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Pro-p and cohomological aspects of anabelian geometry of hyperbolic polycurves

By

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Abstract

In this article, we study the étale fundamental groups of hyperbolic polycurves, i.e., successive extensions of families of hyperbolic curves. Among others, we show that the isomorphism class of a hyperbolic polycurve of dimension ≤ 4 defined over a sub-p-adic field is completely determined by its geometrically pro-p fundamental group under a certain group-theoretic condition. Moreover, we show that the dimension of a hyperbolic polycurve over a field of characteristic zero can be reconstructed group-theoretically from its geometric fundamental group. This article is based on author's works [11],[12].

§ 1. Introduction

Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, $G_k := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k, and X a variety over k. (In this article, a variety over k is defined to be a scheme that is of finite type, separated, and geometrically connected over k.) Write Π_X for the étale fundamental group of X (for some choice of basepoint). Then the structure morphism $X \to \operatorname{Spec} k$ induces a natural surjection $\Pi_X \twoheadrightarrow G_k$ (note that $\Pi_{\operatorname{Spec} k}$ is isomorphic to G_k). Write $\Delta_{X/k}$ for the kernel of the surjection $\Pi_X \twoheadrightarrow G_k$. It is well-known that $\Delta_{X/k}$ is isomorphic to $\Pi_{X \times_k \overline{k}}$, which is often called the geometric fundamental group.

A. Grothendieck proposed the following philosophy (cf. [1],[2]):

Conjecture 1.1 (Grothendieck conjecture). For certain types of k, if X is "an anabelian variety" over k, then the isomorphism class of X is completely determined by

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the étale fundamental group Π_X as a profinite group equipped with a surjection onto G_k.

Although we do not have any general definition of the notion of "an anabelian variety", successive extensions of families of hyperbolic curves (hereinafter called "hyperbolic polycurves") have been regarded as typical examples of anabelian varieties:

Definition 1.2 (cf. [3] Definition 2.1)**.** Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *hyperbolic curve* (*of type* (g, r)) over S if there exist

- a pair of nonnegative integers (g, r);
- a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S;
- a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

such that

- $2g - 2 + r > 0$;
- any geometric fiber of $X^{\mathrm{cpt}} \to S$ is (a necessarily smooth proper curve) of genus g;
- the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
- X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

(ii) We shall say that X is a *hyperbolic polycurve* (*of relative dimension* n) over S if there exist a positive integer n and a (not necessarily unique) factorization of the structure morphism $X \to S$

$$X = X_n \to X_{n-1} \to \cdots \to X_2 \to X_1 \to S = X_0$$

such that, for each $i = 1, \ldots, n,\ X_i \to X_{i-1}$ is a hyperbolic curve[1].

The Grothendieck conjecture for hyperbolic polycurves of dimension ≤ 2 was proved in [7] (cf. [7] Theorems 16.5, a2.4), and thereafter, in [3], it is extended to the case of hyperbolic polycurves of dimension ≤ 4:

[1] In this article, when we call X/S a hyperbolic polycurve, we always fix this factorization.

Theorem 1.3 (cf. [3] Corollary 3.18). *Let p be a prime number, k a sub-p-adic field*[2]*, and X, Y hyperbolic polycurves over k. Suppose that either X or Y is of dimension ≤ 4. Then the natural map*

$$\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\Pi_Y, \Pi_X)/\,\mathrm{Inn}(\Delta_{X/k})$$

is bijective. In particular, the isomorphism class of a hyperbolic polycurve X of dimension ≤ 4 over k is completely determined by $\Pi_X \twoheadrightarrow G_k$.

One of the main ingredients of this theorem is the "Hom version" of the Grothendieck conjecture for hyperbolic curves (cf. [7] Theorem A). But in [7], a similar result for "pro-p fundamental groups" is also proved. Thus, it seems that the "pro-p version" of the Grothendieck conjecture for hyperbolic polycurves of dimension ≤ 4 is also true. However, this conjecture is not expected to hold without any assumptions, because we cannot apply an argument similar to [3]. Before explaining the reason for this, we define "pro-p fundamental groups", and formulate the pro-p Grothendieck conjecture.

