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Recent progress in higher-dimensional anabelian geometry

By

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Abstract

The purpose of this article is to report contents of the author's lecture delivered in the workshop "Algebraic Number Theory and Related Topics 2016" in Kyoto University. We survey some known facts about the Grothendieck conjecture for algebraic varieties of dimension greater than one, and then present the author's recent works on hyperbolic polycurves – algebraic varieties in the form of successive fiberations by hyperbolic curves. In particular, we focus on a certain hyperbolic polycurve obtained as a finite étale cover of $\mathcal{M}_{2,r}$, the moduli space of curves of genus two with ordered r marked points.

§ 1. Review of previous results in higher-dimensional anabelian geometry

§ 1.1. Grothendieck's anabelian conjecture

In the early 1980's, Grothendieck propounded the anabelian geometry. The fundamental idea of anabelian geometry is that there is a class of varieties called anabelian varieties such that, roughly speaking, the geometry and arithmetic of an anabelian variety is encoded in its étale fundamental group and that any point of a smooth variety has a fundamental system of Zariski open neighborhoods consisting of anabelian varieties (e.g. [G]).

In the case of dimension ≥ 2, the definition of anabelian varieties is unknown. But several conditions to be satisfied are formulated. One of the formulations is as follows:

Let k be a field, $\bar{k}$ a separable closure of k. We denote by G_k the absolute Galois group $\mathrm{Gal}\left(\bar{k}/k\right)$. Let X, Y be geometrically connected schemes (or, more generally,

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stacks) of finite type over $\mathrm{Spec}(k)$. We denote by $\pi_1(X)$ the étale fundamental group of X with respect to a suitable geometric point, and $\pi_1(Y)$ the étale fundamental group of Y with respect to a suitable geometric point. We write $\mathrm{Isom}_k(X, Y)$ for the set of all k-isomorphisms from X to Y, and $\mathrm{Isom}_{\mathrm{G}_k}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}\left(\pi_1(Y \otimes_k \bar{k})\right)$ for the set of all continuous isomorphisms over G_k from $\pi_1(X)$ to $\pi_1(Y)$, divided by the right action of $\pi_1(Y \otimes_k \bar{k})$ on $\pi_1(Y)$ by the conjugation. For k, X and Y, we consider the condition $\mathbf{IGC}_k(X, Y)$ that the natural map

$$\mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{\mathrm{G}_k}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}\left(\pi_1(Y \otimes_k \bar{k})\right),$$

is bijective. Then the following formulation will be natural:

If k-schemes (or, more generally, stacks) X, Y are anabelian, then $\mathbf{IGC}_k(X, Y)$ holds.

It has been pointed out that the following higher-dimensional schemes (or, more generally, stacks) should be anabelian (cf. [G]):

(1) Hyperbolic polycurves, (2) Moduli stacks of hyperbolic curves.

Here, a hyperbolic polycurve is a successive extension of families of a hyperbolic curve (cf. Definition 1.3 (2)). We are now led to the following conjecture:

Conjecture 1.1. *Let k be a finitely generated field over $\mathbb{Q}$.*
(1) If X, Y are hyperbolic polycurves, then $\mathbf{IGC}_k(X, Y)$ holds.
(2) If X, Y are the moduli stacks of hyperbolic curves, then $\mathbf{IGC}_k(X, Y)$ holds.

In this article, we call Conjecture 1.1 the Isom-version of Grothendieck anabelian conjecture for higher-dimensional varieties (or the Grothendieck conjecture for higher-dimensional varieties, for short). We survey known results on the Grothendieck conjecture for higher-dimensional varieties in §1 and introduce author's recent result on the Grothendieck conjecture for the moduli stacks of curves of genus 2 in §2.

Remark 1.2. (1) For k, X and Y, we write $\mathrm{Hom}_k^{dom}(X, Y)$ for the set of all dominant k-morphisms from X to Y, and $\mathrm{Hom}_{\mathrm{G}_k}^{open}(\pi_1(X), \pi_1(Y))$ for the set of all open homomorphisms over G_k from $\pi_1(X)$ to $\pi_1(Y)$. We can consider the condition $\mathbf{HGC}_k(X, Y)$ that the natural map

$$\mathrm{Hom}_k^{dom}(X, Y) \to \mathrm{Hom}_{\mathrm{G}_k}^{open}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}\left(\pi_1(Y \otimes_k \bar{k})\right).$$

is bijective. We call the conjecture that $\mathbf{HGC}_k(X, Y)$ holds for anabelian schemes (or, more generally, stacks) the Hom-version of the Grothendieck conjecture in this article.

