RIMS Kôkyûroku Bessatsu
B77 (2020), 183–189

Arithmetic and dynamical degrees of rational self-maps on algebraic varieties

By

Yohsuke Matsuzawa*

Abstract

In this note, we introduce the Silverman's conjecture on asymptotic growth of the height function along the orbit of dynamical systems of algebraic varieties. We report the recent results related to the conjecture.

§ 1. Introduction

Let X be a smooth projective variety defined over the field of algebraic numbers $\overline{\mathbb{Q}}$. For a dominant rational self-map $f\colon X \dashrightarrow X$, Silverman has formulated a conjecture which relates the asymptotic growth of the height along the f-orbit to the dynamical degree of f. In this note, we introduce the conjecture and summarize recent results about this conjecture.

§ 2. Arithmetic and Dynamical degrees

Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$. Let $f\colon X \dashrightarrow X$ be a dominant rational map. Silverman introduced the arithmetic degree which measures the growth rate of the height function along the f-orbit.

Definition 2.1. Fix a Weil height function h_X associated with an ample divisor on X. We write $h_X^+ = \max\{h_X, 1\}$. Let

$$X_f = \{P \in X(\overline{\mathbb{Q}}) \mid \text{for every } n \geq 0,\ f \text{ is defined at } f^n(P)\}$$

Received March 24, 2017. Revised November 30, 2017.
2010 Mathematics Subject Classification(s): 37P15
Key Words: *Arithmetic dynamics, dynamical degree, arithmetic degree.*
Supported by the Program for Leading Graduate Schools, MEXT, Japan
*Graduate school of mathematical sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.
e-mail: `myohsuke@ms.u-tokyo.ac.jp`

be the set of points whose f-orbit is well-defined. For $P \in X_f$, we define upper and lower arithmetic degrees as follows.

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_X^+(f^n(P))^{1/n}$$
$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} h_X^+(f^n(P))^{1/n}.$$

If these coincide, we write the limit $\alpha_f(P)$ and call arithmetic degree of f at P.

Example 2.2 ([9])**.** Let $f\colon X \longrightarrow X$ be a polarized endomorphism. Here, polarized means there exists an ample $\mathbb{R}$-divisor H on X such that f^*H is numerically equivalent to δH for some $\delta > 1$. Basic examples of such morphisms are non-isomorphic surjective endomorphisms on projective spaces and multiplication by n map ($n > 1$) on abelian varieties. In this case, $\alpha_f(P)$ exists and

$$\alpha_f(P) = 1 \iff \text{the } f\text{-orbit of } P \text{ is finite}$$
$$\alpha_f(P) = \delta \iff \text{the } f\text{-orbit of } P \text{ is infinite.}$$

In the study of algebraic dynamics, the asymptotic behavior of "degrees" of $(f^n)_n$ is important. The first dynamical degree is especially important in the relation with the arithmetic degree.

Definition 2.3. Let $N^1(X)$ be the group of divisors on X modulo numerical equivalence and $N^1(X)_\mathbb{R} = N^1(X)\otimes_\mathbb{Z}\mathbb{R}$. For a dominant rational map f, take a resolution of indeterminacy $p\colon Y \longrightarrow X, q\colon Y \longrightarrow X, f\circ p = q$ and define $f^* = p_* \circ q^*\colon N^1(X)_\mathbb{R} \longrightarrow N^1(X)_\mathbb{R}$. This is independent of the choice of resolution. Fix a norm $||\ ||$ on the finite dimensional vector space $\mathrm{Hom}_\mathbb{R}(N^1(X)_\mathbb{R}, N^1(X)_\mathbb{R})$. The first dynamical degree of f is

$$\delta_f = \lim_{n\to\infty} ||(f^n)^*||^{1/n}.$$

This is independent of the choice of $||\ ||$.

Main references about dynamical degrees are, for example, [3, 1, 16, 17].