Let G be a profinite group. Then we shall write G^p for the maximal pro-p quotient of G. Let $X \to Y$ be a morphism of varieties over k. Then we shall write $\Delta_{X/Y} := \ker(\Pi_X \to \Pi_Y)$, and $\Pi^p_{X/Y} := \Pi_X/\ker(\Delta_{X/Y} \twoheadrightarrow \Delta^p_{X/Y})$. Note that if X is a variety over k, then the natural sequence

$$1 \to \Delta^p_{X/k} \to \Pi^p_{X/k} \to G_k \to 1$$

is exact. $\Pi^p_{X/k}$ is sometimes called the geometrically pro-p fundamental group. Now we can state the pro-p Grothendieck conjecture:

Conjecture 1.4. For certain types of k, if X is "an anabelian variety" over k, then the isomorphism class of X is completely determined by the geometrically pro-p fundamental group $\Pi^p_{X/k}$ as a profinite group equipped with a surjection onto G_k.

In this article, we consider the pro-p Grothendieck conjecture for hyperbolic polycurves. But, as mentioned above, we cannot apply an argument similar to [3] without any assumptions. Indeed, let S be a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. Then for any triplet of integers (i, j, l) such that $0 \leq i < j < l \leq n$, we have a natural exact sequence

$$1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1$$

(cf. [11] Remark 3.8), which is one of the main ingredients of [3]. Nevertheless, since the operation of taking the maximal pro-p quotient of a profinite group is not exact, the sequence

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

[2] We shall say that k is sub-p-adic if k is isomorphic to a subfield of a finitely generated extension of $\mathbb{Q}_p$.

is not exact in general. For this reason, we introduce a condition that the above sequence is exact, and consider the Grothendieck conjecture for such hyperbolic polycurves:

Definition 1.5. Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S. We shall say that X/S satisfies condition $(*)_p$ if for any triplet of integers (i,j,l) such that $0 \leq i < j < l \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_l/X_j} \to \Delta^p_{X_l/X_i} \to \Delta^p_{X_j/X_i} \to 1$$

is exact.

Remark. For example, it is well-known that if X/S is a configuration space of a hyperbolic curve over S (cf. e.g., [8] Definition 2.1), then X/S satisfies condition $(*)_p$ (cf. [8] Proposition 2.2). Moreover, we can prove the following (cf. [11] Proposition 3.20):

For any X, S as in Definition 1.5, there exists a connected finite étale Galois covering $Y \to X$ such that $Y/\operatorname{Nor}(Y/S)$ (where $\operatorname{Nor}(Y/S)$ is the normalization of Y in the finite extension of the function field of Y obtained by forming the algebraic closure of the function field of Y in the function field of X) is a hyperbolic polycurve (with respect to the factorization induced by the factorization of X/S as in [3] Proposition 2.3) satisfying condition $(*)_p$.

The following is one of the main results:

Theorem 1.6 (cf. [11] Theorem 1.2)**.** *Let p be a prime number, k a sub-p-adic field, and X, Y hyperbolic polycurves over k satisfying condition $(*)_p$. Suppose that either X or Y is of dimension ≤ 4. Then the natural map*

$$\operatorname{Isom}_k(Y,X) \to \operatorname{Isom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})/\operatorname{Inn}(\Delta^p_{X/k})$$

is bijective. In particular, the isomorphism class of a hyperbolic polycurve X of dimension ≤ 4 over k satisfying condition $(*)_p$ is completely determined by $\Pi^p_{X/k} \twoheadrightarrow G_k$.

Remark. If X/k and Y/k are configuration spaces of hyperbolic curves X', Y' over k, respectively, then Theorem 1.6 is true without the assumption "either X or Y is of dimension ≤ 4" (cf. [5] Corollary 2.6. Note that in [5] Corollary 2.6, it was assumed that $\{(g_{X'}, r_{X'}); (g_{Y'}, r_{Y'})\} \cap \{(0,3); (1,1)\} \neq \emptyset$ (where $(g_{X'}, r_{X'})$, $(g_{Y'}, r_{Y'})$ are the types of X'/k, Y'/k, respectively), but it seems that we can remove this assumption by using [4] Theorem 2.5. See also [9] Corollary B and *Note added*, which dealt with the case $X = Y$).

