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Rational orbits of prehomogeneous vector spaces

By

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§ 1. Pairs of exceptional Jordan algebras

This paper is an announcement of recent results of the author ([2], [3] with R. Kato and [5]). Let k be a field of characteristic not equal to $2, 3$. We denote the separable closure of k and the algebraic closure of k by $k^{\mathrm{sep}}, \overline{k}$ respectively.

If G is an algebraic group over k, and R is a k-algebra, G_R is the group of R-rational points of G. For algebraic groups, we only consider representations which are algebraic. If G is an algebraic group, we use the notation G° for its identity component. If V is a representation of G over k and $x \in V$, we denote the G-orbit of x by $G \cdot x$ to avoid confusion with the stabilizer G_x. We denote the group of $n \times n$ invertible matrices by $\mathrm{GL}(n)$. Let $\mathrm{SL}(n) = \{g \in \mathrm{GL}(n) \mid \det g = 1\}$. We denote the $n \times n$ unit matrix by I_n.

Let $\widetilde{\mathbb{O}}$ be the split octonion over k. It is the normed algebra over k obtained by the Cayley–Dickson process (see [1, pp.101–110]). If $A = \mathrm{M}(2)$ is the algebra of 2×2 matrices with the determinant as the norm, then $\widetilde{\mathbb{O}}$ is $A(+)$ with the notation of [1]. An *octonion* is, by definition, a normed algebra which is a k-form of $\widetilde{\mathbb{O}}$. Let $\mathbb{O}$ be an octonion. We use the notation $\|x\|$ for the norm of $x \in \mathbb{O}$. For $x, y \in \mathbb{O}$, let

$$Q(x, y) = \tfrac{1}{2}(\|x + y\| - \|x\| - \|y\|).$$

This is a non-degenerate symmetric bilinear form such that $Q(x, x) = \|x\|$. Let $L \subset \mathbb{O}$ be the orthogonal complement of $k \cdot 1$ with respect to Q. If $x = x_1 + x_2$ where $x_1 \in k \cdot 1$, $x_2 \in L$, then we define

$$\overline{x} = x_1 - x_2$$

and call it the *conjugate* of x. For $x \in \mathbb{O}$, we define the trace $\mathrm{tr}(x)$ by $\mathrm{tr}(x) = x + \overline{x}$.

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Let $\mathcal{J}$ be the exceptional Jordan algebra over k. Any element $X \in \mathcal{J}$ is of the form:

$$X = \begin{pmatrix} s_1 & x_3 & \overline{x_2} \\ \overline{x_3} & s_2 & x_1 \\ x_2 & \overline{x_1} & s_3 \end{pmatrix}, \quad s_i \in k,\ x_i \in \mathbb{O} \quad (i = 1, 2, 3).$$

For elements of $\mathcal{J}$, the notion of the determinant is well-defined and is given by

$$\det(X) = s_1 s_2 s_3 + \operatorname{tr}(x_1 x_2 x_3) - s_1\|x_1\| - s_2\|x_2\| - s_3\|x_3\|.$$

The multiplication of $\mathcal{J}$ is defined as follows:

$$X \circ Y = \frac{1}{2}(XY + YX)$$

where the multiplication used on the right-hand side is the multiplication of matrices.

The algebraic group GE_6 is defined by

$$GE_6 = \{L \in \mathrm{GL}(\mathcal{J}) \mid {}^{\forall}X \in \mathcal{J}, \det(LX) = c(L)\det(X) \text{ for some } c(L) \in \mathrm{GL}(1)\}.$$

Let

$$G = GE_6 \times \mathrm{GL}(2),\ V = \mathcal{J} \otimes \mathrm{Aff}^2$$

where Aff^2 is the 2-dimensional affine space regarded as a vector space. Then V is a representation of G.

