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The Fourier coefficients of the McKay-Thompson series and the traces of CM values, an announcement

By

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Abstract

This is an announcement of our results [8] on the arithmetic formulas for the Fourier coefficients of the McKay-Thompson series of square-free level N.

The elliptic modular function $j(\tau)$ enjoys many beautiful properties. Its Fourier coefficients are related to the Monster group, and its CM values generate abelian extensions over imaginary quadratic fields. Kaneko established a relation between the Fourier coefficients and CM values. In this article, we are concerned with analogues of Kaneko's result for the McKay-Thompson series.

§ 1. Introduction

The elliptic modular function $j(\tau)$ plays important roles in many different fields. In particular, the Fourier coefficients of $j(\tau)$ have a mysterious connection with the degrees of irreducible representations of the Monster group $\mathbb{M}$. This is known as the monstrous moonshine formulated by Conway and Norton [3], and proved by Borcherds [1]. This connection says that the Fourier coefficients of $j(\tau)$ can be expressed as simple linear combinations of the degrees of irreducible representations of the Monster group. On the other hand, Kaneko [5] found the arithmetic formulas for the Fourier coefficients of $j(\tau)$, that is, the coefficients could also be expressed as a finite sum of CM values of $j(\tau)$.

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Here the CM value is a special value of $j(\tau)$ at an imaginary quadratic point, which is algebraic and generates a certain abelian extension over the imaginary quadratic field.

More generally, Borcherds showed that there exists a hauptmodul $T_g(\tau)$ on a certain genus 0 subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for each element $g \in \mathbb{M}$, and these hauptmoduln are constructed by the monster module. We call these hauptmoduln $T_g(\tau)$ the McKay-Thompson series. In the particular case of the identity element $e \in \mathbb{M}$, the corresponding hauptmodul $T_e(\tau)$ is equal to $j(\tau) - 744$. Then for some McKay-Thompson series and sporadic simple groups, a similar connection is observed. (See [4, Section 7.3: More Monstrous Moonshine]).

Observation 1.1. The Fourier coefficients of some McKay-Thompson series are close to the degrees of irreducible representations of some sporadic simple groups as follows.

- $T_{1A}(\tau) = j(\tau) - 744 = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$
  Degrees of irreducible representations (DIR) of the Monster group:
  $\{1, 196883, 21296876, 842609326, \dots\}$.

- $T_{2A}(\tau) = j_2^*(\tau) = q^{-1} + 4372q + 96256q^2 + 1240002q^3 + \cdots,$
  DIR of the Baby Monster group: $\{1, 4371, 96255, 1139374, \dots\}$.

- $T_{3A}(\tau) = j_3^*(\tau) = q^{-1} + 783q + 8672q^2 + 65367q^3 + \cdots,$
  DIR of the Fischer group Fi_{23}: $\{1, 782, 3588, 5083, 25806, 30888, \dots\}$.

- $T_{5A}(\tau) = j_5^*(\tau) = q^{-1} + 134q + 760q^2 + 3345q^3 + \cdots,$
  DIR of the Harada-Norton group: $\{1, 133, 133, 760, 3344, \dots\}$.

- $T_{6A}(\tau) = j_6^*(\tau) = q^{-1} + 79q + 352q^2 + 1431q^3 + \cdots,$
  DIR of the Fischer group Fi_{22}: $\{1, 78, 429, 1001, \dots\}$.

- $T_{7A}(\tau) = j_7^*(\tau) = q^{-1} + 51q + 204q^2 + 681q^3 + \cdots,$
  DIR of the Held group: $\{1, 51, 51, 153, 153, 680, \dots\}$.

- $T_{10A}(\tau) = j_{10}^*(\tau) = q^{-1} + 22q + 56q^2 + 177q^3 + \cdots,$
  DIR of the Higman-Sims group: $\{1, 22, 77, 154, \dots\}$.

Here $\tau \in \mathfrak{H}$ (the upper half-plane) and $q = e^{2\pi i\tau}$.

In this article, we express these Fourier coefficients in terms of the traces of CM values of the McKay-Thompson series as analogues of Kaneko's arithmetic formulas.