Although the Grothendieck conjecture for hyperbolic polycurves of dimension ≤ 4 is already proved, the case of dimension ≥ 5 is still open. However, even if the dimension is greater than 4, the étale fundamental group of a hyperbolic polycurve may have various geometric information. In this article, we reconstruct the relative dimension of a hyperbolic polycurve X over S (resp. a hyperbolic polycurve X over S satisfying condition $(*)_p$) from the cohomology groups of $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$), which is the other of the main results:

Theorem 1.7 (cf. [12] Theorem A, Example 2.6.1). *Let S be a connected noetherian separated normal scheme over k, X a hyperbolic polycurve over S (resp. a hyperbolic polycurve over S satisfying condition $(*)_p$) and m a nonnegative integer. Write n for the relative dimension of X/S. Then the following conditions are equivalent:*

(1) $m = n$.

(2) *For any positive real number M, there exists an open subgroup $V \subset \Delta_{X/S}$ (resp. $V \subset \Delta^p_{X/S}$) such that, for any open subgroup $U \subset V$ of V, any nonzero finite (resp. finite p-primary) U-module A, and any nonnegative integer i such that $i \neq m$, it holds that* $\log(\sharp H^m(U, A)) > M \log(\sharp H^i(U, A))$.

In particular, the relative dimension of X over S can be reconstructed group-theoretically from $\Delta_{X/S}$ (resp. $\Delta^p_{X/S}$).

Remark. In the special case that X/k is a configuration space of a hyperbolic curve over k, it has already been verified in [4] Theorem 1.6 that the dimension can be reconstructed group-theoretically from $\Delta_{X/k}, \Delta^p_{X/k}$. (The reconstruction algorithm in [4] is different from our algorithm.)

§ 2. Pro-p Grothendieck Conjecture

In this section, we give a sketch of the proof of Theorem 1.6. The injectivity of the map discussed in Theorem 1.6 is not so hard to prove:

Proposition 2.1. *Let p be a prime number, X a hyperbolic polycurve over k satisfying condition $(*)_p$, and Y a geometrically integral variety over k. Write $\mathrm{Hom}^{\mathrm{dom}}_k(Y, X) \subset \mathrm{Hom}_k(Y, X)$ for the subset of dominant morphisms from Y to X over k and $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) \subset \mathrm{Hom}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k})$ for the subset of open homomorphisms from $\Pi^p_{Y/k}$ to $\Pi^p_{X/k}$ over G_k. Then there exists a natural map*

$$\mathrm{Hom}^{\mathrm{dom}}_k(Y, X) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi^p_{Y/k}, \Pi^p_{X/k}) / \mathrm{Inn}(\Delta^p_{X/k}),$$

and, moreover, this map is injective.

In this article, we consider the surjectivity of the map discussed in Theorem 1.6 in the case where the dimension of X is equal to 4 and the dimension of Y is greater than or equal to 4 (for simplicity). In [3], the surjectivity of the map discussed in Theorem 1.3 is proved by using the following two facts:

Theorem 2.2 (cf. [3] Corollary 3.12). *Let p be a prime number, k a sub-p-adic field, S a normal variety over k, X a hyperbolic polycurve of relative dimension 2 over S, Y a hyperbolic polycurve over k, and $\phi : \Pi_Y \to \Pi_X$ a homomorphism over G_k. Suppose that the following conditions are satisfied:*

(1) *The composite $\Pi_Y \xrightarrow{\phi} \Pi_X \to \Pi_S$ arises from a morphism $Y \to S$ over k.*

(2) *ϕ is an open injection.*

(3) $\dim(X)$ $(= \dim(S) + 2) \leq \dim(Y)$.

Then ϕ arises from a quasi-finite dominant morphism $Y \to X$ over k. In particular, $\dim(X) = \dim(Y)$.

Theorem 2.3 (cf. [3] Theorem 3.14). *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 2 over k, Y a normal variety over k, and $\phi : \Pi_Y \to \Pi_X$ an open homomorphism over G_k. Suppose that the kernel of ϕ is topologically finitely generated. Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k. In particular,* $\dim(Y) \geq 2$.

