RIMS Kôkyûroku Bessatsu
B78 (2020), 001–027

Bilinearization of the q-Sasano system of type $D_7^{(1)}$ and special polynomials associated with its rational solutions

By

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Abstract

We propose a formulation in terms of bilinear relations satisfied by τ-variables for the q-Sasano system of type $D_7^{(1)}$, and derive a family of special polynomials associated with its rational solutions. Each of these polynomials are invariant under the action of the Weyl group $W(D_5)$.

§ 1. Introduction

It is known that many of (discrete) Painlevé equations and their higher order generalizations admit two classes of special solutions. One is transcendental classical solutions expressible in terms of functions of hypergeometric type. Another one is algebraic or rational solutions.

A typical way for constructing rational solutions to the (discrete) Painlevé systems is as follows. First one constructs a particular solution on the fixed points with respect to an automorphism of the Dynkin diagram that specifies the Weyl group as a group of Bäcklund transformations for the system under consideration. Applying the Bäcklund transformations to the solution, one can generate a family of rational solutions. Moreover, it is known that such a family of rational solutions can be expressed in terms of Schur functions or the universal characters [7, 1, 2, 3, 4, 6, 10].

In this article we deal with a q-analogue of the Sasano system of type $D_7^{(1)}$ [5]. We present a formulation in terms of bilinear relations satisfied by so-called τ-variables. Further we illustrate a refined way to construct a class of rational solutions to the

Received January 8, 2019. Revised March 14, 2019.
2010 Mathematics Subject Classification(s): 05A30, 33D99, 33E17, 33E30
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(discrete) Painlevé systems and its polynomial factors of τ-functions. Applying the way to the q-Sasano system, we derive a family of special polynomials associated with rational solutions.

§ 2. A birational realization of the Weyl group of type $D_7^{(1)}$

The author has derived a q-analogue of the Sasano system of type $D_{2N+3}^{(1)}$ [5], starting with a realization of the Weyl group as a group of birational transformations proposed by Tsuda and Takenawa [11]. In this section we give a brief review of derivation of the q-Sasano system of type $D_7^{(1)}$.

§ 2.1. Rational variety and root system

Let $f = (f_0, f_1, \ldots, f_5)$ be the inhomogeneous coordinates of $(\mathbb{P}^1)^6$. Consider the following subvarieties:

$$\begin{aligned} C_n^{+1} &= \{f_{n-1} = 0, f_n = -u_n, f_{n+1} = \infty\}, \\ C_n^{-1} &= \{f_{n-1} = \infty, f_n = -1/v_n, f_{n+1} = 0\}, \end{aligned} \qquad (n = 1, 2, 3, 4),$$

and

$$\begin{aligned} C_1^{+2} &= \{f_0 = 0, f_1 = -\tilde{u}_1, f_2 = \infty\}, \ C_1^{-2} = \{f_0 = \infty, f_1 = -1/\tilde{v}_1, f_2 = 0\}, \\ C_4^{+2} &= \{f_3 = 0, f_4 = -\tilde{u}_4, f_5 = \infty\}, \ C_4^{-2} = \{f_3 = \infty, f_4 = -1/\tilde{v}_4, f_5 = 0\}. \end{aligned}$$

Let $\epsilon : X \to (\mathbb{P}^1)^6$ be the blowing-up along the above 12 subvarieties.

The second cohomology group of X is given by

$$H^2(X, \mathbb{Z}) \cong \mathrm{NS}\,(X) = \bigoplus_{n=0}^{5} \mathbb{Z}H_n \oplus \bigoplus_{i \in \{\pm 1, \pm 2\}} \mathbb{Z}E_1^i \oplus \bigoplus_{i \in \{\pm 1\},\, n=2,3} \mathbb{Z}E_n^i \oplus \bigoplus_{i \in \{\pm 1, \pm 2\}} \mathbb{Z}E_4^i,$$

where $\mathrm{NS}\,(X)$ is the Néron-Severi group of X; we denote by H_n the divisor class of hyperplanes $\{f_n = \text{const.}\}$ and by E_n^i the class of exceptional divisors $\epsilon^{-1}(C_n^i)$. The second homology group can be described by

$$H_2(X, \mathbb{Z}) = \bigoplus_{n=0}^{5} \mathbb{Z}h_n \oplus \bigoplus_{i \in \{\pm 1, \pm 2\}} \mathbb{Z}e_1^i \oplus \bigoplus_{i \in \{\pm 1\},\, n=2,3} \mathbb{Z}e_n^i \oplus \bigoplus_{i \in \{\pm 1, \pm 2\}} \mathbb{Z}e_4^i,$$

where h_n corresponds to a line of degree $(0, \ldots, 0, \overset{n}{\check{1}}, 0, \ldots, 0)$ and e_n^i to a line restricted in a fibre of the exceptional divisor $\epsilon^{-1}(C_n^i)$. Thus the intersection pairing $\langle\, , \rangle : H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$ is defined by $\langle H_m, h_n \rangle = \delta_{m,n}, \langle E_m^i, e_n^j \rangle = -\delta_{m,n}\delta_{i,j}$ and $\langle \text{otherwise} \rangle = 0$. Introduce the root lattice Q and coroot lattice $\check{Q}$ as follows:

$$Q = \bigoplus_{n=0}^{7} \mathbb{Z}\alpha_n \subset H^2(X, \mathbb{Z}), \quad \check{Q} = \bigoplus_{n=0}^{7} \mathbb{Z}\check{\alpha}_n \subset H_2(X, \mathbb{Z}),$$

where α_n and $\check{\alpha}_n$ are given by

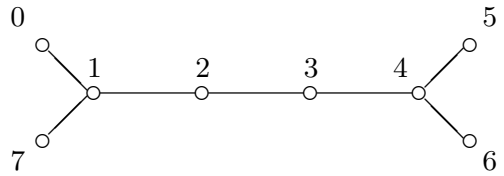
$$\begin{aligned}
&\alpha_0 = E_1^{+1} - E_1^{+2}, \quad \alpha_7 = E_1^{-1} - E_1^{-2},\\
&\alpha_n = H_n - E_n^{+1} - E_n^{-1} \qquad (n = 1, 2, 3, 4),\\
&\alpha_5 = E_4^{+1} - E_4^{+2}, \quad \alpha_6 = E_4^{-1} - E_4^{-2},
\end{aligned}$$

and

$$\begin{aligned}
&\check{\alpha}_0 = e_1^{+1} - e_1^{+2}, \quad \check{\alpha}_7 = e_1^{-1} - e_1^{-2},\\
&\check{\alpha}_n = h_{n-1} + h_{n+1} - e_n^{+1} - e_n^{-1} \qquad (n = 1, 2, 3, 4),\\
&\check{\alpha}_5 = e_4^{+1} - e_4^{+2}, \quad \check{\alpha}_6 = e_4^{-1} - e_4^{-2},
\end{aligned}$$

respectively. The Dynkin diagram of the canonical root basis is given by

(2.1)



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The simple reflection $s_n := s_{\alpha_n}$ associated with a root α_n naturally acts on $H^2(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ by

$$\begin{aligned}
&s_n(\Lambda) = \Lambda + \langle \Lambda, \check{\alpha}_n \rangle \alpha_n, \ \ \Lambda \in H^2(X, \mathbb{Z}),\\
&s_n(\lambda) = \lambda + \langle \alpha_n, \lambda \rangle \check{\alpha}_n, \quad \lambda \in H_2(X, \mathbb{Z}).
\end{aligned}$$

These generate the affine Weyl group $W(D_7^{(1)}) = \langle s_0, \ldots, s_7 \rangle$. In addition, one can introduce involutions ι_k $(k = 1, 2, 3)$ by

$$\begin{aligned}
&\iota_1 : E_1^{+i} \leftrightarrow E_1^{-i}, \ E_3^{+i} \leftrightarrow E_3^{-i}, \ e_1^{+i} \leftrightarrow e_1^{-i}, \ e_3^{+i} \leftrightarrow e_3^{-i},\\
&\iota_2 : E_2^{+i} \leftrightarrow E_2^{-i}, \ E_4^{+i} \leftrightarrow E_4^{-i}, \ e_2^{+i} \leftrightarrow e_2^{-i}, \ e_4^{+i} \leftrightarrow e_4^{-i},\\
&\iota_3 : E_1^{i} \leftrightarrow E_4^{i}, \quad E_2^{i} \leftrightarrow E_3^{i}, \quad e_1^{i} \leftrightarrow e_4^{i}, \quad e_2^{i} \leftrightarrow e_3^{i},\\
&\quad H_1 \leftrightarrow H_4, \quad H_2 \leftrightarrow H_3, \quad h_1 \leftrightarrow h_4, \quad h_2 \leftrightarrow h_3.
\end{aligned}$$

These realize the automorphisms of the above Dynkin diagram. In fact, the action on the root vectors is given by

$$\begin{aligned}
&\iota_1 : \alpha_0 \leftrightarrow \alpha_7,\\
&\iota_2 : \alpha_5 \leftrightarrow \alpha_6,\\
&\iota_3 : \alpha_0 \leftrightarrow \alpha_5, \quad \alpha_7 \leftrightarrow \alpha_6, \quad \alpha_1 \leftrightarrow \alpha_4, \quad \alpha_2 \leftrightarrow \alpha_3.
\end{aligned}$$

They satisfy the following relations

$$\begin{aligned}
&\iota_1 s_0 = s_7 \iota_1, \quad \iota_1 s_7 = s_0 \iota_1, \quad \iota_1 s_i = s_i \iota_1 \, (i \neq 0, 7),\\
&\iota_2 s_5 = s_6 \iota_2, \quad \iota_2 s_6 = s_5 \iota_2, \quad \iota_2 s_i = s_i \iota_2 \, (i \neq 5, 6),\\
&\iota_3 s_0 = s_5 \iota_3, \quad \iota_3 s_5 = s_0 \iota_3, \quad \iota_3 s_7 = s_6 \iota_3, \quad \iota_3 s_6 = s_7 \iota_3, \quad \iota_3 s_k = s_{5-k} \iota_3 \, (k = 1, 2, 3, 4).
\end{aligned}$$

Note that $\delta := \alpha_0 + \alpha_7 + 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \alpha_5 + \alpha_6$ and its dual element $\check{\delta} := \check{\alpha}_0 + \check{\alpha}_7 + 2(\check{\alpha}_1 + \check{\alpha}_2 + \check{\alpha}_3 + \check{\alpha}_4) + \check{\alpha}_5 + \check{\alpha}_6$ are invariant under the action of $W(D_7^{(1)}) \rtimes \Omega$, $\Omega = \langle \iota_1, \iota_2, \iota_3 \rangle$.