This formulation seems natural. Indeed, Mochizuki proved ([M1] Theorem A) that $\mathbf{HGC}_k(X, Y)$ holds when k is a sub-p-adic field, X is a smooth variety over k and Y is a hyperbolic curve over k. However, even if both X and Y are anabelian, $\mathbf{HGC}_k(X,Y)$ does not always hold. For example, let $Y \subset \mathbb{P}^N$ be an anabelian surface (say, a proper hyperbolic polycurve of dimension 2) and X a smooth hyperplane section of Y, which automatically turns out to be an anabelian curve. Then, there are not any dominant k-morphisms form X to Y, but the natural closed immersion from X to Y induces a surjective homomorphism over G_k from $\pi_1(X)$ to $\pi_1(Y)$. Thus, it is hard to formulate a suitable Hom-version of the Grothendieck conjecture for higher-dimensional varieties.

(2) When k is of positive characteristic, it is hard to formulate even Isom-version, since there exist the Frobenius morphisms. Moreover, it is not easy to reduce the proof to the case of dimension one as well, because, for a family of curves, the fundamental group of a fiber is not always a subgroup of the fundamental group of the total space.

(3) Grothendieck pointed out that the Siegel modular varieties should be also anabelian. But Y. Ihara and H. Nakamura resolved the problem negatively by showing that their geometric fundamental groups were not center-trivial ([IN] §3 Example (S)).

§ 1.2. Hyperbolic polycurves

In this section, we introduce known results on the Grothendieck conjecture for hyperbolic polycurves.

Definition 1.3 (cf. [H] Definition 2.1, [SS] Definition 6.1, [MT] Definition 2.1.(i))**.** (1) Let S be a scheme and X a scheme over S. For a pair of nonnegative integers (g,r) with $2g-2+r>0$, we call $X \to S$ a *hyperbolic curve of type* (g,r) if the following three conditions hold:

(i) there exists a proper smooth geometrically connected scheme Y over S of relative dimension 1 such that all the geometric fibers are curves of genus g,

(ii) there exists a closed subscheme D of Y which is finite and étale over S of degree r,

(iii) X is isomorphic to $Y \setminus D$ over S.

(2) Let S be a scheme and X a scheme over S. If there exist a positive integer n and a [not necessary unique] factorization $X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = S$ of the structure morphism $X \to S$ such that X_i/X_{i-1} is a hyperbolic curve for each $i \in \{1, \cdots, n\}$, then X is called a *hyperbolic polycurve* over S and $X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = S$ is called *a sequence of parameterising morphisms.*

(3) Let X be a hyperbolic polycurve over a field k. If there is a sequence of parameterising morphisms $X = X_n \to X_{n-1} \to \cdots \to X_0 = \mathrm{Spec}(k)$ such that X_i/X_{i-1} is an elementary fibration and X_i admits an embedding into a product of hyperbolic curves ($i = 1, \cdots, n$), then X is referred to as a *strongly hyperbolic Artin neighborhood.*

(4) Let S be a scheme and X a scheme over S. For a positive integer m, we write

P_m for $X \times_S \cdots \times_S X$ (the product of m copies of $X \to S$). Let n be a positive integer ≥ 2. For positive integers $i, j \leq n$ such that $i < j$, write $p_{i,j} : P_n \to P_2$ for the projection of the product P_n to the i-th and j-th factors. Write Δ for the image of the diagonal embedding $X \to X \times_S X$, and write $X_n \to S$ for the open subscheme of P_n obtained by forming the complement of the union $\bigcup_{1 \leq i < j \leq n} p_{i,j}^{-1}(\Delta)$. Then we shall refer to the S-scheme $X_n \to S$ as the n-th *(pure) configuration space* associated to $X \to S$.

Theorem 1.4. ([H] Theorem B) *Let k be a sub-p-adic field (i.e., a field isomorphic to a subfield of a finitely generated field extension of $\mathbb{Q}_p$) for some prime number p and $\bar{k}$ an algebraic closure of k. Let X and Y be hyperbolic polycurves over k. If* $\dim(X) \leq 4$ *or* $\dim(Y) \leq 4$*, then* $\mathbf{IGC}_k(X, Y)$ *holds.*

Remark 1.5. The case where $\dim(X) \leq 2$ and $\dim(Y) \leq 2$ was proved by S. Mochizuki ([M1] Theorem D).