Thus, to prove Theorem 1.6, we want to prove a pro-p version of the above two theorems. However, even though we assume that the hyperbolic polycurves in question satisfy condition $(*)_p$, there remains some difficulty. The difference between the pro-p version and the original (profinite) version is the necessity of considering base schemes. In other words, $\Pi^p_{X/Y}$ depends on the base scheme Y, although Π_X does not depend on Y. It seems (to the author) that choosing a suitable base scheme to complete the proof is very difficult. In this article, to avoid this problem, first we assume a certain condition stronger than $(*)_p$ and use the maximal pro-p quotient Π^p_X of Π_X (which is independent of the base scheme):

Definition 2.4. Let p be a prime number, n a positive integer, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S. We shall say that X/S satisfies condition $(**)_p$ if for any pair of integers (i, j) such that $0 \leq i < j \leq n$, the sequence of profinite groups

$$1 \to \Delta^p_{X_j/X_i} \to \Pi^p_{X_j} \to \Pi^p_{X_i} \to 1$$

is exact.

This condition is suitable for using the group Π_X^p. It follows from an argument similar to [3], we can prove the following:

Theorem 2.5. *Let p be a prime number, k a sub-p-adic field, S a normal variety over k, X a hyperbolic polycurve of relative dimension 2 over S satisfying condition $(**)_p$, Y a hyperbolic polycurve over k satisfying condition $(**)_p$, and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *The composite $\Pi_Y^p \xrightarrow{\phi} \Pi_X^p \to \Pi_S^p$ arises from a morphism $Y \to S$ over k.*

(2) *ϕ is an open injection.*

(3) $\dim(X)$ $(= \dim(S) + 2) \leq \dim(Y)$.

Then ϕ arises from a quasi-finite dominant morphism $Y \to X$ over S. In particular, $\dim(X) = \dim(Y)$.

Theorem 2.6. *Let p be a prime number, k a sub-p-adic field, X a hyperbolic polycurve of dimension 2 over k satisfying condition $(**)_p$, Y a normal variety over k, and $\phi : \Pi_Y^p \to \Pi_X^p$ an open homomorphism over G_k^p. Suppose that $\Pi_{Y \times_k \bar{k}}^p \to \Pi_Y^p$ is injective, and, moreover, that the kernel of ϕ is topologically finitely generated. Then ϕ arises from a uniquely determined dominant morphism $Y \to X$ over k. In particular,* $\dim(Y) \geq 2$.

The following corollary follows from Theorems 2.5, 2.6, together with the fact that $\ker(\Pi_X^p \twoheadrightarrow \Pi_{X_2}^p) \cong \Delta_{X/X_2}^p$ is topologically finitely generated (cf. e.g., [11] Proposition 3.16(iii)):

Corollary 2.7. *Let p be a prime number, k a sub-p-adic field, X, Y hyperbolic polycurves over k satisfying condition $(**)_p$, and $\phi : \Pi_Y^p \to \Pi_X^p$ a homomorphism over G_k^p. Suppose that the following conditions are satisfied:*

(1) *ϕ is an isomorphism.*

(2) $\dim(X) = 4$.

(3) $\dim(Y) \geq 4$.

Then ϕ arises from a uniquely determined isomorphism $Y \to X$ over k. In particular, $\dim(Y) = 4$.

Now, we shall prove Theorem 1.6 by reducing it to Corollary 2.7. Note that condition $(**)_p$ is stronger than condition $(*)_p$. Moreover, Π_X^p is a quotient of $\Pi_{X/k}^p$. Thus, we need the following lemmas:

Lemma 2.8. *Let p be a prime number, S a connected noetherian separated normal scheme over k, and X a hyperbolic polycurve of relative dimension n over S satisfying condition $(*)_p$. Then there exists a connected finite étale Galois covering $T \to S$ of S such that $X \times_S T/T$ satisfies condition $(**)_p$ (we consider $X \times_S T/T$ as a hyperbolic polycurve with natural factorization induced by a factorization of X/S).*

Lemma 2.9. *Let p be a prime number, (m, n) a pair of integers such that $0 \leq m < n$, S a connected noetherian separated normal scheme over k, X a hyperbolic polycurve of relative dimension n over S satisfying condition $(**)_p$, and Y a connected noetherian scheme over X_m. Let us fix a homomorphism $\Pi^p_{Y/S} \to \Pi^p_{X_m/S}$ arising from $Y \to X_m$. Then there exists a natural bijection*

$$\mathrm{Hom}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X) \xrightarrow{1:1} \mathrm{Hom}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}).$$

If, moreover, $\Pi^p_{Y/S} \to \Pi^p_{X_m/S}$ is surjective, then the above bijection determines a bijection

$$\mathrm{Hom}^{\mathrm{open}}_{\Pi^p_{X_m}}(\Pi^p_Y, \Pi^p_X) \xrightarrow{1:1} \mathrm{Hom}^{\mathrm{open}}_{\Pi^p_{X_m/S}}(\Pi^p_{Y/S}, \Pi^p_{X/S}).$$

We are now in a position to give a sketch of the proof of (the surjectivity of the map discussed in) Theorem 1.6 (in the case where $\dim(X) = 4$ and $\dim(Y) \geq 4$).