Example 2.4. Since $f^*\colon N^1(X)_\mathbb{R} \longrightarrow N^1(X)_\mathbb{R}$ is defined over $\mathbb{Z}$, δ_f is equals to or greater than 1.

Example 2.5. When f is a morphism, δ_f is the spectral radius of f^*, that is, the supremum among the absolute values of eigenvalues of f^*. Indeed, since f is a morphism, we have $(f^n)^* = (f^*)^n$. Thus $\delta_f = \lim_{n\to\infty} ||(f^*)^n||^{1/n}$ and elementary linear algebra shows that this is the spectral radius of f^*.

Example 2.6 (Another definition of the dynamical degree)**.** Let H be any ample divisor on X. Then we have

$$\delta_f = \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n}.$$

Example 2.7. Let f be a polarized endomorphism as in Example 2.2. Then we have $\delta_f = \delta$.

Example 2.8. When the ground field is the complex number field $\mathbb{C}$, the first dynamical degree and the topological entropy are closely related. In the same manner as in Definition 2.3, we can define higher dynamical degrees $\delta_{f,p}, 0 \le p \le \dim X$. (Replace $N^1(X)$ by $N^p(X)$. Thus $\delta_f = \delta_{f,1}$ in this notation.) Then, the fundamental results by Gromov, Yomdin say that if f is a morphism, then we have $h_f = \max_{0\le p\le \dim X}\{\log \delta_{f,p}\}$ where h_f is the topological entropy of f with respect to the complex topology of X. Moreover, the first dynamical degree is greater than 1 if and only if the topological entropy is positive.

Remark. In the above example, we algebraically define the dynamical degrees, i.e. we use the action on $N^p(X)$. Actually, Gromov and Yomdin considered the actions on the cohomologies. By [16, Theorem 1.1], our p-th dynamical degree is equal to

$$\delta_{f,p} = \lim_{n\to\infty} ((f^n)^* H^p \cdot H^{\dim X - p})^{1/n}.$$

When X is a complex projective manifold, this is equal to

$$\lim_{n\to\infty} \left(\int_X (f^n)^* \omega_X^p \wedge \omega_X^{\dim X - p} \right)^{1/n}$$

where ω_X is a Kahler form on X. This is the complex geometrical definition of p-th dynamical degree (cf. [3]). When f is a surjective morphism, this is equal to the spectral radius of $f^*\colon H^{2p}(X) \longrightarrow H^{2p}(X)$ (cf. [2, §3], [4, §4]).

To study the behavior of the height function along the orbit, the following conjecture is a guiding problem.

Conjecture 2.9 (Silverman, Kawaguchi-Silverman [14, 9])**.**

(1) *For every $P \in X_f$, $\alpha_f(P)$ exists; $\overline{\alpha}_f(P) = \underline{\alpha}_f(P)$.*

(2) *For $P \in X_f$ with Zariski dense f-orbit, we have $\alpha_f(P) = \delta_f$.*

Example 2.10. By Example 2.2 and 2.7, Conjecture 2.9 is true for polarized endomorphisms.

Kawaguchi and Silverman proved that the first part of the conjecture is true when f is a morphism [8]. They also proved the second part in several cases including the followng.

- X is a surface and f is an automorphism [5].
- $X = \mathbb{P}^n$ and f is a rational map defined by monomials [14].
- $X = \mathbb{P}^n$ and f is a regular affine automorphism [6].
- X is an abelian variety [8, 15].

We refer to [7, 8, 9, 10, 11, 13, 14, 15] for results related to the conjecture.

§ 3. Main theorems

In general, the arithmetic degree is less than or equals to the first dynamical degree.

Theorem 3.1 ([10]). *Let $f\colon X \dashrightarrow X$ be a dominant rational map. Fix an ample height function h_X on X. For any $\epsilon > 0$, there exists $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C(\delta_f + \epsilon)^n h_X^+(P)$$

for all $P \in X_f$ and $n \geq 0$. In particular $\overline{\alpha}_f(P) \leq \delta_f$.