The half of the anti-canonical class $-\frac{1}{2}K_X = H_0 + \cdots + H_5 - \sum_{n,i} E_n^i \in H^2(X, \mathbb{Z})$ can be decomposed in two ways

$$-\frac{1}{2}K_X = \sum_{n=0}^{5} D_n^0 = \sum_{n=0}^{5} D_n^\infty,$$

where

$$\begin{aligned}
D_0^0 &= H_0 - E_1^{+1} - E_1^{+2}, & D_0^\infty &= H_0 - E_1^{-1} - E_1^{-2},\\
D_1^0 &= H_1 - E_2^{+1}, & D_1^\infty &= H_1 - E_2^{-1},\\
D_2^0 &= H_2 - E_1^{-1} - E_1^{-2} - E_3^{+1}, & D_2^\infty &= H_2 - E_1^{+1} - E_1^{+2} - E_3^{-1},\\
D_3^0 &= H_3 - E_2^{-1} - E_4^{+1} - E_4^{+2}, & D_3^\infty &= H_3 - E_2^{+1} - E_4^{-1} - E_4^{-2},\\
D_4^0 &= H_4 - E_3^{-1}, & D_4^\infty &= H_4 - E_3^{+1},\\
D_5^0 &= H_5 - E_4^{-1} - E_4^{-2}, & D_5^\infty &= H_5 - E_4^{+1} - E_4^{+2}.
\end{aligned}$$

Note that divisor classes D_n^0 and D_n^∞ are effective and are represented by the strict transforms of hyperplanes $\{f_n = 0\}$ and $\{f_n = \infty\}$, respectively. In parallel, we formally define an element $-\frac{1}{2}k_X \in H_2(X, \mathbb{Z})$ by

$$-\frac{1}{2}k_X = 2h_0 + \cdots + 2h_5 - \sum_{n,i} e_n^i = \sum_{n=0}^{5} d_n^0 = \sum_{n=0}^{5} d_n^\infty,$$

where

$$\begin{aligned}
d_0^0 &= h_5 + h_1 - e_1^{+1} - e_1^{+2}, & d_0^\infty &= h_5 + h_1 - e_1^{-1} - e_1^{-2},\\
d_1^0 &= h_0 + h_2 - e_2^{+1}, & d_1^\infty &= h_0 + h_2 - e_2^{-1},\\
d_2^0 &= h_1 + h_3 - e_1^{-1} - e_1^{-2} - e_3^{+1}, & d_2^\infty &= h_1 + h_3 - e_1^{+1} - e_1^{+2} - e_3^{-1},\\
d_3^0 &= h_2 + h_4 - e_2^{-1} - e_4^{+1} - e_4^{+2}, & d_3^\infty &= h_2 + h_4 - e_2^{+1} - e_4^{-1} - e_4^{-2},\\
d_4^0 &= h_3 + h_5 - e_3^{-1}, & d_4^\infty &= h_3 + h_5 - e_3^{+1},\\
d_5^0 &= h_4 + h_0 - e_4^{-1} - e_4^{-2}, & d_5^\infty &= h_4 + h_0 - e_4^{+1} - e_4^{+2}.
\end{aligned}$$

Note that we have the relations

$$\begin{aligned}
D_0^0 - D_2^\infty + D_4^0 &= D_0^\infty - D_2^0 + D_4^\infty, & D_1^0 - D_3^\infty + D_5^0 &= D_1^\infty - D_3^0 + D_5^\infty,\\
d_0^0 - d_2^\infty + d_4^0 &= d_0^\infty - d_2^0 + d_4^\infty, & d_1^0 - d_3^\infty + d_5^0 &= d_1^\infty - d_3^0 + d_5^\infty.
\end{aligned}$$

Let us introduce the lattice D and $\check{D}$ by

$$D = \bigoplus_{n=0}^{5} \mathbb{Z} D_n^0 \oplus \bigoplus_{n=0}^{5} \mathbb{Z} D_n^\infty, \quad \check{D} = \bigoplus_{n=0}^{5} \mathbb{Z} d_n^0 \oplus \bigoplus_{n=0}^{5} \mathbb{Z} d_n^\infty.$$

It is easy to see that $Q \subset \check{D}^{\perp}$ and $\check{Q} \subset D^{\perp}$. Due to

$$\operatorname{rank} H^2(X;\mathbb{Z}) = 18, \quad \operatorname{rank} Q = 8, \quad \operatorname{rank} \check{D} = 12 - 2 = 10,$$

we see that $Q = \check{D}^{\perp}$ and $\check{Q} = D^{\perp}$. The action of the involutions $\iota_k\,(k = 1,2,3)$ on $D_n^*, d_n^*\,(* = 0,\infty)$ is given by

$$\begin{aligned}
&\iota_1 : D_n^0 \leftrightarrow D_n^\infty, \quad d_n^0 \leftrightarrow d_n^\infty \quad (n = 0,2,4),\\
&\iota_2 : D_n^0 \leftrightarrow D_n^\infty, \quad d_n^0 \leftrightarrow d_n^\infty \quad (n = 1,3,5),\\
&\iota_3 : D_n^0 \leftrightarrow D_{5-n}^\infty, \; d_n^0 \leftrightarrow d_{5-n}^\infty \quad (n = 0,1,\dots,5).
\end{aligned}$$

§ 2.2. Birational representation of the Weyl group

Let us lift the above linear action of the Weyl group $W = W(D_7^{(1)}) \rtimes \Omega$ to that of birational transformations on the rational variety X. First we introduce the multiplicative root variables $a_i\,(i = 0,1,\dots,7)$ attached to the canonical roots α_i. The action of W is given by

$$s_i(a_j) = a_j a_i^{-C_{ij}} \tag{2.2}$$

and

$$\begin{aligned}
&\iota_1(a_0) = 1/a_7, \quad \iota_1(a_7) = 1/a_0, \quad \iota_1(a_k) = 1/a_k\,(k \neq 0,7),\\
&\iota_2(a_5) = 1/a_6, \quad \iota_2(a_6) = 1/a_5, \quad \iota_2(a_k) = 1/a_k\,(k \neq 5,6),\\
&\iota_3(a_0) = 1/a_5, \quad \iota_3(a_5) = 1/a_0, \quad \iota_3(a_7) = 1/a_6, \quad \iota_3(a_6) = 1/a_7,\\
&\iota_3(a_1) = 1/a_4, \quad \iota_3(a_2) = 1/a_3, \quad \iota_3(a_3) = 1/a_2, \quad \iota_3(a_4) = 1/a_1,
\end{aligned} \tag{2.3}$$

where $C = (C_{ij})_{i,j=0}^7$ is the generalized Cartan matrix of type $D_7^{(1)}$. Using the variables a_i, we fix the parameterization of subvarieties C_n^i by

$$\begin{aligned}
&u_1 = a_1 a_0^{-1/2} a_7^{1/2}, \; v_1 = a_1 a_0^{1/2} a_7^{-1/2},\\
&u_n = v_n = a_n \qquad (n = 2,3),\\
&u_4 = a_4 a_5^{-1/2} a_6^{1/2}, \; v_4 = a_4 a_5^{1/2} a_6^{-1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{u}_1 = a_1 a_0^{3/2} a_7^{1/2} = s_0(u_1), \; \tilde{v}_1 = a_1 a_0^{1/2} a_7^{3/2} = s_7(v_1),\\
&\tilde{u}_4 = a_4 a_5^{3/2} a_6^{1/2} = s_5(u_4), \; \tilde{v}_4 = a_4 a_5^{1/2} a_6^{3/2} = s_6(v_4).
\end{aligned}$$

Let $\mathbb{K}(f)$ be the field of rational functions in $f = (f_0,\dots,f_5)$, where the coefficient field $\mathbb{K} = \mathbb{C}(a^{1/4})$ is generated by $a_n^{1/4}\,(n = 0,1,\dots,7)$.

Proposition 2.1. Let us define the birational transformations s_i $(i = 0, 1, \dots, 7)$ by

$$s_1(f_0) = a_1 f_0 \frac{f_1 + v_1^{-1}}{f_1 + u_1}, \qquad s_1(f_2) = a_1^{-1} f_2 \frac{f_1 + u_1}{f_1 + v_1^{-1}},$$
$$s_n(f_{n-1}) = f_{n-1} \frac{a_n f_n + 1}{f_n + a_n}, \quad s_n(f_{n+1}) = f_{n+1} \frac{f_n + a_n}{a_n f_n + 1}, \qquad (n = 2, 3),$$
$$s_4(f_3) = a_4 f_3 \frac{f_4 + v_4^{-1}}{f_4 + u_4}, \qquad s_4(f_5) = a_4^{-1} f_5 \frac{f_4 + u_4}{f_4 + v_4^{-1}},$$

and ι_k $(k = 1, 2, 3)$ by

$$\iota_1(f_{2n}) = 1/f_{2n} \quad (n = 0, 1, 2), \qquad \iota_2(f_{2n+1}) = 1/f_{2n+1} \quad (n = 0, 1, 2),$$
$$\iota_3(f_n) = 1/f_{5-n} \quad (n = 0, 1, \dots, 5).$$

Then these transformations, together with (2.2) and (2.3), realize the extended affine Weyl group $W = W(D_7^{(1)}) \rtimes \Omega$ over the field $\mathbb{K}(f)$.

Note that $q := a_0 a_7 (a_1 a_2 a_3 a_4)^2 a_5 a_6$ is invariant under the action of $W(D_7^{(1)})$. We also see that the quantities $f_0 f_2 f_4$ and $f_1 f_3 f_5$ are $W(D_7^{(1)})$-invariant. We impose bellow the normalization conditions $f_0 f_2 f_4 = f_1 f_3 f_5 = 1$ so that these quantities are invariant under the action of W.

Let us consider the translation operator $T = \iota_1 \iota_2 s_{210712345643} \in W$, where $s_{i \cdots j} = s_i \cdots s_j$. One can regard the iteration of T as a discrete time evolution. Since the action on the multiplicative root variables is given by $T : (a_2, a_3) \mapsto (q a_2, a_3/q)$, we see that $u := \dfrac{(a_0 a_7)^{1/2} a_1 a_2}{a_3 a_4 (a_5 a_6)^{1/2}}$ plays a role of the independent variable. One can obtain the system of q-difference equations

$$T(f_n) = R_n(u; a'; f), \quad (n = 0, 1, \dots, 5), \tag{2.4}$$

where $a' = (a_0^{1/4}, a_7^{1/4}, a_1^{1/4}, (a_2 a_3)^{1/4}, a_4^{1/4}, a_5^{1/4}, a_6^{1/4})$ and R_n is a rational function in indicated variables. Action of a subgroup $\langle s_0, s_7, s_1, s_{232}, s_4, s_5, s_6 \rangle$, which is isomorphic to $W(D_6^{(1)})$, commutes with that of T, and describes the symmetry of the above system of q-difference equations [5]. This system is reduced to the Sasano system [8, 9] of type $D_6^{(1)}$ in a continuous limit.