Sketch of the proof of Theorem 1.6. Let $\phi : \Pi^p_{Y/k} \xrightarrow{\sim} \Pi^p_{X/k}$ be an isomorphism over G_k. Then it follows from Lemma 2.8 that there exists a finite Galois extension k' of k such that $X \times_k k'/k'$ and $Y \times_k k'/k'$ satisfy condition $(**)_p$. Note that $\Pi^p_{X\times_k k'/k'}, \Pi^p_{Y\times_k k'/k'}$ are the inverse image of the normal open subgroup $G_{k'} \subset G_k$ by the surjections $\Pi^p_{X/k} \twoheadrightarrow G_k, \Pi^p_{Y/k} \twoheadrightarrow G_k$, respectively. Thus, ϕ induces an isomorphism $\phi' : \Pi^p_{Y\times_k k'/k'} \xrightarrow{\sim} \Pi^p_{X\times_k k'/k'}$. Write $\psi : \Pi^p_{Y\times_k k'} \xrightarrow{\sim} \Pi^p_{X\times_k k'}$ for the isomorphism induced by ϕ'. Then it follows from Corollary 2.7 that ψ arises from a uniquely determined isomorphism $Y \times_k k' \to X \times_k k'$ over k'. In light of Lemma 2.9, ϕ' arises from the above isomorphism $Y \times_k k' \to X \times_k k'$. The uniqueness of $Y \times_k k' \to X \times_k k'$ implies that $Y \times_k k' \to X \times_k k'$ is compatible with the natural actions of $\mathrm{Gal}(k'/k)$. Thus, by descending the morphism, we obtain an isomorphism $Y \xrightarrow{\sim} X$ over k. We can prove that ϕ arises from the above isomorphism $Y \xrightarrow{\sim} X$. This completes the proof of the surjectivity of the map discussed in Theorem 1.6. □

§ 3. Reconstruction of the Dimension of Hyperbolic Polycurves

In this section, we give a sketch of the proof of Theorem 1.7. From now on, we only treat $\Delta_{X/S}$, but we can apply the same argument for $\Delta^p_{X/S}$. The main ingredients of Theorem 1.7 are the calculation of the cohomology groups of étale fundamental group of hyperbolic curves, and the Hochschild-Serre spectral sequence. First, we calculate the cohomology groups of étale fundamental group of hyperbolic curves:

Lemma 3.1. *Let S be a connected noetherian separated normal scheme over k and X a hyperbolic curve of type (g, r) over S. Then the following hold:*

(i) *For any prime number p,* $\mathrm{cd}_p(\Delta_{X/S}) = \begin{cases} 1 & (r > 0) \\ 2 & (r = 0). \end{cases}$

(ii) *For any finite $\Delta_{X/S}$-module A, it holds that*

$$\sharp H^0(\Delta_{X/S}, A) \leq \sharp A,\ \sharp H^2(\Delta_{X/S}, A) \leq \sharp A,$$
$$\sum_{i=0}^{2} (-1)^i \log(\sharp H^i(\Delta_{X/S}, A)) = (2 - 2g - r) \cdot \log(\sharp A).$$

Lemma 3.1 follows from the fact that $\Delta_{X/S}$ is isomorphic to the profinite completion of the group

$$G := \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r = 1 \rangle$$

(cf. e.g., [13] Proposition (1.1)(i), [3] Proposition 2.4(ii)), together with a finite free resolution of the $\mathbb{Z}[G]$-module $\mathbb{Z}$ with trivial G-action (cf. [6] §11). Let us observe that Lemma 3.1(ii) implies that $H^0(\Delta_{X/S}, A)$ and $H^2(\Delta_{X/S}, A)$ are not so large, but that $H^1(\Delta_{X/S}, A)$ is large if $2g - 2 + r$ is large.