§ 2. Main theorem

Let N be a positive square-free integer such that the genus of the congruence subgroup $\Gamma_0(N)$ is 0, that is, $N = 1, 2, 3, 5, 6, 7, 10$, and 13. For these N, we write $j_N(\tau)$ and $j_N^*(\tau)$ for the hauptmoduln (McKay-Thompson series) on $\Gamma_0(N)$ and the Fricke group $\Gamma_0^*(N)$, respectively. Here $\Gamma_0^*(N)$ is generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for e with $e|N$ and $(e, N/e) = 1$, where W_e is a matrix of the form $\frac{1}{\sqrt{e}}\left[\begin{smallmatrix} xe & y \\ zN & we \end{smallmatrix}\right]$ with $\det W_e = 1$ and $x, y, z, w \in \mathbb{Z}$. In the case of $N = 1$, we put $j_1^*(\tau) := j(\tau) - 744$ and $\Gamma_0^*(1) = \mathrm{SL}_2(\mathbb{Z})$. For a positive integer d such that $-d$ is congruent to a square modulo $4N$, $\mathcal{Q}_{d,N}$ denotes the set of positive definite binary quadratic forms $Q(X,Y) = [a,b,c] := aX^2 + bXY + cY^2$ $(a, b, c \in \mathbb{Z},\ a \equiv 0 \pmod{N})$ of discriminant $-d$ on which $\Gamma_0^*(N)$ acts. For each $Q \in \mathcal{Q}_{d,N}$, we define the corresponding CM point α_Q by the unique root in $\mathfrak{H}$ of $Q(X,1) = 0$. For a positive integer m and the hauptmodul $j_N^*(\tau)$, let $\varphi_m(j_N^*)$ be the unique polynomial in j_N^* satisfying $\varphi_m(j_N^*(\tau)) = q^{-m} + O(q)$. We define the modular trace by

$$\mathbf{t}_m^{(N*)}(d) := \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^*(N)} \frac{1}{|\overline{\Gamma_0^*(N)}_Q|} \varphi_m(j_N^*(\alpha_Q)),$$

where $|\overline{\Gamma_0^*(N)}_Q|$ is the order of the stabilizer of Q in $\overline{\Gamma_0^*(N)} := \Gamma_0^*(N)/\{\pm I\}$. Then, we can express the Fourier coefficients of $j_N(\tau)$ and $j_N^*(\tau)$ in terms of the modular traces.

Theorem 2.1. *For $N \in \{1, 2, 3, 5, 6, 7, 10, 13\}$, let $c_n^{(N)}$ and $c_n^{(N*)}$ be the nth Fourier coefficients of $j_N(\tau)$ and $j_N^*(\tau)$, respectively. For any $n \geq -1$, we have*

$$\begin{aligned}
2nc_n^{(p)} &= \sum_{r \in \mathbb{Z}} \boldsymbol{t}_2^{(p*)}(4n - r^2) + \frac{24(3 - p\sigma_1(2/p))}{p-1} \sigma_1^{(p)}(n) \qquad (p = 2, 3, 5, 7, 13), \\
2nc_n^{(6)} &= \sum_{r \in \mathbb{Z}} \boldsymbol{t}_2^{(6*)}(4n - r^2) + 7\sigma_1^{(6)}(n) + 26\sigma_1^{(3)}(n/2) - 3\sigma_1^{(2)}(n/3), \\
2nc_n^{(10)} &= \sum_{r \in \mathbb{Z}} \boldsymbol{t}_2^{(10*)}(4n - r^2) + 4\sigma_1^{(10)}(n) + 12\sigma_1^{(5)}(n/2), \\
2nc_n^{(1*)} &= \sum_{r \in \mathbb{Z}} \boldsymbol{t}_2^{(1*)}(4n - r^2) \quad \cdots \textit{Kaneko's formula}, \\
2nc_n^{(p*)} &= \sum_{r \in \mathbb{Z}} \left\{ \boldsymbol{t}_2^{(p*)}(4n - r^2) - \boldsymbol{t}_2^{(p*)}(4pn - r^2) \right\} \qquad (p = 2, 3, 5, 7, 13), \\
2nc_n^{(p_1p_2*)} &= \sum_{r \in \mathbb{Z}} \left\{ \boldsymbol{t}_2^{(p_1p_2*)}(4n - r^2) - \boldsymbol{t}_2^{(p_1p_2*)}(4p_1n - r^2) \right. \\
&\qquad \left. - \boldsymbol{t}_2^{(p_1p_2*)}(4p_2n - r^2) + \boldsymbol{t}_2^{(p_1p_2*)}(4p_1p_2n - r^2) \right\} \quad (p_1p_2 = 6, 10),
\end{aligned}$$

