)^\nu}{\nu!}\right)\\ &\qquad\times\left(\sum_{l=0}^{\infty}\frac{(4y)^l}{l!}\right)\left(\sum_{\mu=j}^{\infty}\left\{{\mu \atop j}\right\}\frac{(4y)^\nu}{\nu!}\right)\\ &= \sum_{j=0}^{\infty}(j!)^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\widehat{E}_m^{(0)}\left\{{n-m \atop j}\right\}4^{n-m}\frac{x^n}{n!}\\ &\qquad\times\sum_{k=0}^{\infty}\sum_{\mu=0}^{k}\binom{k}{\mu}\left\{{\mu \atop j}\right\}4^k\frac{y^k}{k!}\,.\end{aligned}$$

Since the left hand side is equal to

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\widehat{E}_n^{(-k)}\frac{x^n}{n!}\frac{(4y)^k}{k!}\,,$$

comparing the coefficients on both sides, we get the desired result. □

Remark. When $k=1$ in Theorem 4.3, we have

$$\widehat{E}_n^{(-1)} = \begin{cases}0 & \text{if } n=0;\\ \sum_{m=0}^{n}\binom{n}{m}\left\{{n-m \atop 1}\right\}4^{n-m}\widehat{E}_m^{(0)} & \text{if } n\ge 1\,.\end{cases}$$

It matches the result in Lemma 3.4, as

$$\widehat{E}_n^{(-1)} = \frac{7^n+5^n}{2}(n\ge 0)\,.$$

§ 5. Congruence relations

Poly-Euler numbers of the second kind with positive indices are rational numbers, but those with negative indices are integers. Hence, it is worthwhile considering congruence relations.

In [6], we determined the parity and the divisibility of poly-Euler numbers of the second kind as follows.

Lemma 5.1. *For any nonnegative integer k, we have*

$$\widehat{E}_n^{(-k)} \equiv \begin{cases} 0 \pmod 2 & \text{if } n \text{ is odd,} \\ 1 \pmod 2 & \text{if } n \text{ is even.} \end{cases}$$

Lemma 5.2. *For an odd prime p with $p > 3$ and a nonnegative integer k, we have*

$$\widehat{E}_p^{(-k)} \equiv 2^{k+2} - 2 \pmod p\,.$$

For a nonnegative integer k, we have

$$\widehat{E}_3^{(-k)} \equiv (-1)^k + 1 \pmod 3\,,$$
$$\widehat{E}_2^{(-k)} \equiv 0 \pmod 2\,.$$

In this section, we shall give some more congruence relations of poly-Euler numbers of the second kind with negative indices.

Proposition 5.3. *Let p be an odd prime, and k be a fixed nonnegative integer. Then for integers n and m with $n, m \ge 0$ and $n \equiv m \pmod{p-1}$, we have*

$$\widehat{E}_n^{(-k)} \equiv \widehat{E}_m^{(-k)} \pmod p\,.$$

Proof. By Fermat's Little Theorem, if $n \equiv m \pmod{p-1}$, then

$$(4l+3)^n \equiv (4l+3)^m \quad \text{and} \quad (4l+1)^n \equiv (4l+1)^m \pmod p\,.$$

By Lemma 3.2 with the fact that $(4l+3)^n + (4l+1)^n$ is even for $n \ge 0$, we get the desired result. □

Example 5.4. Let $p = 5$ and $k = 3$. As $6 \equiv 2 \pmod 4$,

$$\widehat{E}_6^{(-3)} - \widehat{E}_2^{(-3)} = 41809933 - 613 = 5 \cdot 8361864 .$$

Let $p = 3$ and $k = 4$. As $7 \equiv 5 \pmod 2$,

$$\widehat{E}_7^{(-4)} - \widehat{E}_5^{(-4)} = 11615023034 - 27930182 = 3 \cdot 3862364284 .$$

Theorem 5.5. *Let p be an odd prime. If $k \equiv p-2 \pmod{p-1}$ for odd integers n and k, then we have*

$$\widehat{E}_n^{(-k)} \equiv 0 \pmod p .$$

Proof. Notice that if $l \geq p$ then $l! \equiv 0 \pmod p$, and if $(p-1) \not| k$, $\left\{ {k \atop p-1} \right\} \equiv 0 \pmod p$. Hence, by Lemma 3.3,

$$\begin{aligned}\widehat{E}_n^{(-k)} &= (-1)^k \sum_{l=0}^{k} (-1)^l l! \left\{ {k \atop l} \right\} B(n,l) \\ &= (-1)^k \sum_{l=0}^{p-2} (-1)^l l! \left\{ {k \atop l} \right\} B(n,l) ,\end{aligned}$$

where

$$\begin{aligned}B(n,l) &:= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} (4l+2)^{n-2m} \\ &= \begin{cases} 0 \pmod p & \text{if } l = \frac{p-1}{2}; \\ -B(n,p-l-1) \pmod p & \text{if } l = 1,2,\ldots,\frac{p-3}{2} . \end{cases}\end{aligned}$$

Thus, we get

$$\begin{aligned}\widehat{E}_n^{(-k)} &\equiv -\sum_{l=1}^{p-2} (-1)^l l! \left\{ {p-2 \atop l} \right\} B(n,l) \\ &\equiv -\sum_{l=1}^{(p-3)/2} (-1)^l l! \left\{ {p-2 \atop l} \right\} \big(B(n,l) + B(n,p-l-1) \big) \\ &\equiv 0 \pmod p .\end{aligned}$$

Therefore, we have the desired result. $\square$

Example 5.6. Let $p = 5$. Then for any odd number n we have $\widehat{E}_n^{(-3)} \equiv 0 \pmod 5$. Together with Lemma 5.1, we can know that $\widehat{E}_n^{(-3)} \equiv 0 \pmod{10}$. As seen in Table 2, all of

$$\widehat{E}_1^{(-3)} = 30, \quad \widehat{E}_3^{(-3)} = 10770, \quad \widehat{E}_5^{(-3)} = 2741670 \quad \text{and} \quad \widehat{E}_7^{(-3)} = 628464090$$

are divided by 5.

Let $p = 7$. Then for any odd number n we have $E_n^{(-5)} \equiv 0 \pmod 7$.

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