Next, we review the Hochschild-Serre spectral sequence:

Theorem 3.2 (Hochschild-Serre spectral sequence (cf. e.g., [10] Theorem (2.4.1))). *Let G be a profinite group, $H \subset G$ a normal closed subgroup of G, and A a G-module. Then there exists a spectral sequence*

$$E_2^{ij} = H^i(G/H, H^j(H, A)) \Rightarrow H^{i+j}(G, A).$$

It is called the Hochschild-Serre spectral sequence.

Now let us consider the cohomology groups of étale fundamental groups of hyperbolic polycurves. Let S be a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. For $1 \leq j \leq n$, let us write (g_j, r_j) for the type of the hyperbolic curve X_j/X_{j-1}. Then, since there is a natural exact sequence $1 \to \Delta_{X_l/X_j} \to \Delta_{X_l/X_i} \to \Delta_{X_j/X_i} \to 1$, we can understand inductively the behavior of the cohomology groups of $\Delta_{X/S}$ to some extent by using Lemma 3.1 and Theorem 3.2. In particular, if $2g_j - 2 + r_j$ is sufficiently large for each $1 \leq j \leq n$, we can prove that $H^n(\Delta_{X/S}, A)$ is "large" by showing that $H^1(\Delta_{X_1/S}, H^{n-1}(\Delta_{X/X_1}, A))$ is "large":

Lemma 3.3. *Let M be a positive real number and A a nonzero finite $\Delta_{X/S}$-module. Suppose that for each $1 \leq j \leq n$, it holds that $2g_j - 2 + r_j \geq \mathrm{cd}(\Delta_{X_j/X_{j-1}}) \cdot 3^j M$. Then for each nonnegative integer i such that $i \neq n$, it holds that $\log(\sharp H^n(\Delta_{X/S}, A)) > M \log(\sharp H^i(\Delta_{X/S}, A))$.*

Now we can prove Theorem 1.7:

Proof of Theorem 1.7. First, let us observe that there exists at most one nonnegative integer m that satisfies condition (2). Thus, it suffices to prove the implication (1) $\Rightarrow$ (2). Next, let us recall that an open subgroup V of $\Delta_{X/S}$ corresponds to a connected finite étale covering $Y \to X$. It is known that if $Y \to X$ is a connected finite étale covering, then Y has a natural structure of a hyperbolic polycurve

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_2 \to Y_1 \to Y_0$$

induced by X/S, and, moreover, for each $0 \leq j \leq n-1$, we have $\Delta_{Y/Y_j} = \Delta_{X/X_j} \cap V$ (cf. [3] Proposition 2.3, [11] Lemma 3.9(ii)). Thus, in light of Lemma 3.1(ii), if we write (g'_j, r'_j) for the type of Y_j/Y_{j-1}, then the value $2g'_j - 2 + r'_j$ is determined by $(\Delta_{X/X_{j-1}} \cap V)/(\Delta_{X/X_j} \cap V)$, and, by taking V sufficiently small, we can take this value arbitrarily large. In particular, there exists an open subgroup $V \subset \Delta_{X/S}$ such that $2g'_j - 2 + r'_j > 2 \cdot 3^j M$. Then it follows from Lemma 3.3 that, for any nonzero finite V-module A and any nonnegative integer i such that $i \neq n$, it holds that $\log(\sharp H^n(V, A)) > M \log(\sharp H^i(V, A))$. Thus, for any open subgroup $U \subset V$ of V, any nonzero finite U-module A, and any nonnegative integer i such that $i \neq n$, it holds that

$$\log(\sharp H^n(U, A)) = \log(\sharp H^n(V, \mathrm{Ind}_V^U A)) > M \log(\sharp H^i(V, \mathrm{Ind}_V^U A)) = M \log(\sharp H^i(U, A)).$$

This completes the proof of the implication (1) $\Rightarrow$ (2), hence also of Theorem 1.7. $\square$

§ 4. Remarks on the Cohomology Groups of Étale Fundamental Groups of Hyperbolic Polycurves

In this section, we reconstruct some more invariants of hyperbolic polycurves. Moreover, we give an application. In this section, let S be a connected noetherian separated normal scheme over k and X a hyperbolic polycurve of relative dimension n over S. For $1 \leq j \leq n$, let us write (g_j, r_j) for the type of the hyperbolic curve X_j/X_{j-1}. Write $t := \sharp\{j \mid 1 \leq j \leq n,\ r_j = 0\}$.