where $\sigma_1(n) := \sum_{d|n} d$ *and* $\sigma_1^{(N)}(n) := \sum_{\substack{d|n \\ d \not\equiv 0 (N)}} d$. *If* $x \notin \mathbb{Z}_{\geq 0}$, *the value of* $\sigma_1^{(N)}(x)$ *is 0, and we put* $\sigma_1^{(N)}(0) := (N-1)/24$. *In addition, we define some additional values as follows.*

$$\boldsymbol{t}_2^{(N*)}(0) := \begin{cases} 6 & N = 1, \\ 5 & N = 2, \\ 3 & N = 3, 5, 7, 13, \\ 5/2 & N = 6, 10, \end{cases} \quad \boldsymbol{t}_2^{(N*)}(-1) := -1, \ \boldsymbol{t}_2^{(N*)}(-4) := -2.$$

Remark. (1) Each sum in this theorem is finite.
(2) The cases of $N = 2, 3$, and 5 were established by Ohta [9] and the author and Osanai [7] by using the Riemann-Roch theorem.

Zagier [10] showed the modularity of the generating function of the modular traces $\mathbf{t}_m^{(1*)}(d)$, and Bruinier and Funke [2] generalized Zagier's result by using the Kudla-Millson theta lift. By virtue of their works, C. H. Kim [6] and the author [8] obtained weakly holomorphic Jacobi forms.

Theorem 2.2. ($N = 1, 2, 3, 5, 7, 13$*: Kim [6],* $N = 6, 10$*: M. [8]*) *For* $N \in \{1, 2, 3, 5, 6, 7, 10, 13\}$, *we put* $\boldsymbol{t}_2^{(N)}(d) := 2^{\mu_N(d)} \boldsymbol{t}_2^{(N*)}(d)$, *where* $\mu_N(d)$ *is the number of prime factors of* (N, d). *If* $(N, d) = 1$, *we put* $\mu_N(d) := 0$. *Then the generating function*

$$g_2^{(N)}(\tau, z) := \sum_{\substack{n \gg -\infty \\ r \in \mathbb{Z}}} \boldsymbol{t}_2^{(N)}(4Nn - r^2) q^n \zeta^r$$

is a weakly holomorphic Jacobi form of weight 2 *and index* N. *In the cases of* $N = 2, 3, 5, 6, 7, 10$, *and* 13, *this function becomes a weak Jacobi form, that is, the condition* $n \gg -\infty$ *is replaced with* $n \geq 0$.

Then the function

$$\tilde{g}_2^{(N)}(\tau) := \frac{1}{\tau^2} \sum_{\ell=0}^{N-1} g_2^{(N)}\left(-\frac{1}{N\tau}, \frac{\ell}{N}\right) = N^2 2^{\mu_N(N)} \sum_{n=-1}^{\infty} \left(\sum_{r \in \mathbb{Z}} \mathbf{t}_2^{(N*)}(4n - r^2) \right) q^n$$

is a weakly holomorphic modular form of weight 2 on $\Gamma_0(N)$ with a pole only at $\tau = i\infty$. Comparing with $j_N'(\tau)$ and ${j_N^*}'(\tau)$, we can prove Theorem 2.1.

Remark. The results of Bruinier and Funke works for an arbitrary modular function of general square-free level N. Hereby we can construct weakly holomorphic Jacobi forms of weight 2 and index N.

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