§ 2.3. Representation over the field of τ-variables

Let us introduce the τ-variables τ_n^i attached to the centers C_n^i of the blowing-up, and consider the decomposition of the variables

$$f_0 = \frac{\tau_1^{+1} \tau_1^{+2}}{\tau_1^{-1} \tau_1^{-2}}, \quad f_1 = \frac{\tau_2^{+1}}{\tau_2^{-1}}, \quad f_2 = \frac{\tau_1^{-1} \tau_1^{-2} \tau_3^{+1}}{\tau_1^{+1} \tau_1^{+2} \tau_3^{-1}},$$
$$f_3 = \frac{\tau_2^{-1} \tau_4^{+1} \tau_4^{+2}}{\tau_2^{+1} \tau_4^{-1} \tau_4^{-2}}, \quad f_4 = \frac{\tau_3^{-1}}{\tau_3^{+1}}, \quad f_5 = \frac{\tau_4^{-1} \tau_4^{-2}}{\tau_4^{+1} \tau_4^{+2}}.$$

Proposition 2.2. Define the birational transformations s_i $(i=0,1,\dots,7)$ by

$$s_0:\tau_1^{+1}\leftrightarrow\tau_1^{+2},\qquad s_7:\tau_1^{-1}\leftrightarrow\tau_1^{-2},\qquad s_5:\tau_4^{+1}\leftrightarrow\tau_4^{+2},\qquad s_6:\tau_4^{-1}\leftrightarrow\tau_4^{-2},$$

$$\begin{aligned} s_1(\tau_1^{+1})&=\frac{v_1^{1/2}\tau_2^{+1}+v_1^{-1/2}\tau_2^{-1}}{\tau_1^{-1}},\ s_1(\tau_1^{-1})=\frac{u_1^{-1/2}\tau_2^{+1}+u_1^{1/2}\tau_2^{-1}}{\tau_1^{+1}},\\ s_4(\tau_4^{+1})&=\frac{v_4^{1/2}\tau_3^{-1}+v_4^{-1/2}\tau_3^{+1}}{\tau_4^{-1}},\ s_4(\tau_4^{-1})=\frac{u_4^{-1/2}\tau_3^{-1}+u_4^{1/2}\tau_3^{+1}}{\tau_4^{+1}}, \end{aligned}\tag{2.5}$$

$$\begin{aligned} s_2(\tau_2^{+1})&=\frac{v_2^{1/2}\tau_3^{+1}\tau_1^{-1}\tau_1^{-2}+v_2^{-1/2}\tau_1^{+1}\tau_1^{+2}\tau_3^{-1}}{\tau_2^{-1}},\\ s_2(\tau_2^{-1})&=\frac{u_2^{-1/2}\tau_3^{+1}\tau_1^{-1}\tau_1^{-2}+u_2^{1/2}\tau_1^{+1}\tau_1^{+2}\tau_3^{-1}}{\tau_2^{+1}},\\ s_3(\tau_3^{+1})&=\frac{v_3^{1/2}\tau_4^{+1}\tau_4^{+2}\tau_2^{-1}+v_3^{-1/2}\tau_2^{+1}\tau_4^{-1}\tau_4^{-2}}{\tau_3^{-1}},\\ s_3(\tau_3^{-1})&=\frac{u_3^{-1/2}\tau_4^{+1}\tau_4^{+2}\tau_2^{-1}+u_3^{1/2}\tau_2^{+1}\tau_4^{-1}\tau_4^{-2}}{\tau_3^{+1}}, \end{aligned}\tag{2.6}$$

and ι_k $(k=1,2,3)$ by

$$\begin{aligned} &\iota_1:\tau_1^{\pm i}\leftrightarrow\tau_1^{\mp i},\ \tau_3^{\pm1}\leftrightarrow\tau_3^{\mp1},\\ &\iota_2:\tau_4^{\pm i}\leftrightarrow\tau_4^{\mp i},\ \tau_2^{\pm1}\leftrightarrow\tau_2^{\mp1},\\ &\iota_3:\tau_1^{i}\leftrightarrow\tau_4^{i},\qquad \tau_2^{i}\leftrightarrow\tau_3^{i}. \end{aligned}$$

Then these transformations, together with (2.2) and (2.3), realize the extended affine Weyl group $W=W(D_7^{(1)})\rtimes\Omega$ over the field $\mathbb{K}(\tau)$ of rational functions in τ_n^i.

Finally we remark a sort of regularity of τ-variables. Let us consider the Weyl group orbit $M=W.\{E_1^{\pm1},E_1^{\pm2},E_2^{\pm1},E_3^{\pm1},E_4^{\pm1},E_4^{\pm2}\}\subset\mathrm{NS}\,(X)$ and define the τ-variables on the lattice M by

$$\tau_{E_n^i}=\tau_n^i,\qquad w(\tau_\Lambda)=\tau_{w(\Lambda)},\quad \Lambda\in M,\quad w\in W.$$

As is shown in [11], τ_Λ $(\Lambda\in M)$ is a Laurent polynomial in τ_n^i, though one can only state that τ_Λ $(\Lambda\in M)$ is a rational function of τ_n^i from the above proposition.

§ 3. τ-variables on the lattice and bilinearization

In this section we present a formulation in terms of bilinear relations satisfied by τ-variables. It is shown that the orbit $M^{[1]}:=W.\{E_1^{+1}\}$ can be identified with the weight

lattice $P(D_7)$. Bilinear relations for τ-variables on $M^{[1]}$ are characterized in terms of the lattice.

§ 3.1. Bilinearization

Let us present a formulation for the q-Sasano system of type $D_7^{(1)}$ in terms of bilinear relations for τ-variables, starting with the realization of $W = W(D_7^{(1)}) \rtimes \Omega$ given in Proposition 2.2.

From the action (2.5) and (2.6) of W, one can get the relations for the τ-variables on the lattice M

$$\begin{aligned}
\tau_{E_1^{-1}}\tau_{H_1-E_1^{-1}} &= v_1^{1/2}\tau_{E_2^{+1}} + v_1^{-1/2}\tau_{E_2^{-1}},\\
\tau_{E_2^{-1}}\tau_{H_2-E_2^{-1}} &= v_2^{1/2}\tau_{E_3^{+1}}\tau_{E_1^{-1}}\tau_{E_1^{-2}} + v_2^{-1/2}\tau_{E_1^{+1}}\tau_{E_1^{+2}}\tau_{E_3^{-1}},
\end{aligned}$$

for instance. These are homogeneous with respect to τ-variables if we define the degree of them by

$$\deg \tau_\Lambda = \begin{cases} 1 \left(\Lambda \in W.\{E_1^i, E_4^i\}\right) \\ 2 \left(\Lambda \in W.\{E_2^i, E_3^i\}\right) \end{cases}.$$

In this subsection we derive bilinear relations for the τ-variables of degree 1. The τ-variables of degree 2 are expressed by homogeneous polynomials of degree 2 of the τ-variables of degree 1.

We introduce the notation σ_i $(i = 0, 7, 5, 6)$ by

$$\sigma_0 = \tau_1^{+2}, \quad \sigma_7 = \tau_1^{-2}, \quad \sigma_5 = \tau_4^{+2}, \quad \sigma_6 = \tau_4^{-2},$$

and $\sigma_{j\cdots i,i} = s_{j\cdots i}(\sigma_i)$. It is easy to see the following from (2.5).

Lemma 3.1. The τ-variables $\tau_n^{\pm 1}$ $(n = 2, 3)$ can be expressed in terms of the τ-variables of degree 1 by

$$\begin{aligned}
\langle a_1\rangle\tau_2^{+1} &= u_1^{1/2}\sigma_{10,0}\sigma_{7,7} - v_1^{-1/2}\sigma_{0,0}\sigma_{17,7},\\
\langle a_1\rangle\tau_2^{-1} &= v_1^{1/2}\sigma_{0,0}\sigma_{17,7} - u_1^{-1/2}\sigma_{10,0}\sigma_{7,7},\\
\langle a_4\rangle\tau_3^{+1} &= v_4^{1/2}\sigma_{5,5}\sigma_{46,6} - u_4^{-1/2}\sigma_{45,5}\sigma_{6,6},\\
\langle a_4\rangle\tau_3^{-1} &= u_4^{1/2}\sigma_{45,5}\sigma_{6,6} - v_4^{-1/2}\sigma_{5,5}\sigma_{46,6},
\end{aligned} \tag{3.1}$$

where $\langle x\rangle = x - x^{-1}$.

The variables $\tau_n^{\pm 1}$ $(n = 2, 3)$ admit another expression in terms of the τ-variables of degree 1.

Lemma 3.2. From (2.5) and (2.6), the variables $\tau_n^{\pm 1}$ $(n = 2, 3)$ can be also expressed by

$$\begin{aligned}
\langle a_3\rangle\langle a_6\rangle\tau_2^{+1} &= \sigma_{6345,5}\sigma_{45,5} - \sigma_{345,5}\sigma_{645,5},\\
\langle a_3\rangle\langle a_5\rangle\tau_2^{-1} &= \sigma_{5346,6}\sigma_{46,6} - \sigma_{346,6}\sigma_{546,6},\\
\langle a_2\rangle\langle a_7\rangle\tau_3^{+1} &= \sigma_{7210,0}\sigma_{10,0} - \sigma_{210,0}\sigma_{710,0},\\
\langle a_2\rangle\langle a_0\rangle\tau_3^{-1} &= \sigma_{0217,7}\sigma_{17,7} - \sigma_{217,7}\sigma_{017,7}.
\end{aligned} \tag{3.2}$$

Proof. Applying s_2 to the first relation of (2.5) and eliminating $s_2(\tau_2^{\pm 1})$ by using the first two relations of (2.6), we get

$$\sigma_{210,0} = a_2 v_1^{1/2}\sigma_7\frac{\tau_3^{+1}}{\tau_2^{-1}} + v_1^{1/2}\frac{\sigma_0\sigma_{0,0}}{\sigma_{7,7}}\frac{\tau_3^{-1}}{\tau_2^{-1}} + a_2^{-1}v_1^{-1/2}\sigma_7\frac{\tau_3^{+1}}{\tau_2^{+1}} + v_1^{-1/2}\frac{\sigma_0\sigma_{0,0}}{\sigma_{7,7}}\frac{\tau_3^{-1}}{\tau_2^{+1}}.$$

Applying s_7 to the first relation of (2.5), we have

$$\sigma_{710,0} = a_7 v_1^{1/2}\frac{\tau_2^{+1}}{\sigma_7} + a_7^{-1}v_1^{-1/2}\frac{\tau_2^{-1}}{\sigma_7}.$$

Then we see that $s_7(\sigma_{210,0}\sigma_{710,0}) - \sigma_{210,0}\sigma_{710,0} = \langle a_2\rangle\langle a_7\rangle\tau_3^{+1}$, which is the third expression of (2.5). ∎