Remark.

1. This theorem holds over any ground field where the Weil height machine is well-defined.
2. When the characteristic of the ground field is zero, this theorem is appeared in [9], but unfortunately it turned out that their proof did not work.

When f is a morphism, we have slightly stronger estimates.

Theorem 3.2 ([10]). *Let $f : X \to X$ be a surjective morphism. Let $\rho = \dim N^1(X)_{\mathbb{R}}$ be the Picard number of X.*

(1) *When $\delta_f = 1$, there exists a constant $C > 0$ such that*

$$h_X^+(f^n(P)) \leq C n^{2\rho+2} h_X^+(P)$$

for all $P \in X(\overline{\mathbb{Q}})$ and $n \geq 1$.

(2) *Assume that* $\delta_f > 1$. *Then there exists a constant* $C > 0$ *such that*

$$h_X^+(f^n(P)) \leq C n^\rho \delta^n h_X^+(P)$$

for all $P \in X(\overline{\mathbb{Q}})$ *and* $n \geq 1$.

A canonical height function is a dynamical analogue of the Néron-Tate height. It can be used to study the dynamics of the self-map on the rational points of X. Very few are known, however, about canonical heights of dynamical systems of rational map. The following theorem says that under some technical (may be essential) conditions, the canonical height is well-defined.

Theorem 3.3 ([10]). *Let* $f : X \dashrightarrow X$ *be a dominant rational map. Assume* $\delta_f > 1$ *and there exists a nef* $\mathbb{R}$*-divisor* H *on* X *such that* $f^*H \equiv \delta_f H$. *Fix a height function* h_H *associated with* H. *Then for any* $P \in X_f$, *if the sequence* $(h_H(f^n(P))/\delta_f^n)_{n\geq 0}$ *is bounded below, the limit*

$$\lim_{n\to\infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

exists.

As a special case of this theorem, we have the following.

Corollary 3.4 ([10]). *Assume* $\dim N^1(X)_{\mathbb{R}} = 1$, $\delta_f > 1$ *and* f *is algebraically stable;* $(f^n)^* = (f^*)^n \colon N^1(X)_{\mathbb{R}} \longrightarrow N^1(X)_{\mathbb{R}}$. *Let* H *be any ample divisor on* X *and fix a height* h_H *associated with* H. *Then, the limit*

$$\lim_{n\to\infty} \frac{h_H(f^n(P))}{\delta_f^n}$$

exists for every $P \in X_f$.

It seems difficult to solve Conjecture 2.9 in full generality. For the case where X is a surface, Kawaguchi proved Conjecture 2.9 holds for automorphisms [5]. K. Sano, T. Shibata and the author proved the following.

Theorem 3.5 ([11]). *Let* X *be a smooth projective surface.*

(1) *For any surjective morphism* $f \colon X \longrightarrow X$, *Conjecture 2.9 is true.*

(2) *If* X *is non-rational, then for any birational self-map* $f \colon X \dashrightarrow X$, *Conjecture 2.9 is true.*

Independent of Conjecture 2.9, it is natural to ask that whether there exists a point whose arithmetic degree equals to the dynamical degree. (Note that there are no Zariski dense orbits if the Kodaira dimension of X is positive.) The following theorem answers this question affirmatively when the self-map is a morphsim.

Theorem 3.6 ([11]). *Let X be a smooth projective variety.*

(1) *For any surjective morphism $f\colon X \longrightarrow X$, there exists a point $P \in X$ such that $\alpha_f(P) = \delta_f$.*

(2) *If $f\colon X \longrightarrow X$ is an automorphism, then there exists a subset $S \subset X(\overline{\mathbb{Q}})$ satisfying the following properties.*

- *S is Zariski dense in X.*
- *For $P, Q \in S$ with $P \neq Q$, the f-orbit of P and Q is disjoint.*
- *$\alpha_f(P) = \delta_f$ for any $P \in S$.*

So far, we are working over $\overline{\mathbb{Q}}$. We can consider similar problems over, for example, the one dimensional function field $\overline{k(t)}$ of characteristic zero (we assume $k = \overline{k}$ for simplicity). Over the function field, K. Sano, T. Shibata and the author proved the following.