For $x = (x_1, x_2) \in V$ $(x_1, x_2 \in \mathcal{J})$ and variables $v = (v_1, v_2)$, we put

$$F_x(v) = \det(v_1 x_1 + v_2 x_2).$$

This is a binary cubic form of v. Let $\Delta(x)$ be the discriminant of $F_x(v)$. The pair (G, V) is a prehomogeneous vector space and $\Delta(x)$ is a relative invariant polynomial. For details on this prehomogeneous vector space, see Sections 1,2 of [3]. Let $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$ (the set of semi-stable points).

Let $\mathrm{JIC}(k)$ be the set of equivalence classes of pairs $(\mathcal{M}, \mathfrak{n})$ where $\mathcal{M}$'s are isotopes of $\mathcal{J}$ and $\mathfrak{n}$'s are cubic étale subalgebras of $\mathcal{M}$ (see [4, pp.154–158] for the definition of isotopes of $\mathcal{J}$). Two such pairs $(\mathcal{M}_1, \mathfrak{n}_1)$ $(\mathcal{M}_2, \mathfrak{n}_2)$ are equivalent if there exists an isomorphism $\mathcal{M}_1 \to \mathcal{M}_2$ over k which induces an isomorphism from $\mathfrak{n}_1$ to $\mathfrak{n}_2$.

The following theorem is [3, Theorem 1.18].

Theorem 1.1. *The set $G_k \backslash V_k^{\mathrm{ss}}$ corresponds bijectively with the set* $\mathrm{JIC}(k)$.

In [3], to each element of V_k^{ss}, an isotope of $\mathcal{J}$ and its cubic étale subalgebra are explicitly constructed using an equivariant map from V to $\mathcal{J}$. More detailed discussion regarding the construction of $\mathcal{M}$ is in [2].

If $\alpha \in \overline{k}$ and $F_x(\alpha, 1) = 0$, we call α a root of $F_x(v)$. Let $\mathrm{H}^1(k, \mathfrak{S}_3)$ be the first Galois cohomology set where the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\mathfrak{S}_3$ is trivial. It is easy to see that the set $\mathrm{H}^1(k, \mathfrak{S}_3)$ is in bijective correspondence with conjugacy classes of anti-homomorphisms from $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ to $\mathfrak{S}_3$. So $\mathrm{H}^1(k, \mathfrak{S}_3)$ is in bijective correspondence with isomorphism classes of cubic étale algebras over k also.

If k is a finite field or the octonion is split, we can obtain a more precise statement. The following theorem is [3, Theorem 1.19].

Theorem 1.2. *If k is a finite field or $\mathbb{O}$ is the split octonion, then there is a bijective correspondence between $G_k \backslash V_k^{\mathrm{ss}}$ and $\mathrm{H}^1(k, \mathfrak{S}_3)$. Moreover, if $x \in V_k^{\mathrm{ss}}$, then the corresponding cohomology class in $\mathrm{H}^1(k, \mathfrak{S}_3)$ is the element determined by the action of the Galois group on the set of roots of $F_x(v)$.*

§ 2. Primitive trivectors in dimension six

We consider another prehomogeneous vector space. Let $W = \mathrm{Aff}^6$ and $W_1 = \wedge^3 W$. These are irreducible representations of GL(6). Let $\mathbb{e}_1, \dots, \mathbb{e}_6$ be the coordinate vectors of W. If $1 \le i_1, \dots, i_t \le 6$ are distinct, we use the notation $e_{i_1 \cdots i_t} = \mathbb{e}_{i_1} \wedge \cdots \wedge \mathbb{e}_{i_t}$. For $x, y \in W_1$, we define $B(x, y) = x \wedge y \in \wedge^6 W \cong k$. This is a non-degenerate alternating bilinear form on W_1. It is easy to see that $B(gx, gy) = \det g\, B(x, y)$ for $g \in \mathrm{GL}(6)$. Let $\omega = e_{14} + e_{25} + e_{36}$. We put

$$\begin{aligned} \mathrm{Sp}(6) &= \{g \in \mathrm{GL}(6) \mid g\omega = \omega\}, \\ \mathrm{GSp}(6) &= \{g \in \mathrm{GL}(6) \mid {}^{\exists}c(g) \in \mathrm{GL}(1), g\omega = c(g)\omega\}. \end{aligned}$$

These are connected algebraic subgroups of GL(6). It is well-known that Sp(6) is a simple group, GSp(6) is a reductive group and $c(g)$ is a rational character of GSp(6) with kernel Sp(6).