Proposition 3.3. We have the bilinear relations

$$\begin{aligned}
\langle a_0\rangle\sigma_{10,0}\sigma_{7,7} - \langle a_0a_1\rangle\sigma_{0,0}\sigma_{17,7} + \langle a_1\rangle\sigma_0\sigma_{017,7} &= 0,\\
\langle a_7\rangle\sigma_{0,0}\sigma_{17,7} - \langle a_7a_1\rangle\sigma_{10,0}\sigma_{7,7} + \langle a_1\rangle\sigma_{710,0}\sigma_7 &= 0,\\
\langle a_5\rangle\sigma_{45,5}\sigma_{6,6} - \langle a_5a_4\rangle\sigma_{5,5}\sigma_{46,6} + \langle a_4\rangle\sigma_5\sigma_{546,6} &= 0,\\
\langle a_6\rangle\sigma_{5,5}\sigma_{46,6} - \langle a_6a_4\rangle\sigma_{45,5}\sigma_{6,6} + \langle a_4\rangle\sigma_{645,5}\sigma_6 &= 0,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\langle a_2\rangle\langle a_7\rangle\left(v_4^{1/2}\sigma_{5,5}\sigma_{46,6} - u_4^{-1/2}\sigma_{45,5}\sigma_{6,6}\right) &= \langle a_4\rangle\left(\sigma_{7210,0}\sigma_{10,0} - \sigma_{210,0}\sigma_{710,0}\right),\\
\langle a_2\rangle\langle a_0\rangle\left(u_4^{1/2}\sigma_{45,5}\sigma_{6,6} - v_4^{-1/2}\sigma_{5,5}\sigma_{46,6}\right) &= \langle a_4\rangle\left(\sigma_{0217,7}\sigma_{17,7} - \sigma_{217,7}\sigma_{017,7}\right),\\
\langle a_3\rangle\langle a_6\rangle\left(u_1^{1/2}\sigma_{10,0}\sigma_{7,7} - v_1^{-1/2}\sigma_{0,0}\sigma_{17,7}\right) &= \langle a_1\rangle\left(\sigma_{6345,5}\sigma_{45,5} - \sigma_{345,5}\sigma_{645,5}\right),\\
\langle a_3\rangle\langle a_5\rangle\left(v_1^{1/2}\sigma_{0,0}\sigma_{17,7} - u_1^{-1/2}\sigma_{10,0}\sigma_{7,7}\right) &= \langle a_1\rangle\left(\sigma_{5346,6}\sigma_{46,6} - \sigma_{346,6}\sigma_{546,6}\right),
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\langle a_3\rangle\langle a_6\rangle\sigma_{46345,5}\sigma_{5,5} - \langle a_3a_4\rangle\langle a_4a_6\rangle\sigma_{6345,5}\sigma_{45,5} + \langle a_4\rangle\langle a_3a_4a_6\rangle\sigma_{345,5}\sigma_{645,5} &= 0,\\
\langle a_3\rangle\langle a_5\rangle\sigma_{45346,6}\sigma_{6,6} - \langle a_3a_4\rangle\langle a_4a_5\rangle\sigma_{5346,6}\sigma_{46,6} + \langle a_4\rangle\langle a_3a_4a_5\rangle\sigma_{346,6}\sigma_{546,6} &= 0,\\
\langle a_2\rangle\langle a_7\rangle\sigma_{17210,0}\sigma_{0,0} - \langle a_1a_2\rangle\langle a_1a_7\rangle\sigma_{7210,0}\sigma_{10,0} + \langle a_1\rangle\langle a_1a_2a_7\rangle\sigma_{210,0}\sigma_{710,0} &= 0,\\
\langle a_2\rangle\langle a_0\rangle\sigma_{10217,7}\sigma_{7,7} - \langle a_1a_2\rangle\langle a_0a_1\rangle\sigma_{0217,7}\sigma_{17,7} + \langle a_1\rangle\langle a_0a_1a_2\rangle\sigma_{217,7}\sigma_{017,7} &= 0.
\end{aligned} \tag{3.5}$$

Proof. The variables $\tau_2^{\pm 1}$ and $\tau_3^{\pm}$ are invariant under the action of $s_i\,(i \neq 2)$ and $s_i\,(i \neq 3)$, respectively. Then one can get the bilinear relations (3.3) and (3.5) from the expressions (3.1) and (3.2), respectively. The bilinear relations (3.4) are directly obtained from (3.1) and (3.2). ∎

By tedious calculation we see that the relations (2.5) and (2.6) are recovered from the bilinear relations (3.3) and (3.4). We also get the following by direct computation.

Proposition 3.4. Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be a set of all the bilinear relations obtained by the action of $W = W(D_7^{(1)}) \rtimes \Omega$ on the relations (3.3),(3.4) and (3.5), respectively. Then any bilinear relation that belongs to $\mathcal{B}_1$ or $\mathcal{B}_3$ can be derived from those of $\mathcal{B}_2$.

§ 3.2. τ-variables on the lattice

The lattice $M = W.\{E_1^i, E_2^i, E_3^i, E_4^i\}$ is decomposed into two orbits by the action of the Weyl group W as

$$M = M^{[1]} \sqcup M^{[2]}, \qquad M^{[1]} = W.\{E_1^{+1}\}, \quad M^{[2]} = W.\{E_2^{+1}\}.$$

Note that $\deg \tau_\Lambda = d$ if $\Lambda \in M^{[d]}$. Each of $M^{[d]}$ is further decomposed into four orbits by the action of $W(D_7^{(1)})$ as

$$M^{[1]} = M_1^+ \sqcup M_1^- \sqcup M_4^+ \sqcup M_4^-, \quad M^{[2]} = M_2^+ \sqcup M_2^- \sqcup M_3^+ \sqcup M_3^-,$$

where $M_n^\pm$ is the $W(D_7^{(1)})$-orbit of $E_n^{\pm 1}$.

Proposition 3.5. Each of eight orbits can be expressed by

$$\begin{aligned}
M_1^+ &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (1,0,0,0,0,0), I^\infty = (0,0,1,0,0,0), \langle \Lambda, \lambda \rangle = -1\},\\
M_1^- &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,1,0,0,0), I^\infty = (1,0,0,0,0,0), \langle \Lambda, \lambda \rangle = -1\},\\
M_4^+ &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,0,1,0,0), I^\infty = (0,0,0,0,0,1), \langle \Lambda, \lambda \rangle = -1\},\\
M_4^- &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,0,0,0,1), I^\infty = (0,0,0,1,0,0), \langle \Lambda, \lambda \rangle = -1\},
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
M_2^+ &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,1,0,0,0,0), I^\infty = (0,0,0,1,0,0), \langle \Lambda, \lambda \rangle = -1\},\\
M_2^- &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,0,1,0,0), I^\infty = (0,1,0,0,0,0), \langle \Lambda, \lambda \rangle = -1\},\\
M_3^+ &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,1,0,0,0), I^\infty = (0,0,0,0,1,0), \langle \Lambda, \lambda \rangle = -1\},\\
M_3^- &= \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (0,0,0,0,1,0), I^\infty = (0,0,1,0,0,0), \langle \Lambda, \lambda \rangle = -1\},
\end{aligned}$$

where $I^* = I^*(\Lambda) = (i_0^*, \ldots, i_5^*), i_n^* = i_n^*(\Lambda) = \langle \Lambda, d_n^* \rangle\,(* = 0, \infty)$ and λ is a dual element of Λ.

Let us introduce a map $\vec{m} : \mathrm{NS}\,(X) \to \mathbb{Z}^8$ by

$$\vec{m}(\Lambda) = (m_0, m_7, m_1, m_2, m_3, m_4, m_5, m_6), \qquad m_i := \langle \Lambda, \check{\alpha}_i \rangle \quad (i = 0, 1, \ldots, 7),$$

and $|\vec{m}(\Lambda)|$ by $|\vec{m}(\Lambda)| := m_0 + m_7 + 2(m_1 + m_2 + m_3 + m_4) + m_5 + m_6 = \langle \Lambda, \check{\delta} \rangle$.

Proof. Here we show the proposition on M_1^+. Let us denote by N_1^+ the set of right-hand side;

$$N_1^+ := \{\Lambda \in \mathrm{NS}(X) \,|\, I^0 = (1,0,0,0,0,0), I^\infty = (0,0,1,0,0,0), \langle \Lambda, \lambda \rangle = -1\}.$$

It is obvious that $M_1^+ \subset N_1^+$. Noticing that $\check{\delta} = \sum_{n=1}^{4} (d_n^0 + d_n^\infty)$, for any $\Lambda \in N_1^+$ there exists $w \in W(D_7^{(1)})$ such that $\Lambda' = w(\Lambda)$ satisfies $\vec{m}(\Lambda') = (1,0,0,0,0,0,0,0)$, $(0,1,0,0,0,0,0,0)$, $(0,0,0,0,0,0,1,0)$ or $(0,0,0,0,0,0,0,1)$ due to $|\vec{m}(\Lambda)| = 1$. Since $\Lambda' \in N_1^+$, we see that $\Lambda' = E_1^{+2}$. This means that $\Lambda \in M_1^+$. ∎

Let us investigate the lattice $M^{[1]}$ more precisely. Consider the restriction of $\vec{m}$ to $M^{[1]}$ and introduce a set $L \subset \mathbb{Z}^8$ by

$$L = \left\{ \mu \in \mathbb{Z}^8 \,\middle|\, |\mu| := \mu_0 + \mu_7 + 2(\mu_1 + \mu_2 + \mu_3 + \mu_4) + \mu_5 + \mu_6 = 1 \right\}.$$

Lemma 3.6. The restriction $\vec{m}\,|_{M^{[1]}}$ is an injective, and $\mathrm{Im}\,\vec{m}\,|_{M^{[1]}} = L$.

The above lemma can be proved by noticing that all the element $w \in W = W(D_7^{(1)}) \rtimes \Omega$ leave the intersection pairing $\langle\,,\rangle$ invariant. Therefore the lattice $M^{[1]}$ ($\cong L$) can be identified with the weight lattice $P(D_7) = \bigoplus_{n=1}^{7} \mathbb{Z}\varpi_n$ by the correspondence $L \ni \mu \mapsto \sum_{n=1}^{7} \mu_n \varpi_n \in P(D_7)$.

An infinite number of τ-variables assigned to $M^{[1]} \cong P(D_7)$ satisfies all the bilinear relations that belong to $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$. Hereafter we consider this overdetermined system for the τ-variables assigned to $M^{[1]}$.

§ 3.3. Characterization of bilinear relations

Here we characterize bilinear relations that belong to $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ in terms of the lattice. We consider the first and second relations of (3.3), the third of (3.4) and the first of (3.5) as representatives.

At first, let us discuss the first and second relations of (3.3). The pairs of the τ-variables appeared there have a common barycenter, twice of which is $H_1 \in \mathrm{NS}\,(X)$. Note that $\Lambda = H_1$ is the unique element of $\mathrm{NS}(X)$ that satisfies the conditions

$$I^0(\Lambda) = I^\infty(\Lambda) = (1,0,1,0,0,0), \quad \langle \Lambda, \lambda \rangle = 0, \quad \vec{m}(\Lambda) = (0,0,0,1,0,0,0,0),$$

where λ is a dual element of Λ. Consider decompositions $H_1 = \Lambda + \Lambda'$ ($\Lambda, \Lambda' \in M^{[1]}$). It is easy to see from (3.6) that $\Lambda \in M^{+1}$, $\Lambda' \in M^{-1}$. Further we have a following:

Lemma 3.7. The possible decompositions $H_1 = \Lambda + \Lambda' \, (\Lambda, \Lambda' \in M^{[1]})$ are given by

$$\begin{aligned} H_1 &= E_1^{+2} + (H_1 - E_1^{+2}) \\ &= E_1^{+1} + (H_1 - E_1^{+1}) \\ &= (H_1 - E_1^{-2}) + E_1^{-2} \\ &= (H_1 - E_1^{-1}) + E_1^{-1}. \end{aligned}$$

For each decomposition we have $\langle \Lambda' - \Lambda, \lambda' - \lambda \rangle = -4$.

Lemma 3.8. We have $\dim |H_1| = 2$, where $|H_1|$ is a linear system defined by the element $H_1 \in \mathrm{NS}\,(X)$.