Remark 1.6. For $U = X, Y$, we write $\pi_1(U \otimes_k \bar{k})_p$ for the maximal pro-p quotient of $\pi_1(U \otimes_k \bar{k})$, and $\pi_1(U)_{(p)}$ for $\pi_1(U)/\mathrm{Ker}(\pi_1(U) \to \pi_1(U \otimes_k \bar{k})_p)$. We can consider the condition $\mathbf{IGC}_k^p(X, Y)$ that the natural map

$$\mathrm{Isom}_k(X, Y) \xrightarrow{\sim} \mathrm{Isom}_{G_k}\left(\pi_1(X)_{(p)}, \pi_1(Y)_{(p)}\right) / \mathrm{Inn}\left(\pi_1(Y \otimes_k \bar{k})_p\right).$$

is bijective. We call the conjecture that $\mathbf{IGC}_k^p(X, Y)$ holds for anabelian schemes the pro-p Isom-version of the Grothendieck conjecture. K. Sawada proved the pro-p Isom-version of the Grothendieck conjecture under the same assumption as Theorem 1.4 and some mild conditions ([S]). See also his article in this volume. Then, very recently he proved that there are only finitely many isomorphism classes of hyperbolic polycurves, whose étale fundamental group is isomorphic to a prescribed profinite group.

Theorem 1.7. ([SS] Corollary 1.6) *Let k be a finitely generated field over $\mathbb{Q}$ and $\bar{k}$ an algebraic closure of k. If X and Y are strongly hyperbolic Artin neighborhoods over k, then* $\mathbf{IGC}_k(X, Y)$ *holds.*

Remark 1.8. This result implies that any point of any smooth variety over a finitely generated field over $\mathbb{Q}$ has a fundamental system of Zariski open neighborhoods consisting of anabelian varieties ([SS] Corollary 1.7).

The Grothendieck conjecture for the configuration spaces of hyperbolic curves over a sub-p-adic field holds.

Theorem 1.9. (cf. [HMM] Theorem A, [HM] Corollary 2.6, [M1] Theorem A) *Let k be a sub-p-adic field for some prime number p and $\bar{k}$ an algebraic closure of k. If X and Y are the configuration spaces of hyperbolic curves over k, then* $\mathbf{IGC}_k(X, Y)$ *holds.*

Remark 1.10. On anabelian properties of the configuration spaces of hyperbolic curves, there are many partial or related studies by so many Japanese researchers. Among other things, we should give a special mention to the following studies. The first one is the study on the Isom-version of the Grothendieck conjecture for the case that X and Y are equal and the configuration spaces of the hyperbolic curve of type (0, 3), and k is a number field (cf. [N], [Tam]). The second one is the study on the pro-p Isom-version of the Grothendieck conjecture for the case where X and Y are equal and the configuration spaces of (general) hyperbolic curves, and k is a number field (cf. [NT], [M1]). The third one is the study on the Isom-version of the Grothendieck conjecture for configuration spaces of hyperbolic curves of type $(g,\ r)$ $((g,r) \neq (0,3),(1,1))$ (cf. [MT], [HM]).

§ 1.3. The moduli stacks of hyperbolic curves of genus ≤ 1

In this section, we introduce known results on the Grothendieck conjecture for the moduli stacks of hyperbolic curves of genus ≤ 1.

Let $\mathcal{M}_{g,r}$ be the moduli stack of ordered r-pointed proper smooth curves of genus g over a field k. The Grothendieck conjecture for $\mathcal{M}_{0,r}$ $(r \geq 4)$ is an immediate consequence of Theorem 1.9, because $\mathcal{M}_{0,r}$ is the configuration space of the hyperbolic curve of type (0, 3).

Theorem 1.11. *Let k be a sub-p-adic field for some prime number p and $\bar{k}$ an algebraic closure of k. Then $\mathbf{IGC}_k(\mathcal{M}_{0,r^\alpha}, \mathcal{M}_{0,r^\beta})$ holds for $r^\alpha \geq 4$ and $r^\beta \geq 4$.*

Remark 1.12. When k is a number field, H. Nakamura proved a pro-p version of this theorem ([N] Theorem A). When k is a finitely generated field over $\mathbb{Q}$, this theorem is a direct consequence of Theorem 1.7.