Let $G = \mathrm{GL}(1) \times \mathrm{GSp}(6)$. We define an action of GL(1) on W_1 assuming that $\alpha \in \mathrm{GL}(1)$ acts by multiplication by α. This makes W_1 a representation of G. The subspace $U = \{v \wedge \omega \mid v \in W\} \subset W_1$ is invariant by the action of G. So $V = W_1/U$ is a representation of G defined over k.

Obviously, $\dim V = 14$. We use the same notation $e_{i_1 \cdots i_t}$ for its image in V by abuse of notation. We put

$$w = e_{123} + e_{456} \in V, \quad \nu = \begin{pmatrix} 0 & \mathrm{I}_3 \\ \mathrm{I}_3 & 0 \end{pmatrix} \in \mathrm{GSp}(6).$$

Note that w is orthogonal to U and that $(1, \nu) \in G_w$.

Let Ex(2) be the set of isomorphism classes of extensions of k of degree up to two (which is in bijective correspondence with $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$). If k_1/k is a quadratic extension, $\sigma(k_1)$ is the non-tivial element of $\mathrm{Gal}(k_1/k)$ and A is a 3×3 matrix with entries in k_1, we define $A^* = {}^tA^{\sigma(k_1)}$ where $A^{\sigma(k_1)}$ is obtained by applying $\sigma(k_1)$ to all entries. If $A^* = A$, A is said to be Hermitian. Let $\mathrm{SH}_3(k_1)$ be the set of 3×3 Hermitian matrices with entries in k_1 and with determinant 1. The group $\mathrm{SL}(3)_{k_1}$ acts on $\mathrm{SH}_3(k_1)$ by $Q \mapsto gQg^*$ for $g \in \mathrm{SL}(3)_{k_1}$, $Q \in \mathrm{SH}_3(k_1)$. If $Q \in \mathrm{SH}_3(k_1)$, $\mathrm{SU}(k_1, Q)$ is the special unitary group with respect to Q. The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{SL}(3)$ by assuming that the action of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is $g \mapsto {}^tg^{-1}$. This action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{SL}(3)$ defines a semi-direct product structure $\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z})$. Then $\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z})$ acts on $\mathrm{SH}_3(k_1)$ by assuming that the action of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ is by $\mathrm{SH}_3(k_1) \ni Q \mapsto {}^tQ^{-1} \in \mathrm{SH}_3(k_1)$.

The following theorem is [5, Theorem 1.10].

Theorem 2.1. *There is a map $\gamma_V : G_k\backslash V_k^{\mathrm{ss}} \to \mathrm{Ex}(2)$ with the following properties (1)–(4).*

(1) $\gamma_V^{-1}(k) = G_k \cdot w$.

(2) $G_w^\circ \cong \mathrm{GL}(1) \times \mathrm{SL}(3)$, $G_w/G_w^\circ \cong \mathbb{Z}/2\mathbb{Z}$ *and* G_w/G_w° *is represented by* $(1, \nu)$.

(3) If k_1 is a quadratic extension of k, $\gamma_V^{-1}(k_1)$ is in bijective correspondence with $(\mathrm{SL}(3)_{k_1} \rtimes (\mathbb{Z}/2\mathbb{Z}))\backslash\mathrm{SH}_3(k_1)$.

(4) If $G \cdot x \in \gamma_V^{-1}(k_1)$ corresponds to the orbit of $Q \in \mathrm{SH}_3(k_1)$, $G_x^\circ \cong \mathrm{GL}(1) \times \mathrm{SU}(k_1, Q)$. Also G_x/G_x° is represented by an element of $G_{x\,k}$ of order two.

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