Proof. Let $\Phi(\Lambda)$ be the defining polynomial of the hypersurface corresponding to the element $\Lambda = \sum_{n=1}^{4} d_n H_n - \sum_{n,i} \mu_n^i E_n^i \in \mathrm{NS}(X)$. This hypersurface is of degree $d = (d_1, d_2, d_3, d_4)$ in the coordinates $f = (f_1, f_2, f_3, f_4)$ and passes through C_n^i with multiplicity μ_n^i. Then we have $\Phi(H_1) = af_1 + b \, (a, b \in \mathbb{C})$, which means $\dim |H_1| = 2$. ∎

From the decompositions of H_1 given in Lemma 3.7 we get four pairs of τ-variables

$$\sigma_0 \sigma_{017,7}, \quad \sigma_{0,0} \sigma_{17,7}, \quad \sigma_{710,0} \sigma_7, \quad \sigma_{10,0} \sigma_{7,7}. \tag{3.7}$$

Lemma 3.8 implies that there exists a linear relation among any three of four pairs. The first and second relations of (3.3) are typical ones.

Next we discuss the third relation of (3.4). The pairs of the τ-variables appeared in left-hand side have a common barycenter, twice of which is H_1. Since twice of the common barycenter of the pairs of the τ-variables appeared in right-hand side is $H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2}$, the four pairs of τ-variables do not have a common barycenter apparently. However, we see

$$H_1 + D_3^{\infty} + D_4^0 + D_4^{\infty} = H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} + D_1^0, \tag{3.8}$$

then two barycenters are equivalent to each other modulo an element of the lattice D. We see that $\Lambda = H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2}$ is the unique element of $\mathrm{NS}(X)$ that satisfies the conditions

$$\begin{aligned} &I^0(\Lambda) = (0,0,0,2,0,0), \quad I^{\infty} = (0,0,0,0,0,2), \quad \langle \Lambda, \lambda \rangle = 0, \\ &\vec{m}(\Lambda) = (0,0,0,1,0,0,0,0), \end{aligned}$$

where λ is a dual element of Λ.

Lemma 3.9. The possible decompositions of $H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} =$

$\Lambda + \Lambda' \, (\Lambda, \Lambda' \in M^{[1]})$ are given by

$$\begin{aligned} &H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} \\ &\quad = E_4^{+2} + (H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} - E_4^{+2}) \\ &\quad = E_4^{+1} + (H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} - E_4^{+1}) \\ &\quad = (H_4 - E_4^{-1}) + (H_3 + H_4 - E_3^{+1} - E_3^{-1} - E_4^{-2}) \\ &\quad = (H_4 - E_4^{-2}) + (H_3 + H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1}). \end{aligned}$$

For each decomposition we have $\langle \Lambda' - \Lambda, \lambda' - \lambda \rangle = -4$.

From the above decompositions we get four pairs of τ-variables

$$\sigma_5 \sigma_{546345,5}, \quad \sigma_{5,5} \sigma_{46345,5}, \quad \sigma_{45,5} \sigma_{6345,5}, \quad \sigma_{645,5} \sigma_{345,5}. \tag{3.9}$$

Let us consider a linear system $|\Lambda|$ defined by $\Lambda = H_1 + H_3 + 2H_4 - E_2^{+1} - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2} \in \mathrm{NS}\,(X)$, where Λ is equivalent to both sides of (3.8). Then we have $\dim |\Lambda| = 3$. This means that there exists a linear relation among any four pairs of τ-variables, two pairs of which come from (3.7) and others come from (3.9). The third relation of (3.4) is typical one.

Finally let us mention the first relation of (3.5). Since we have $\dim |\Lambda| = 2$, where $\Lambda = H_3 + 2H_4 - E_3^{+1} - E_3^{-1} - E_4^{-1} - E_4^{-2}$, there exists a linear relation among any three of four pairs of (3.9). The first relation of (3.5) is typical one.

§ 4. Construction of special polynomials associated with rational solutions

Let us consider the transformation $\pi := \iota_1 \iota_2 \in W = W(D_7^{(1)}) \rtimes \Omega$ which is an automorphism of the Dynkin diagram (2.1). Note that this commutes with the translation T, which means the transformation π is a symmetry of the system of q-difference equations (2.4). It is obvious that the system has a particular solution $f_n = 1 \, (n = 0, 1, \ldots, 5)$ when the parameters satisfy $a_0 = a_7$ and $a_5 = a_6$, which lies on the fixed points with respect to π. Applying Bäcklund transformations to this solution, one can get a family of solutions expressed by rational functions of six parameters including q. Each of solutions possesses internal symmetry described by an extention of the affine Weyl group of type $D_5^{(1)}$, which can be realized as the centralizer of π of the Weyl group $W = W(D_7^{(1)}) \rtimes \Omega$. The remaining symmetry, which is isomorphic to the Weyl group of type $(A_1 + A_1)^{(1)}$, generates a family of solutions. In this section we construct the τ-functions associated with the rational solutions in a refined way.

§ 4.1. Preliminaries

The centralizer of $\pi = \iota_1 \iota_2$ of the Weyl group $W = W(D_7^{(1)}) \rtimes \Omega$ is given by

$$C_W(\pi) = \langle s_{07170}, s_1, s_2, s_3, s_4, s_{56465}, s_{07}, s_{56}, \iota_1, \iota_2, \iota_3 \rangle,$$

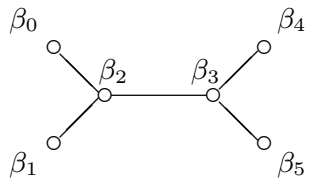
which is isomorphic to an extended affine Weyl group of type $D_5^{(1)}$. The simple roots $\beta_i\,(i=0,1,\dots,5)$ and coroots $\check{\beta}_i\,(i=0,1,\dots,5)$ are given by

$$\begin{aligned}
&\beta_0=\alpha_0+\alpha_1+\alpha_7, \quad \beta_i=\alpha_i\,(i=1,2,3,4), \quad \beta_5=\alpha_4+\alpha_5+\alpha_6,\\
&\check{\beta}_0=\check{\alpha}_0+\check{\alpha}_1+\check{\alpha}_7, \quad \beta_i=\check{\alpha}_i\,(i=1,2,3,4), \quad \check{\beta}_5=\check{\alpha}_4+\check{\alpha}_5+\check{\alpha}_6.
\end{aligned}$$

Note that we have

$$\beta_0+\beta_1+2(\beta_2+\beta_3)+\beta_4+\beta_5=\alpha_0+\alpha_7+2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6=\delta,$$

and the Dynkin diagram is given by



.

We denote the simple reflection associated with a root $\beta_i\,(i=0,1,\dots,5)$ by $w_i\,(i=0,1,\dots,5)$, which can be expressed by

$$w_0=s_{07170}, \quad w_i=s_i\,(i=1,2,3,4), \quad w_5=s_{56465}.$$

Each of the transformations s_{07}, s_{56} and $\iota_k\,(k=1,2,3)$ behaves itself as an automorphism of the above Dynkin diagram. To be more precise, ι_1 and ι_2 keep the simple roots $\beta_i\,(i=0,1,\dots,5)$ invariant, and the action of the other transformations is given by

$$\begin{aligned}
s_{07}&:\beta_0\leftrightarrow\beta_1,\\
s_{56}&:\beta_4\leftrightarrow\beta_5,\\
\iota_3\ &:\beta_k\leftrightarrow\beta_{5-k} \quad (k=0,1,\dots,5).
\end{aligned}$$

They satisfy the following relations

$$\begin{aligned}
&s_{07}w_0=w_1s_{07}, \quad s_{07}w_1=w_0s_{07}, \quad s_{07}w_i=w_is_{07}\,(i\neq 0,1),\\
&s_{56}w_4=w_5s_{56}, \quad s_{56}w_5=w_4s_{56}, \quad s_{56}w_i=w_is_{56}\,(i\neq 4,5),\\
&\iota_3w_i=w_{5-i}\iota_3 \quad (i=0,1,\dots,5).
\end{aligned}$$

Next we introduce $\gamma_i^{(k)}\,(i=0,1\,;\,k=1,2)$ by

$$\begin{aligned}
&\gamma_0^{(1)}=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6, \quad \gamma_1^{(1)}=\alpha_7+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\\
&\gamma_0^{(2)}=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \quad \gamma_1^{(2)}=\alpha_7+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6,
\end{aligned}$$

which are orthogonal to $\bigoplus_{i=0}^{5}\mathbb{Z}\beta_i\cong Q(D_5^{(1)})$, and satisfy the relations $\gamma_0^{(1)}+\gamma_1^{(1)}=\gamma_0^{(2)}+\gamma_1^{(2)}=\delta$. They generate two root lattices $\bigoplus_{i=0}^{1}\mathbb{Z}\gamma_i^{(k)}\cong Q(A_1^{(1)})\,(k=1,2)$, which

are orthogonal to each other. Denote the reflection associated with a root $\gamma_i^{(k)}$ by $r_i^{(k)}$. The group generated by $r_i^{(k)}$ $(i=0,1\,;\,k=1,2)$ is isomorphic to the Weyl group of type $(A_1+A_1)^{(1)}$, and commutes with $\langle w_0,w_1,\dots,w_5\rangle \cong W(D_5^{(1)})$. The action of the transformations s_{07}, s_{56} and ι_k $(k=1,2,3)$ on the simple roots $\gamma_i^{(k)}$ $(i=0,1\,;\,k=1,2)$ is given by

$$\begin{aligned}
s_{07}&:\gamma_0^{(1)}\leftrightarrow\gamma_1^{(2)},\quad \gamma_0^{(2)}\leftrightarrow\gamma_1^{(1)},\\
s_{56}&:\gamma_0^{(1)}\leftrightarrow\gamma_0^{(2)},\quad \gamma_1^{(1)}\leftrightarrow\gamma_1^{(2)},\\
\iota_1&:\gamma_0^{(1)}\leftrightarrow\gamma_1^{(2)},\quad \gamma_0^{(2)}\leftrightarrow\gamma_1^{(1)},\\
\iota_2&:\gamma_0^{(1)}\leftrightarrow\gamma_0^{(2)},\quad \gamma_1^{(1)}\leftrightarrow\gamma_1^{(2)},\\
\iota_3&:\gamma_0^{(1)}\leftrightarrow\gamma_1^{(1)}.
\end{aligned}$$

We decompose the lattice $M^{[1]}$ into a family of five-dimensional lattices according to the values of the intersection pairing with the coroots $\check{\gamma}_1^{(1)}$ and $\check{\gamma}_1^{(2)}$;

$$M^{[1]}=\coprod_{l,m\in\mathbb{Z}} M_{l,m},\quad M_{l,m}=\{\Lambda\in M^{[1]}\,|\,\langle\Lambda,\check{\gamma}_1^{(1)}\rangle=-l,\langle\Lambda,\check{\gamma}_1^{(2)}\rangle=-m\}.$$

Let us introduce a map $j\,:\,M^{[1]}\to\mathbb{Z}^6$ by

$$j(\Lambda)=(j_0,j_1,j_2,j_3,j_4,j_5),\quad j_i:=-\langle\Lambda,\check{\beta}_i\rangle,\quad \Lambda\in M^{[1]}.$$