The Grothendieck conjecture for the moduli stacks of hyperbolic curves of genus one has not been proven yet. However the Grothendieck conjecture for some finite étale coverings of $\mathcal{M}_{1,r}$ $(r \geq 1)$ was proved by Y. Hoshi, R. Kinoshita and C. Nakayama.

Theorem 1.13. ([HKN] Theorem 2.2) *Let k be a sub-p-adic field for some prime number p and $\bar{k}$ an algebraic closure of k. For $\xi = \alpha, \beta$, let n^ξ be an integer ≥ 1, X^ξ a finite étale covering of $\mathcal{M}_{1,1}$ that is a hyperbolic curve over k, $X_i^\xi = X^\xi \times_{\mathcal{M}_{1,1}} \mathcal{M}_{1,i}$ $(i = 1, 2, \cdots, n^\xi)$. Write $(g^\xi,\ r^\xi)$ for the type of the hyperbolic curve X^ξ. Assume that the following two conditions (1) and (2) hold:*

(1) $g^\alpha \geq 1$ or $r^\alpha \geq n^\beta$. (2) $g^\beta \geq 1$ or $r^\beta \geq n^\alpha$.

Then $\mathbf{IGC}_k(X^\alpha_{n^\alpha}, X^\beta_{n^\beta})$ holds.

The key point is to prove the lemma that the Grothendieck conjecture for the configuration spaces of a hyperbolic curve over a hyperbolic curve is true. The key ingredient for the proof of the lemma is the elasticity (see Definition 1.14) of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic 0.

Definition 1.14 ([M2] Definition 1.1 (ii)). Let G be a topological group. The topological group G is said to be *elastic* if the following condition holds: every topologically finitely generated closed normal subgroup of an open subgroup of G is either trivial or of finite index in G.

Fact 1.15 ([M2] Proposition 2.3 (i)). *Let C be a hyperbolic curve over an algebraically closed field of characteristic* 0*. Then $\pi_1(C)$ is elastic.*

This fact plays an essential role also in the proof the key lemma (Lemma 2.8) for the main result (Theorem 2.5).

§ 2. Grothendieck's anabelian conjecture for the moduli spaces of curves of genus 2

§ 2.1. Preliminaries

We study the Grothendieck conjecture for finite étale coverings of $\mathcal{M}_{2,r}$. Behind this study, there is a prediction that anabelian properties of a scheme is strongly related to anabelian properties of its finite étale coverings.

We shall begin with preliminaries.

Definition 2.1. Let X be an algebraic stack over a field k, and Y a geometrically connected finite étale covering of X.

(1) If there is a dense open substack U of Y such that U is a geometrically connected scheme over k, then Y is called an *orbi-covering* of X over k.

(2) If Y is a hyperbolic polycurve over k, then Y is called an *HPC-covering* of X.

When $r = 0, 1$, the Grothendieck conjecture holds for any orbi-coverings of $\mathcal{M}_{2,r}$.

Proposition 2.2. *Let k be a sub-p-adic field for some prime number p and $\bar{k}$ an algebraic closure of k. For $\xi = \alpha, \beta$, let r^ξ be a nonnegative integer such that $2g^\xi - 2 + r^\xi > 0$, and X^ξ an orbi-covering of $\mathcal{M}_{g^\xi, r^\xi}$. If $3g^\alpha - 3 + r^\alpha \leq 4$ or $3g^\beta - 3 + r^\beta \leq 4$, then X^α, X^β satisfy $\mathbf{IGC}_k(X^\alpha, X^\beta)$. Especially, if $g^\alpha = g^\beta = 2$ and either $0 \leq r^\alpha \leq 1$ or $0 \leq r^\beta \leq 1$, then $\mathbf{IGC}_k(X^\alpha, X^\beta)$ holds.*

Remark 2.3. There exists an orbi-covering of $\mathcal{M}_{2,r}$. For example, we have a finite étale (non-canonical) covering $cl : \mathcal{M}_{0,6} \to \mathcal{M}_{2,0}$ corresponding to $(2,0)$-curve over

$$\mathcal{M}_{0,6} \simeq \mathrm{Spec}\Big(k\big[x, y, z, x^{-1}, y^{-1}, z^{-1}, (x-1)^{-1}, (y-1)^{-1}, (z-1)^{-1}, (x-y)^{-1}, (y-z)^{-1}, (z-x)^{-1}\big]\Big)$$

defined by "$s^2 = t(t-1)(t-x)(t-y)(t-z)$". Since $\mathcal{M}_{0,6}$ is a scheme, $\mathcal{M}'_{2,r} := \mathcal{M}_{0,6} \times_{\mathcal{M}_{2,0}} \mathcal{M}_{2,r}$ is a scheme for $r \geq 0$, that is to say, $\mathcal{M}'_{2,r}$ is an orbi-covering of $\mathcal{M}_{2,r}$.