Theorem 3.7 ([12]). *Let $f\colon X \dashrightarrow X$ be a dominant rational self-map on a smooth projective variety X defined over $\overline{k(t)}$.*

(1) *We can prove the inequality $\overline{\alpha}_f(P) \leq \delta_f, P \in X_f$ geometrically.*

(2) *If the coefficient field k is uncountable, Theorem 3.6(2) holds for f.*

The key is the following lemma which allows us to work over a model over a curve.

Lemma 3.8. *Let $f\colon X \dashrightarrow X$ be as above. Let C be a smooth projective curve over k so that there exists a smooth projective variety $\mathcal{X}$ over k, a surjective morphism $\pi\colon \mathcal{X} \longrightarrow C$ and a dominant rational map $F\colon \mathcal{X} \dashrightarrow \mathcal{X}$ such that*

1. *$\pi \circ F = \pi$ and*
2. *the geometric generic fiber of π is X and F induces f.*

Then we have $\delta_f = \delta_F$.

Acknowledgement. The author would like to thank the referee for comments and suggestions.

References

[1] Diller, F., Favre, C., *Dynamics of bimeromorphic maps of surfaces* , Amer. J. Math. **123** (2001), no. 6, 1135–1169.

[2] Dinh, T.-C., Nguyên, V-A., *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. **86** (2011), 817–840.

[3] Dinh, T.-C., Sibony, N., *Une borne supérieure de l'entropie topologique d'une application rationnelle*, Ann. of Math. (2) **161** (2005), no. 3, 1637–1644.

[4] Dinh, T-C., Sibony, N., *Equidistribution problems in complex dynamics of higher dimension*, Internat. J. Math. **28**, 1750057 (2007).

[5] Kawaguchi, S., *Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint*, Amer. J. Math. **130** (2008), no. 1, 159–186.

[6] Kawaguchi, S., *Local and global canonical height functions for affine space regular automorphisms*, Algebra Number Theory **7** (2013), no. 5, 1225–1252.

[7] Kawaguchi, S., Silverman, J. H., *Examples of dynamical degree equals arithmetic degree*, Michigan Math. J. **63** (2014), no. 1, 41–63.

[8] Kawaguchi, S., Silverman, J. H., *Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties*, Trans. Amer. Math. Soc. **368** (2016), 5009–5035.

[9] Kawaguchi, S., Silverman, J. H., *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties*, J. Reine Angew. Math. **713** (2016), 21–48.

[10] Matsuzawa, Y., *On upper bounds of arithmetic degrees*, preprint.

[11] Matsuzawa, Y., Sano, K., Shibata, T., Arithmetic degrees and dynamical degrees of endomorphisms on surfaces, preprint.

[12] Matsuzawa, Y., Sano, K., Shibata, T., Arithmetic degree for dynamical systems over function fields of characteristic zero, preprint.

[13] Sano, K., *Dynamical degree and arithmetic degree of endomorphisms on product varieties*, preprint, 2016, https://arxiv.org/abs/1604.04174

[14] Silverman, J. H., *Dynamical degree, arithmetic entropy, and canonical heights for dominant rational self-maps of projective space*, Ergodic Theory Dynam. Systems **34** (2014), no. 2, 647–678.

[15] Silverman, J. H., *Arithmetic and dynamical degrees on abelian varieties*, preprint, 2015, http://arxiv.org/abs/1501.04205

[16] Truong, T.T., *(Relative) dynamical degrees of rational maps over an algebraic closed field*, arXiv:1501.01523.

[17] Truong, T.T., *Relative dynamical degrees of correspondences over a field of arbitrary characteristic*, arXiv:1605.05049.