Then we have $|j(\Lambda)|:=j_0+j_1+2(j_2+j_3)+j_4+j_5=-1$. Further we define a map φ by

$$\begin{array}{rcl}\varphi:M^{[1]}&\to&\widetilde{L}\times\mathbb{Z}^2\\ \cup&&\cup\\ \Lambda&\mapsto&(j(\Lambda),(l,m))\end{array},$$

where $\widetilde{L}:=\left\{j\in\mathbb{Z}^6\,\middle|\,|j(\Lambda)|=-1\right\}$. It is easy to see that the map φ is injective. Then we express τ_Λ $(\Lambda\in M^{[1]})$ by $\tau_\Lambda=\tau_{l,m}^{j(\Lambda)}$ below. In terms of this notation, the action of the subgroup $\widetilde{W}((D_5+2A_1)^{(1)})\subset W$ on the τ-variables can be described by

$$\begin{aligned}
&w_i(\tau_\Lambda)=\tau_{l,m}^{w_i(j(\Lambda))},\quad s_{07}(\tau_\Lambda)=\tau_{-m-1,-l-1}^{(j_1,j_0,j_2,\dots,j_5)},\quad s_{56}(\tau_\Lambda)=\tau_{m,l}^{(j_0,\dots,j_3,j_5,j_4)},\\
&\iota_1(\tau_\Lambda)=\tau_{-m-1,-l-1}^{j(\Lambda)},\iota_2(\tau_\Lambda)=\tau_{m,l}^{j(\Lambda)},\quad \iota_3(\tau_\Lambda)=\tau_{-l-1,m}^{(j_5,j_4,j_3,j_2,j_1,j_0)},\\
&r_1^{(1)}(\tau_\Lambda)=\tau_{-l,m}^{j(\Lambda)},\qquad r_1^{(2)}(\tau_\Lambda)=\tau_{l,-m}^{j(\Lambda)},
\end{aligned}$$

where $w_i(j_k)=j_k-C_{ik}j_i$ with $C=(C_{ik})_{i,k=0}^5$ being the generalized Cartan matrix of type $D_5^{(1)}$.

Let us consider the action of translation operators

$$\begin{aligned}
&T_{1;2}=\pi s_{107123456432},\quad T_{2;3}=\pi s_{210712345643},\quad T_{3;4}=\pi s_{321071234564},\\
&T_{07;1}=\pi s_{071234564321},\quad T_{4;56}=\pi s_{432107123456},\\
&T_{5;7}=\rho s_{017213024136724130217},\quad T_{6;7}=\rho^{-1}s_{017213024135724130217},
\end{aligned}$$

where $\rho = \iota_3\iota_2$ and $\pi = \rho^2\,(= \iota_1\iota_2)$. The subscripts denote the action on the simple roots of $Q(D_7^{(1)})$, namely we have for instance

$$T_{1;2}\,:\alpha \mapsto \alpha + (0,0,1,-1,0,0,0,0)\,\delta,$$
$$T_{07;1}:\alpha \mapsto \alpha + (1,1,-1,0,0,0,0,0)\,\delta,$$

where $\alpha=(\alpha_0,\alpha_7,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$. It is easy to see that $T_{1;2},T_{2;3},T_{3;4},T_{07;1},T_{4;56} \in C_W(\pi)$. The action on the τ-variables $\tau_\Lambda = \tau_{l,m}^{(j_0,j_1,j_2,j_3,j_4,j_5)}$ is described by

$$\begin{aligned} &T_{07;1}(\tau_\Lambda) = \tau_{l,m}^{(j_0+1,j_1-1,j_2,j_3,j_4,j_5)}, \quad T_{4;56}(\tau_\Lambda) = \tau_{l,m}^{(j_0,j_1,j_2,j_3,j_4+1,j_5-1)},\\ &T_{1;2}(\tau_\Lambda) = \tau_{l,m}^{(j_0+1,j_1+1,j_2-1,j_3,j_4,j_5)}, T_{2;3}(\tau_\Lambda) = \tau_{l,m}^{(j_0,j_1,j_2+1,j_3-1,j_4,j_5)},\\ &T_{3;4}(\tau_\Lambda) = \tau_{l,m}^{(j_0,j_1,j_2,j_3+1,j_4-1,j_5-1)}, \end{aligned} \tag{4.1}$$

and

$$T_{5;7}(\tau_\Lambda) = \tau_{l,m-1}^{(j_0-1,j_1,j_2,j_3,j_4,j_5+1)}, \quad T_{6;7}(\tau_\Lambda) = \tau_{l-1,m}^{(j_0-1,j_1,j_2,j_3,j_4,j_5+1)}.$$

It is obvious that $\langle T_{1;2},T_{2;3},T_{3;4},T_{07;1},T_{4;56},T_{5;7},T_{6;7}\rangle.\{E_1^{+2}\} = M^{[1]}$. Since we have

$$T_{5;7},T_{6;7} \in \widetilde{W}((D_5+2A_1)^{(1)}) := \langle w_0,\ldots,w_5,r_0^{(1)},r_1^{(1)},r_0^{(2)},r_1^{(2)},s_{07},s_{56},\iota_1,\iota_2,\iota_3\rangle,$$

we find that $\widetilde{W}((D_5+2A_1)^{(1)}).\{E_1^{+2}\} = M^{[1]}$.

§ 4.2. Classification of bilinear relations

As we stated in Section 3, the q-Sasano system of type $D_7^{(1)}$ under consideration can be regarded as an overdetermined system of all the bilinear relations of $\mathcal{B}_2$ satisfied by the τ-variables $\tau_\Lambda = \tau_{l,m}^{j(\Lambda)}$ assigned to $\Lambda \in M^{[1]}$.

Proposition 4.1. The set of bilinear relations $\mathcal{B}_2$ are decomposed into four orbits by the action of the subgroup $\widetilde{W}((D_5+2A_1)^{(1)}) \subset W$. A representative of each orbit

is given by

$$
\begin{aligned}
&\langle a_7a_1a_2a_3\rangle\langle a_6\rangle a_0^{-1/4}a_7^{-1/4}\\
&\qquad\times\Big(a_2^{1/2}\tau_{0,0}^{(0,-1,1,-1,0,0)}\tau_{0,0}^{(0,1,-1,0,0,0)}-a^{-1/2}\tau_{0,0}^{(1,0,-1,0,0,0)}\tau_{0,0}^{(-1,0,1,-1,0,0)}\Big)\\
&=\langle a_2\rangle\Big(\tau_{0,1}^{(0,0,0,0,-1,0)}\tau_{0,-1}^{(0,0,0,-1,1,0)}-\tau_{1,0}^{(0,0,0,0,0,-1)}\tau_{-1,0}^{(0,0,0,-1,0,1)}\Big),\\
&\langle a_2\rangle\langle a_7\rangle\left(v_4^{1/2}\tau_{0,-1}^{(0,0,0,0,-1,0)}\tau_{-1,0}^{(0,0,0,-1,1,0)}-u_4^{-1/2}\tau_{0,-1}^{(0,0,0,-1,1,0)}\tau_{-1,0}^{(0,0,0,0,-1,0)}\right)\\
&=\langle a_4\rangle\left(\tau_{0,0}^{(0,-1,1,-1,0,0)}\tau_{-1,-1}^{(0,1,-1,0,0,0)}-\tau_{-1,-1}^{(-1,0,1,-1,0,0)}\tau_{0,0}^{(1,0,-1,0,0,0)}\right),\\
&\langle a_3\rangle\langle a_6\rangle\left(u_1^{1/2}\tau_{-1,-1}^{(0,1,-1,0,0,0)}\tau_{0,0}^{(0,-1,0,0,0,0)}-v_1^{-1/2}\tau_{-1,-1}^{(0,-1,0,0,0,0)}\tau_{0,0}^{(0,1,-1,0,0,0)}\right)\\
&=\langle a_1\rangle\left(\tau_{-1,0}^{(0,0,-1,1,-1,0)}\tau_{0,-1}^{(0,0,0,-1,1,0)}-\tau_{0,-1}^{(0,0,-1,1,0,-1)}\tau_{-1,0}^{(0,0,0,-1,0,1)}\right),\\
&\langle a_3\rangle\langle a_1\rangle a_5^{1/4}a_6^{-1/4}\Big((a_7a_1a_2a_3a_4)^{1/2}\tau_{0,-1}^{(0,0,0,0,-1,0)}\tau_{0,1}^{(0,0,0,0,0,-1)}\\
&\qquad\qquad-(a_7a_1a_2a_3a_4)^{-1/2}\tau_{1,0}^{(0,0,0,0,0,-1)}\tau_{-1,0}^{(0,0,0,0,-1,0)}\Big)\\
&=\langle a_7a_1a_2a_3a_4\rangle\Big(\tau_{0,0}^{(0,1,-1,1,-1,-1)}\tau_{0,0}^{(0,-1,1,-1,0,0)}-\tau_{0,0}^{(0,-1,0,1,-1,-1)}\tau_{0,0}^{(0,1,0,-1,0,0)}\Big).
\end{aligned}
\tag{4.2}
$$

In this subsection we give a sketch of our brute-force proof of the above proposition.

It is sufficient to discuss the bilinear relations containing $\tau_{E_1^{+2}}=\tau_{0,0}^{(-1,0,0,0,0,0)}$, since any $\Lambda\in M^{[1]}$ can be transformed to E_1^{+2} by the action of $\widetilde{W}((D_5+2A_1)^{(1)})$. If a bilinear relation of $\mathcal{B}_2$ possesses a term $\tau_\Lambda\tau_{E_1^{+2}}$, then $\Lambda\in M_1^-$ or $\Lambda\in M_1^+$. Note that the stabilizer subgroup S of $C_W(\pi)$ with respect to all the elements $\check{\gamma}_1^{(1)},\check{\gamma}_1^{(2)}$ and E_1^{+2} is given by $S:=\langle w_1,w_2,w_3,w_4,w_5\rangle\rtimes\langle\omega\rangle$ with being $\omega=\iota_2s_{56}$.

First we consider the case where $\Lambda\in M_1^+$. There are 560 elements $\Lambda\in M_1^+$ satisfing $\langle\Lambda+E_1^{+2},\lambda+e_1^{+2}\rangle=0$, where λ is a dual element of Λ. These are classified into the following types:

$$
\begin{aligned}
&\Lambda\in M_{\pm2,0},M_{0,\pm2}:40\times4=160,\\
&\Lambda\in M_{\pm1,\pm1}\qquad\quad:80\times4=320,\\
&\Lambda\in M_{0,0}\qquad\qquad\quad:80.
\end{aligned}
\tag{4.3}
$$

We explain elements of $M_{-1,-1}$ and $M_{0,0}$ in detail.