Outline of the proof of Proposition 2.2. It follows from our assumption that $3g^\xi - 3 + r^\xi \leq 4$, hence $g^\xi \leq 2$. In particular, since X^ξ is an orbi-covering, we conclude that X^ξ is a hyperbolic orbi-polycurve (cf. [H] Definition 3.17) over k (of dimension $3g^\xi - 3 + r^\xi \leq 4$). Thus, this is a direct consequence of [H] Corollary 3.18. □

§ 2.2. Main results

Throughout this subsection, we write k for a sub-p-adic field for some prime number p and $\bar{k}$ for an algebraic closure of k.

We introduce my recent results on the Grothendieck conjecture for finite étale coverings of $\mathcal{M}_{2,r}$ $(r \geq 2)$.

We shall begin with the Grothendieck conjecture for the configuration spaces of a hyperbolic curve over a hyperbolic polycurve under some conditions, which is one of the key ingredient for the proof of Theorem 2.5 (i.e., the Grothendieck conjecture for some finite étale coverings of $\mathcal{M}_{2,r}$).

Theorem 2.4. *For $\xi = \alpha, \beta$, let n^ξ be an integer > 3, X^ξ a hyperbolic polycurve over k of dimension n^ξ, $X^\xi = X^\xi_{n^\xi} \to X^\xi_{n^\xi-1} \to \cdots \to X^\xi_1 \to X^\xi_0 = \mathrm{Spec}(k)$ a sequence of parameterising morphisms, and (g^ξ_i, r^ξ_i) the type of the hyperbolic curve $X^\xi_i \to X^\xi_{i-1}$ $(i = 1, \cdots, n^\xi)$. Assume that $X^\xi_{n^\xi} \to X^\xi_3$ is the $(n^\xi - 3)$-th relative configuration space associated to $X^\xi_4 \to X^\xi_3$ and the following two conditions (1) and (2) holds:*

(1) For any $i = 1, 2, 3$, $g^\alpha_i > g^\beta_4$ or $r^\alpha_i \geq r^\beta_4 + n^\beta - 3$.

(2) For any $i = 1, 2, 3$, $g^\beta_i > g^\alpha_4$ or $r^\beta_i \geq r^\alpha_4 + n^\alpha - 3$.

Then $\mathbf{IGC}_k(X^\alpha, X^\beta)$ *holds.*

Even in the case where $r \geq 2$, the Grothendieck conjecture holds for the pull-back $X \times_{\mathcal{M}_{2,0}} \mathcal{M}_{2,r}$ of an HPC-covering X of $\mathcal{M}_{2,0}$ under some conditions.

Theorem 2.5. *For $\xi = \alpha, \beta$, let n^ξ be an integer ≥ 3, X^ξ an HPC-covering of $\mathcal{M}_{2,0}$, $X^\xi = X^\xi_3 \to X^\xi_2 \to X^\xi_1 \to X^\xi_0 = \mathrm{Spec}(k)$ a sequence of parameterising morphisms of $X^\xi \to \mathrm{Spec}(k)$, (g^ξ_i, r^ξ_i) the type of the hyperbolic curve $X^\xi_i \to X^\xi_{i-1}$ for $i = 1, 2, 3$, and $X^\xi_{n^\xi} := X^\xi \times_{\mathcal{M}_{2,0}} \mathcal{M}_{2,n^\xi-3}$.*

Suppose that the following two conditions (1) and (2) holds:

(1) For any $i = 1, 2, 3,\ g_i^{\alpha} \geq 1$ *or* $r_i^{\alpha} \geq n^{\beta} - 3$.

(2) For any $i = 1, 2, 3,\ g_i^{\beta} \geq 1$ *or* $r_i^{\beta} \geq n^{\alpha} - 3$.