Proposition 4.1 (cf. [12] Corollaries 2.9, 2.17). *The following hold:*

(i) *For any prime number p, $\mathrm{cd}_p(\Delta_{X/S}) = n + t$.*

(ii) *For any finite $\Delta_{X/S}$-module A, it holds that*

$$\sum_{i\geq 0}(-1)^i \log(\sharp H^i(\Delta_{X/S}, A)) = (\prod_{j=1}^{n}(2-2g_j-r_j))\cdot \log(\sharp A)$$

In particular, the values $n+t$ and $\prod_{j=1}^{n}(2-2g_j-r_j)$ (hence also t (cf. Theorem 1.7)) can be reconstructed group-theoretically from $\Delta_{X/S}$.

Proof. Note that it follows from Lemma 3.1(ii) and Theorem 3.2 that $H^i(\Delta_{X/S}, A)$ is finite for any finite $\Delta_{X/S}$-module A and any nonnegative integer i. Thus, assertion (i) follows from Lemma 3.1(i) and [10] Proposition (3.3.8). Assertion (ii) follows from Lemma 3.1(ii) and Theorem 3.2. $\square$

Since (we have already verified that) the values $n, t, \prod_{j=1}^{n}(2-2g_j-r_j)$ can be reconstructed group-theoretically from $\Delta_{X/S}$, the following holds:

Corollary 4.2. *Let k' be a field of characteristic zero, T a connected noetherian separated normal scheme over k' and Y a hyperbolic polycurve of relative dimension n' over T. For $1 \leq j \leq n'$, let us write (g'_j, r'_j) for the type of the hyperbolic curve Y_j/Y_{j-1}. Write $t' := \sharp\{j \mid 1 \leq j \leq n',\ r'_j = 0\}$. Suppose that $\Delta_{Y/T}$ is isomorphic to $\Delta_{X/S}$. Then it holds that $n = n',\ t = t',\ \prod_{j=1}^{n}(2-2g_j-r_j) = \prod_{j=1}^{n'}(2-2g'_j-r'_j)$.*

In fact, our reconstruction algorithm gives us more:

Theorem 4.3 (cf. [12] Corollary 2.20)**.** *Let k' be a field of characteristic zero, T a connected noetherian separated normal scheme over k' and Y a hyperbolic polycurve of relative dimension n' over T. For $1 \leq j \leq n'$, let us write (g'_j, r'_j) for the type of the hyperbolic curve Y_j/Y_{j-1}. Write $t' := \sharp\{j \mid 1 \leq j \leq n',\ r'_j = 0\}$. Suppose that there exists an injective homomorphism $\Delta_{X/S} \hookrightarrow \Delta_{Y/T}$ such that the image is normal in $\Delta_{Y/T}$. Then it holds that $n \leq n',\ t \leq t'$.*

Sketch of the proof. Let us fix a prime number p. Let us regard $\Delta_{X/S}$ as a normal closed subgroup of $\Delta_{Y/T}$ via $\Delta_{X/S} \hookrightarrow \Delta_{Y/T}$. By using [10] Theorem (3.3.9), we can prove that $\mathrm{vcd}_p(\Delta_{Y/T}/\Delta_{X/S}) = \mathrm{cd}_p(\Delta_{Y/T}) - \mathrm{cd}_p(\Delta_{X/S}) = (n'+t')-(n+t)$ (cf. Proposition 4.1(i)), i.e., there exists an open subgroup $U \subset \Delta_{Y/T}/\Delta_{X/S}$ of $\Delta_{Y/T}/\Delta_{X/S}$ such that $\mathrm{cd}_p U = (n'+t')-(n+t)$. On the other hand, since $H^n(\Delta_{X/S}, \mathbb{F}_p)$ and $H^{n'}(\Delta_{Y/T}, \mathbb{F}_p)$ are "large", it follows from Theorem 3.2 that $H^{n'-n}(U, \mathbb{F}_p)$ is "large". More precisely, we can show that there exists an open subgroup $V \subset U$ of U such that $H^{n'-n}(V, \mathbb{F}_p) \neq \{0\}$. This implies that $0 \leq n'-n \leq \mathrm{cd}_p U = (n'+t')-(n+t)$. Thus, we conclude that $n \leq n',\ t \leq t'$. $\square$

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