Any of 80 elements of $M_{-1,-1}$ can be obtained from $2H_1+H_2-E_1^{+2}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}$ by applying S. Then it is sufficient to consider the bilinear relations containing the pair of τ-variables $\tau_{E_1^{+2}}\tau_{2H_1+H_2-E_1^{+2}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}}$, twice of whose barycenter is $2H_1+H_2-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}$. Its possible decompositions are

given by

$$\begin{aligned}
&2H_1 + H_2 - E_1^{-1} - E_1^{-2} - E_2^{+1} - E_2^{-1} \\
&= E_1^{+2} + \left(2H_1 + H_2 - E_1^{+2} - E_1^{-1} - E_1^{-2} - E_2^{+1} - E_2^{-1}\right) \\
&= E_1^{+1} + \left(2H_1 + H_2 - E_1^{+1} - E_1^{-1} - E_1^{-2} - E_2^{+1} - E_2^{-1}\right) \\
&= \left(H_1 - E_1^{-1}\right) + \left(H_1 + H_2 - E_1^{-2} - E_2^{+1} - E_2^{-1}\right) \\
&= \left(H_1 - E_1^{-2}\right) + \left(H_1 + H_2 - E_1^{-1} - E_2^{+1} - E_2^{-1}\right).
\end{aligned}$$

For each decomposition, we denote by

$$(M_{0,0}, M_{-1,-1}), \quad (M_{-1,-1}, M_{0,0}), \quad (M_{-1,-1}, M_{0,0}), \quad (M_{0,0}, M_{-1,-1}) \tag{4.4}$$

the data of sub-lattices to which the corresponding τ-variables are assigned. The first one, for instance, means that $E_1^{+2} \in M_{0,0}$ and $2H_1+H_2-E_1^{+2}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1} \in M_{-1,-1}$. Note that we have

$$H_4 + D_1^0 + D_1^\infty + D_2^0 = 2H_1 + H_2 - E_1^{-1} - E_1^{-2} - E_2^{+1} - E_2^{-1} + D_4^\infty$$

by applying $\iota_3 \in W$ to (3.8). The possible decompositions of H_4 are given by

$$\begin{aligned}
H_4 &= E_4^{+2} + \left(H_4 - E_4^{+2}\right) \\
&= E_4^{+1} + \left(H_4 - E_4^{+1}\right) \\
&= \left(H_4 - E_4^{-2}\right) + E_4^{-2} \\
&= \left(H_4 - E_4^{-1}\right) + E_4^{-1},
\end{aligned}$$

and then we get the data of sub-lattices

$$(M_{-1,0}, M_{0,-1}), \quad (M_{0,-1}, M_{-1,0}), \quad (M_{-1,0}, M_{0,-1}), \quad (M_{0,-1}, M_{-1,0}) \tag{4.5}$$

for each decomposition. There exists a linear relation among any four pairs of τ-variables, two of which come from (4.4) and others come from (4.5). The datum of sub-lattices for resulting relations can be described by

$$(M_{-1,0}, M_{0,-1}) : (M_{-1,0}, M_{0,-1}) \mid (M_{-1,-1}, M_{0,0}) : (M_{-1,-1}, M_{0,0}).$$

A typical relation is given by

$$\langle a_2\rangle\langle a_7\rangle \left(v_4^{1/2}\sigma_{5,5}\sigma_{46,6} - u_4^{-1/2}\sigma_{45,5}\sigma_{6,6}\right) = \langle a_4\rangle \left(\sigma_{7210,0}\sigma_{10,0} - \sigma_{210,0}\sigma_{710,0}\right),$$

which is equivalent to the second relation of (4.2).

Any element of $M_{0,0}$ can be obtained from

$$3H_1 + 2H_2 + H_3 - E_1^{+1} - E_1^{+2} - E_1^{-1} - 2E_1^{-2} - 2E_2^{+1} - 2E_2^{-1} - E_3^{+1} - E_3^{-1}$$

by applying the stabilizer S. Then it is sufficient to consider the bilinear relations containing the term $\tau_{E_1^{+2}}\tau_{3H_1+2H_2+H_3-E_1^{+1}-E_1^{+2}-E_1^{-1}-2E_1^{-2}-2E_2^{+1}-2E_2^{-1}-E_3^{+1}-E_3^{-1}}$. Twice of the barycenter and its possible decompositions are given by

$$\begin{aligned}
&3H_1+2H_2+H_3-E_1^{+1}-E_1^{-1}-2E_1^{-2}-2E_2^{+1}-2E_2^{-1}-E_3^{+1}-E_3^{-1}\\
&=E_1^{+2}+\bigl(3H_1+2H_2+H_3-E_1^{+1}-E_1^{+2}-E_1^{-1}-2E_1^{-2}-2E_2^{+1}-2E_2^{-1}-E_3^{+1}-E_3^{-1}\bigr)\\
&=\bigl(H_1-E_1^{-2}\bigr)+\bigl(2H_1+2H_2+H_3-E_1^{+1}-E_1^{-1}-E_1^{-2}-2E_2^{+1}-2E_2^{-1}-E_3^{+1}-E_3^{-1}\bigr)\\
&=\bigl(H_1+H_2-E_1^{-2}-E_2^{+1}-E_2^{-1}\bigr)\\
&\qquad+\bigl(2H_1+H_2+H_3-E_1^{+1}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}\bigr)\\
&=\bigl(H_1+H_2+H_3-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}\bigr)\\
&\qquad\qquad+\bigl(2H_1+H_2-E_1^{+1}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}\bigr).
\end{aligned}$$

For each decomposition, the data of sub-lattices are given by

$$(M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,. \tag{4.6}$$

We have

$$\begin{aligned}
&H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}+D_1^0+D_1^\infty+D_2^0\\
&=3H_1+2H_2+H_3-E_1^{+1}-E_1^{-1}-2E_1^{-2}-2E_2^{+1}-2E_2^{-1}-E_3^{+1}-E_3^{-1}+D_4^\infty
\end{aligned}$$

by applying $s_{7123}\iota_3\in W$ to (3.8). The possible decompositions of $H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}$ are given by

$$\begin{aligned}
&H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}\\
&=E_4^{+2}+\bigl(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{+2}\bigr)\\
&=E_4^{+1}+\bigl(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{+1}\bigr)\\
&=\bigl(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{-2}\bigr)+E_4^{-2}\\
&=\bigl(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{-1}\bigr)+E_4^{-1},
\end{aligned}$$

and then we get for each decomposition the data of sub-lattices

$$(M_{-1,0},M_{1,0})\,,\quad (M_{0,-1},M_{0,1})\,,\quad (M_{0,1},M_{0,-1})\,,\quad (M_{1,0},M_{-1,0})\,. \tag{4.7}$$

There exists a linear relation among any four pairs of τ-variables, two of which come from (4.6) and others come from (4.7). The datum of sub-lattices for resulting relations is one of the following:

$$\begin{aligned}
&(M_{1,0},M_{-1,0})\,:\,(M_{1,0},M_{-1,0})\mid(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\,,\\
&(M_{1,0},M_{-1,0})\,:\,(M_{0,1},M_{0,-1})\mid(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\,,\\
&(M_{0,1},M_{0,-1})\,:\,(M_{0,1},M_{0,-1})\mid(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\,.
\end{aligned}$$

It is easy to see that the first and third ones can be derived from the second. A typical relation of the second type is given by

$$\begin{aligned}&\langle a_3\rangle\langle a_1\rangle a_5^{1/4}a_6^{-1/4}\Big((a_7a_1a_2a_3a_4)^{1/2}\sigma_{5,5}\sigma_{712346,6}-(a_7a_1a_2a_3a_4)^{-1/2}\sigma_{712345,5}\sigma_{6,6}\Big)\\&=\langle a_7a_1a_2a_3a_4\rangle\Big(\sigma_{137210,0}\sigma_{7210,0}-\sigma_{37210,0}\sigma_{17210,0}\Big),\end{aligned}$$

which is equivalent to the fourth relation of (4.2).

Next we consider the case where $\Lambda\in M_1^-$. There are 280 elements $\Lambda\in M_1^-$ satisfying $\langle\Lambda+E_1^{+2},\lambda+e_1^{+2}\rangle=0$, where λ is a dual element of Λ. These are classified into the following types:

$$\begin{aligned}&\Lambda\in M_{\pm2,0},M_{0,\pm2}:10\times4=40,\\&\Lambda\in M_{\pm1,\pm1}\qquad\quad:40\times4=160,\\&\Lambda\in M_{0,0}\qquad\qquad\quad:80.\end{aligned}\tag{4.8}$$

We explain elements of $M_{-1,-1}$ and $M_{0,0}$ in detail.

Any of 40 elements of $\Lambda\in M_{-1,-1}$ can be obtained from $H_1-E_1^{+2}$ by applying the stabilizer S. Then it is sufficient to consider the bilnear relations containing the pair of τ-variables $\tau_{E_1^{+2}}\tau_{H_1-E_1^{+2}}$, twice of whose barycenter is H_1. Its possible decompositions are given as stated in Lemma 3.7. For each decomposition, the data of sub-lattices are given by

$$(M_{0,0},M_{-1,-1}),\quad(M_{-1,-1},M_{0,0}),\quad(M_{0,0},M_{-1,-1}),\quad(M_{-1,-1},M_{0,0}).\tag{4.9}$$

Recall that we have (3.8). The possible decompositions of $H_3+2H_4-E_3^{+1}-E_3^{-1}-E_4^{-1}-E_4^{-2}$ are given as stated in Lemma 3.9. Then we get for each decomposition the data of sub-lattices

$$(M_{-1,0},M_{0,-1}),\quad(M_{0,-1},M_{-1,0}),\quad(M_{0,-1},M_{-1,0}),\quad(M_{-1,0},M_{0,-1}).\tag{4.10}$$

There exists a linear relation among any four pairs of τ-variables, two of which come from (4.9) and others come from (4.10). The datum of sub-lattices for resulting relations is

$$(M_{-1,-1},M_{0,0}):(M_{-1,-1},M_{0,0})\mid(M_{-1,0},M_{0,-1}):(M_{-1,0},M_{0,-1}).$$

A typical relation is given by

$$\langle a_3\rangle\langle a_6\rangle\left(u_1^{1/2}\sigma_{10,0}\sigma_{7,7}-v_1^{-1/2}\sigma_{0,0}\sigma_{17,7}\right)=\langle a_1\rangle\left(\sigma_{6345,5}\sigma_{45,5}-\sigma_{345,5}\sigma_{645,5}\right),$$

which is equivalent to the third relation of (4.2).