Then $\mathbf{IGC}_k(X^{\alpha}_{n^{\alpha}}, X^{\beta}_{n^{\beta}})$ *holds.*

Remark 2.6. There is a finite étale covering of $\mathcal{M}_{2,r}$ which satisfies the assumption of the theorem. We shall begin with introducing notations. Let X be a geometrically connected finite étale covering of $\mathcal{M}_{0,6}$ over k. Then X is an HPC-covering of $\mathcal{M}_{2,0}$ via $cl : \mathcal{M}_{0,6} \to \mathcal{M}_{2,0}$ (cf. Remark 2.3 and [H] Proposition 2.3).

$$X_i := \begin{cases} X \times_{\mathcal{M}_{2,0}} \mathcal{M}_{2,i-3} & (i = 3, 4, \cdots), \\ \text{the normalization of } \mathcal{M}_{0,3+i} \text{ in } X & (i = 0, 1, 2). \end{cases}$$

Now, for $\xi = \alpha, \beta$, let n^{ξ} be an integer ≥ 3, X^{ξ} a geometrically connected finite étale covering of $\mathcal{M}_{0,6}$, and (g_i^{ξ}, r_i^{ξ}) the type of the hyperbolic curve $X_i^{\xi} \to X_{i-1}^{\xi}$ for $i = 1, 2, 3$. Then it is not difficult to see that there exist X^{α} and X^{β} such that $n^{\xi}, (g_i^{\xi}, r_i^{\xi})$ ($i = 1, 2, 3$ and $\xi = \alpha, \beta$) satisfy the assumption of the theorem. In particular, if $n^{\alpha} \leq 6$ and $n^{\beta} \leq 6$, then $\mathbf{IGC}_k(X^{\alpha}_{n^{\alpha}}, X^{\beta}_{n^{\beta}})$ holds for any geometrically connected finite étale coverings X^{α}, X^{β} of $\mathcal{M}_{0,6}$.

Remark 2.7. If [H] Theorem A (2) can be extended from the case of two dimensional hyperbolic polycurves to the case of higher dimensional hyperbolic polycurves, we can extend our results to higher dimensional cases: Suppose that, if Y is a hyperbolic polycurve of any dimension $\leq n$ over k, Z a normal variety over k and $\varphi : \pi_1(Z) \to \pi_1(Y)$ an open homomorphism over G_k such that $\mathrm{Ker}(\varphi)$ is topologically finitely generated, then φ arises from a uniquely determined dominant morphism $Z \to Y$ over k. Then $\mathbf{IGC}_k(X^{\alpha}, X^{\beta})$ holds if X^{α} and X^{β} are configuration spaces over hyperbolic polycurves of dimension $n + 1$ under some conditions. Consequently, $\mathbf{IGC}_k(X^{\alpha}, X^{\beta})$ holds if X^{α} and X^{β} are finite étale coverings of the moduli stacks of hyperelliptic curves of genus g over k such that $g \leq \dfrac{n}{2} + 1$ under some conditions.

§ 2.3. Outline of the proof

The key ingredient for the proof of Theorem 2.4 and Theorem 2.5 is the elasticity of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic 0 (cf. Fact 1.15) and the Grothendieck conjecture for hyperbolic polycurves of lower dimension ([H] Theorem A and Theorem B).

We give only the outline of the proof of Theorem 2.5. (The outline of the proof of Theorem 2.4 is similar to that of Theorem 2.5.) We shall begin with the key lemma:

Lemma 2.8. *Let* S, T *be connected separated normal schemes of finite type over* k *such that* $\dim(T) \leq 2$, $q : S \to T$ *a dominant* k*-morphism, and* $p_X : X \to S$

(respectively, $p_Y : Y \to T$) be a hyperbolic curve of type $(2,n)$ (respectively, (g,r)). Let ϕ be a surjective continuous homomorphism over G_k which fits into the following diagram

$$\begin{array}{ccccc} & \pi_1(X) & \overset{\phi}{\to} & \pi_1(Y) & \\ \pi_1(p_X) & \downarrow & \circlearrowleft & \downarrow & \pi_1(p_Y) \\ & \pi_1(S) & \overset{\pi_1(q)}{\to} & \pi_1(T). & \end{array}$$

Assume that at least one of the following conditions holds:

(1) $n < r$.

(2) $2 < g$.

(3) the classifying morphism corresponding to $X \to S$ induces a surjection $S \to \mathcal{M}_{2,0}$, and $1 \le g \le 2$.

Then $\phi\left(\mathrm{Ker}(\pi_1(p_X))\right) = \{1\}$, i.e., ϕ factors through a homomorphism $\phi' : \pi_1(S) \to \pi_1(Y)$ so that $\phi = \phi' \circ \pi_1(p_X)$.

Outline of proof. Write Δ_X for $\mathrm{Ker}(\pi_1(p_X))$ and Δ_Y for $\mathrm{Ker}(\pi_1(p_Y))$. Suppose that $\phi(\Delta_X)$ is open in Δ_Y. Then, for any closed point $s \to S$, there exists a dominant morphism $f_{\bar{s}} : X \times_S \bar{s} \to Y \times_T q(\bar{s})$ such that $\pi_1(f_{\bar{s}}) = \phi|_{\Delta_X}$ thanks to the Hom-version of the Grothendieck conjecture for curves ([M1] Theorem A). Therefore, $n \ge r$ and $2 \ge g$. Hence, each of (1) and (2) implies a contradiction immediately. (3) and $g = 2$ imply that there is a surjective morphism $T(\bar{k}) \twoheadrightarrow \mathcal{M}_{2,0}(\bar{k})$, which means $2 \ge \dim(T) \ge 3$. This is a contradiction. (3) and $g = 1$ imply that all the Jacobians of curves of genus 2 are not simple, which is a contradiction. Thus, if (1), (2) or (3) holds, then $\phi(\Delta_X)$ is not open in Δ_Y. The image $\phi(\Delta_X)$ is a topologically finitely generated closed normal subgroup of Δ_Y and Δ_Y is elastic because it is the étale fundamental group of a hyperbolic curve over an algebraically closed field (Fact 1.15). Therefore we conclude that $\phi(\Delta_X) = \{1\}$. This completes the proof of Lemma 2.8 $\square$

Outline of the proof of Theorem 2.5. The injectivity of the map $\mathrm{Isom}_k(X^{\alpha}_{n^{\alpha}}, X^{\beta}_{n^{\beta}}) \to \mathrm{Isom}_{\mathrm{G}_k}\left(\pi_1(X^{\alpha}_{n^{\alpha}}), \pi_1(X^{\beta}_{n^{\beta}})\right)/\mathrm{Inn}\left(\pi_1(X^{\beta}_{n^{\beta}} \otimes_k \bar{k})\right)$ is proved by [H] Proposition 3.2 (ii). We shall show the surjectivity. Let $\psi \in \mathrm{Isom}_{G_k}\left(\pi_1(X^{\alpha}_{n^{\alpha}}), \pi_1(X^{\beta}_{n^{\beta}})\right)$. Then ψ induces a surjetive morphism $\psi_i : \pi_1(X^{\alpha}_{n^{\alpha}}) \twoheadrightarrow \pi_1(X^{\beta}_i)$ $(i = 1, 2, 3)$.

Claim 2.9. *The morphism ψ induces a continuous isomorphism over G_k*

$$\rho : \pi_1(X^{\alpha}_3) \xrightarrow{\sim} \pi_1(X^{\beta}_3).$$

Outline of proof. First, applying Lemma 2.8 to $X/S = X^{\alpha}_{n^{\alpha}}/X^{\alpha}_{n^{\alpha}-1}$, $Y/T = X^{\beta}_1/k$, $q = q_0 : X^{\alpha}_{n^{\alpha}-1} \to \mathrm{Spec}(k)$, and $\phi = \psi_1 : \pi_1(X^{\alpha}_{n^{\alpha}}) \twoheadrightarrow \pi_1(X^{\beta}_1)$, we get a surjective morphism $\rho_1 : \pi_1(X^{\alpha}_{n^{\alpha}-1}) \twoheadrightarrow \pi_1(X^{\beta}_1)$. Thanks to the Hom-version of the

Grothendieck conjecture for curves ([M1] Theorem A), there exists a dominant morphism $q_1 : X^{\alpha}_{n^{\alpha}-1} \to X^{\beta}_1$ which induces ρ_1.

Second, applying Lemma 2.8 to $X/S = X^{\alpha}_{n^{\alpha}}/X^{\alpha}_{n^{\alpha}-1}$, $Y/T = X^{\beta}_2/X^{\beta}_1$, $q = q_1 : X^{\alpha}_{n^{\alpha}-1} \to X^{\beta}_1$, and a surjective homomorphism $\phi = \psi_2 : \pi_1(X^{\alpha}_{n^{\alpha}}) \twoheadrightarrow \pi_1(X^{\beta}_2)$, we get a surjective morphism $\rho_2 : \pi_1(X^{\alpha}_{n^{\alpha}-1}) \twoheadrightarrow \pi_1(X^{\beta}_2)$. Because ψ is an isomorphism and the kernel of $\pi_1(X^{\beta}_{n^{\beta}}) \to \pi_1(X^{\beta}_2)$ is topologically finitely generated ([H] Proposition 2.4 (iii)), $\mathrm{Ker}(\psi_2)$ is topologically finitely generated. Consequently, $\mathrm{Ker}(\rho_2)$ is topologically finitely generated. Therefore, thanks to [H] Theorem A (2), there exists a dominant morphism $q_2 : X^{\alpha}_{n^{\alpha}-1} \to X^{\beta}_2$ which induces ρ_2,

Third, applying Lemma 2.8 to $X/S = X^{\alpha}_{n^{\alpha}}/X^{\alpha}_{n^{\alpha}-1}$, $Y/T = X^{\beta}_3/X^{\beta}_2$, $q = q_2 : X^{\alpha}_{n^{\alpha}-1} \to X^{\beta}_2$, and a surjective homomorphism $\phi = \psi_3 : \pi_1(X^{\alpha}_{n^{\alpha}}) \twoheadrightarrow \pi_1(X^{\beta}_3)$, we get a surjective morphism $\rho_3 : \pi_1(X^{\alpha}_{n^{\alpha}-1}) \twoheadrightarrow \pi_1(X^{\beta}_3)$.

Repeating the same argument for inductive replacements of n_{α} by $n_{\alpha} - i$ ($i = 1, 2, \cdots$), we obtain that $\pi_1(X^{\alpha}_3) \twoheadrightarrow \pi_1(X^{\beta}_3)$.

Applying the same argument to $\psi^{-1} : \pi_1(X^{\beta}_{n^{\beta}}) \to \pi_1(X^{\alpha}_{n^{\alpha}})$, we conclude that $\pi_1(X^{\beta}_3) \twoheadrightarrow \pi_1(X^{\alpha}_3)$. Therefore we get the conclusion of Claim 2.9. □

Since $\dim(X^{\beta}_3) \leq 4$, $\mathbf{IGC}_k(X^{\alpha}_3, X^{\beta}_3)$ holds thanks to the Grothendieck conjecture for hyperbolic polycurves ([H] Theorem B), which means that ρ is induced by a k-isomorphism

$$f_3 : X^{\alpha}_3 \xrightarrow{\sim} X^{\beta}_3.$$

We write η_U for the generic point of an irreducible scheme U. The morphism f_3 induces that $\eta_{X^{\alpha}_3} \xrightarrow{\sim} \eta_{X^{\beta}_3}$, hence $\pi_1(\eta_{X^{\alpha}_3}) \xrightarrow{\sim} \pi_1(\eta_{X^{\beta}_3})$. Therefore, by [H] Proposition 2.4 (ii), we conclude that we have a continuous isomorphism

$$\psi_{\eta} : \pi_1(X^{\alpha}_{n^{\alpha}} \times_{X^{\alpha}_3} \eta_{X^{\alpha}_3}) \xrightarrow{\sim} \pi_1(X^{\beta}_{n^{\beta}} \times_{X^{\beta}_3} \eta_{X^{\beta}_3}),$$

that is compatible with $\pi_1(\eta_{X^{\alpha}_3}) \xrightarrow{\sim} \pi_1(\eta_{X^{\beta}_3})$.

For $\xi = \alpha, \beta$, $X^{\xi}_{n^{\xi}} \times_{X^{\xi}_3} \eta_{X^{\xi}_3}/\eta_{X^{\xi}_3}$ is a configuration space of a hyperbolic curve over a sub-p-adic field. Hence, owing to [MT] Corollary 6.3, see also [HMM] Theorem A, it follows that $n^{\alpha} = n^{\beta}$. By virtue of [HM] Corollary 2.6 (i.e., the Grothendieck conjecture for hyperbolic configuration spaces), ψ_{η} is induced by an isomorphism

$$f_{\eta} : X^{\alpha}_{n^{\alpha}} \times_{X^{\alpha}_3} \eta_{X^{\alpha}_3} \xrightarrow{\sim} X^{\beta}_{n^{\beta}} \times_{X^{\beta}_3} \eta_{X^{\beta}_3},$$

that is compatible with $\eta_{X^{\alpha}_3} \xrightarrow{\sim} \eta_{X^{\beta}_3}$. Hence, using [H] Lemma 2.10, we conclude that ϕ is induced by a k-isomorphism

$$f : X^{\alpha}_{n^{\alpha}} \xrightarrow{\sim} X^{\beta}_{n^{\beta}}.$$

This completes the proof of Theorem 2.5. □

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