Since any element of $M_{0,0}$ can be obtained from $2H_1+H_2-E_1^{+1}-E_1^{+2}-E_1^{-2}-E_2^{+1}-E_2^{-1}$ by applying the stablilizer S, it is sufficient to consider the bilinear relations

containing the term $\tau_{E_1^{+2}}\tau_{2H_1+H_2-E_1^{+1}-E_1^{+2}-E_1^{-2}-E_2^{+1}-E_2^{-1}}$. Twice of the barycenter and its possible decompositions are given by

$$\begin{aligned}
&2H_1+H_2-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}\\
&=E_1^{+2}+\left(2H_1+H_2-E_1^{+1}-E_1^{+2}-E_1^{-2}-E_2^{+1}-E_2^{-1}\right)\\
&=\left(2H_1+H_2-E_1^{+1}-E_1^{-1}-E_1^{-2}-E_2^{+1}-E_2^{-1}\right)+E_1^{-1}\\
&=\left(H_1+H_2-E_1^{-2}-E_2^{+1}-E_2^{-1}\right)+\left(H_1-E_1^{+1}\right)\\
&=\left(H_1-E_1^{-2}\right)+\left(H_1+H_2-E_1^{+1}-E_2^{+1}-E_2^{-1}\right).
\end{aligned}$$

For each decomposition, the data of sub-lattices are given by

$$(M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,,\quad (M_{0,0},M_{0,0})\,. \tag{4.11}$$

Note that we have

$$\begin{aligned}
&2H_1+H_2-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}+D_3^{\infty}+D_4^{0}+D_4^{\infty}\\
&=H_1+H_2+H_3+2H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{-1}-E_4^{-2}+D_1^{0}
\end{aligned}$$

by applying $s_{712}\in W$ to (3.8). The possible decompositions of $H_1+H_2+H_3+2H_4-E_1^{+1}-E_1^{-2}-E_2^{+1}-E_2^{-1}-E_3^{+1}-E_3^{-1}-E_4^{-1}-E_4^{-2}$ are given by

$$\begin{aligned}
&H_1+H_2+H_3+2H_4-E_1^{+1}-E_1^{-2}-E_2^{\pm1}-E_3^{\pm1}-E_4^{-1}-E_4^{-2}\\
&=E_4^{+2}+\left(H_1+H_2+H_3+2H_4-E_1^{+1}-E_1^{-2}-E_2^{\pm1}-E_3^{\pm1}-E_4^{+2}-E_4^{-1}-E_4^{-2}\right)\\
&=E_4^{+1}+\left(H_1+H_2+H_3+2H_4-E_1^{+1}-E_1^{-2}-E_2^{\pm1}-E_3^{\pm1}-E_4^{+1}-E_4^{-1}-E_4^{-2}\right)\\
&=\left(H_4-E_4^{-2}\right)+\left(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{\pm1}-E_3^{\pm1}-E_4^{-1}\right)\\
&=\left(H_4-E_4^{-1}\right)+\left(H_1+H_2+H_3+H_4-E_1^{+1}-E_1^{-2}-E_2^{\pm1}-E_3^{\pm1}-E_4^{-2}\right),
\end{aligned}$$

and then we get for each decomposition the data of sub-lattices

$$(M_{-1,0},M_{1,0})\,,\quad (M_{0,-1},M_{0,1})\,,\quad (M_{-1,0},M_{1,0})\,,\quad (M_{0,-1},M_{0,1})\,. \tag{4.12}$$

There exists a linear relation among any four pairs of τ-variables, two of which come from (4.11) and others come from (4.12). The datum of sub-lattices for resulting relations is one of the following:

$$\begin{aligned}
&(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\mid(M_{-1,0},M_{1,0})\,:\,(M_{-1,0},M_{1,0})\,,\\
&(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\mid(M_{-1,0},M_{1,0})\,:\,(M_{0,-1},M_{0,1})\,,\\
&(M_{0,0},M_{0,0})\,:\,(M_{0,0},M_{0,0})\mid(M_{0,-1},M_{0,1})\,:\,(M_{0,-1},M_{0,1})\,.
\end{aligned}$$

It is easy to see that the first and third ones can be derived from the second. A typical relation of the second type is given by

$$\begin{aligned}
&\langle a_7a_1a_2a_3\rangle\langle a_6\rangle a_0^{-1/4}a_7^{-1/4}\Big(a_2^{1/2}\sigma_{7210,0}\sigma_{17,7}-a^{-1/2}\sigma_{710,0}\sigma_{217,7}\Big)\\
&=\langle a_2\rangle\Big(\sigma_{7126345,5}\sigma_{45,5}-\sigma_{712345,5}\sigma_{645,5}\Big),
\end{aligned}$$

which is equivalent to the first relation of (4.2).

Similar discussions on all the types of (4.3) and (4.8) lead us to Proposition 4.1.

§ 4.3. Formulation of problem

In order to construct rational solutions to the q-Sasano system of type $D_7^{(1)}$ and their τ-functions, we formulate our problem here. Introduce the variables b_i $(i = 0, 1, \ldots, 5)$ attached to the simple roots β_i $(i = 0, 1, \ldots, 5)$ of $D_5^{(1)}$ by

$$b_0^2 = a_0 a_7 a_1, \quad b_i^2 = a_i \, (i = 1, 2, 3, 4), \quad b_5^2 = a_4 a_5 a_6,$$

which satisfy $b_0 b_1 (b_2 b_3)^2 b_4 b_5 = p$, $p^2 = q$.

Problem Construct a family of functions $\tau_\Lambda(b; p) = \tau_\Lambda(b_0, b_1, b_2, b_3, b_4, b_5; p)$ $(\Lambda \in M^{[1]})$ that satisfies all the bilinear relations of $\mathcal{B}_2$ and the condition

$$\tau_{w(\Lambda)}(b; p) = \tau_\Lambda(w(b), w(p)), \quad w \in C_W(\pi). \tag{4.13}$$

As we will see below, the condition (4.13) is too strict to construct functions $\tau_\Lambda(b; p)$, since each of ι_k $(k = 1, 2, 3) \in C_W(\pi)$ acts on p by $\iota_k : p \mapsto p^{-1}$. Then we consider a subgroup $C'_W(\pi) = \langle w_0, w_1, \ldots, w_5, s_{07}, s_{56}, \rho \rangle \subset C_W(\pi)$, all of whose elements leave p invariant, and we require the condition

$$\tau_{w(\Lambda)}(b; p) = \tau_\Lambda(w(b), p), \quad w \in C'_W(\pi) \tag{4.14}$$

instead of (4.13).

Let us investigate the symmetry of a family of functions $\tau_\Lambda(b; p) = \tau_{l,m}^{j(\Lambda)}(b; p)$. First we consider the action of translations (4.1). When $w = T_{1;2}$, for instance, we get

$$\tau_{l,m}^{(j_0+1, j_1+1, j_2-1, j_3, j_4, j_5)}(b_0, b_1, b_2, b_3, b_4, b_5; p) = \tau_{l,m}^{(j_0, j_1, j_2, j_3, j_4, j_5)}(pb_0, pb_1, b_2/p, b_3, b_4, b_5; p)$$

from (4.14). Introducing a family of functions $\tau_{l,m}(b; p) = \tau_{l,m}(b_0, b_1, b_2, b_3, b_4, b_5; p)$ $(l, m \in \mathbb{Z})$ by $\tau_{l,m}(b; p) = \tau_{l,m}^{(j_0, j_1, j_2, j_3, j_4, j_5)}(b_0/p^{j_0}, b_1/p^{j_1}, b_2/p^{j_2}, b_3/p^{j_3}, b_4/p^{j_4}, b_5/p^{j_5}; p)$, one can see that the action of the translations (4.1) is described by a shift of variables b_i $(i = 0, 1, \ldots, 5)$. For instance, we have $T_{1;2}(\tau_{l,m}(b; p)) = \tau_{l,m}(pb_0, pb_1, b_2/p, b_3, b_4, b_5; p)$.

Remark 4.2. The family of functions $\tau_{l,m}(b; p)$ $(l, m \in \mathbb{Z})$ are not assigned to the lattice $M^{[1]}$. When we deal with the functions $\tau_{l,m}(b; p)$, the variables b_i are renormalized by $b_1 b_1 (b_2 b_3)^2 b_4 b_5 = 1$.

Next we consider the element $w = \rho s_{56} w_{4325132435} \in C_W(\pi)$, which acts on $Q(D_5^{(1)}) = \bigoplus_{i=0}^{5} \mathbb{Z}\beta_i$ as a translation. The requirement (4.14) yields to $\tau_{-l-1,m}(b; p) = \tau_{l,m}(b; p)$.

Together with the action of $\pi \in C_W(\pi)$, we see that the family of functions $\tau_{l,m}(b;p)$ $(l,m \in \mathbb{Z})$ satisfies

$$\tau_{-l-1,-m-1}(b;p) = \tau_{-l-1,m}(b;p) = \tau_{l,-m-1}(b;p) = \tau_{l,m}(b;p). \tag{4.15}$$

The action of the simple reflections w_i $(i = 0, 1, \ldots, 5)$ gives us

$$\tau_{l,m}(w_i(b);p) = \tau_{l,m}(b;p), \quad w_i(b_j) = b_j b_i^{-C_{i,j}}, \tag{4.16}$$

where $C = (C_{ij})_{i,j=0}^5$ is the generalized Cartan matrix of type $D_5^{(1)}$. One can also get the symmetry

$$\begin{aligned} &\tau_{m,l}(b_1, b_0, b_2, b_3, b_4, b_5; p) = \tau_{m,l}(b_0, b_1, b_2, b_3, b_5, b_4; p) \\ &= \tau_{m,l}(b_5, b_4, b_3, b_2, b_1, b_0; p) = \tau_{l,m}(b;p) \end{aligned} \tag{4.17}$$

from the action of s_{07}, s_{56} and ρ.

Finally we mention the bilinear relations satisfied by the family of functions $\tau_{l,m}(b;p)$. It is sufficient to consider the relations obtained by applying the translations $T_{6;7}^{-l} T_{5;7}^{-m}$ $(l,m \in \mathbb{Z})$ to (4.2). One can get the relations for $\tau_{l,m}(b;p)$

(4.18)

$$\begin{aligned} &\langle b_2^2\rangle \langle p^{l+m} b_0/b_1\rangle p^{(l-m)/2} \Big(b_4 \tau_{l,m-1}^{[0,0,0,0,-1,0]} \tau_{l-1,m}^{[0,0,0,-1,1,0]} - b_4^{-1} \tau_{l,m-1}^{[0,0,0,-1,1,0]} \tau_{l-1,m}^{[0,0,0,0,-1,0]} \Big) \\ &= \langle b_4^2\rangle \Big(\tau_{l,m}^{[0,-1,1,-1,0,0]} \tau_{l-1,m-1}^{[0,1,-1,0,0,0]} - \tau_{l-1,m-1}^{[-1,0,1,-1,0,0]} \tau_{l,m}^{[1,0,-1,0,0,0]} \Big), \\ &\langle b_3^2\rangle \langle p^{-l+m} b_5/b_4\rangle p^{(l+m)/2} \Big(b_1 \tau_{l-1,m-1}^{[0,1,-1,0,0,0]} \tau_{l,m}^{[0,-1,0,0,0,0]} - b_1^{-1} \tau_{l-1,m-1}^{[0,-1,0,0,0,0]} \tau_{l,m}^{[0,1,-1,0,0,0]} \Big) \\ &= \langle b_1^2\rangle \Big(\tau_{l-1,m}^{[0,0,-1,1,-1,0]} \tau_{l,m-1}^{[0,0,0,-1,1,0]} - \tau_{l,m-1}^{[0,0,-1,1,0,-1]} \tau_{l-1,m}^{[0,0,0,-1,0,1]} \Big), \end{aligned}$$

and

$$\begin{aligned} &\langle p^{l+m+1}/(b_4 b_5)\rangle \langle p^{-l+m} b_5/b_4\rangle (b_1/b_0)^{1/2} \\ &\qquad \times \Big(b_2 \tau_{l,m}^{[0,-1,1,-1,0,0]} \tau_{l,m}^{[0,1,-1,0,0,0]} - b_2^{-1} \tau_{l,m}^{[1,0,-1,0,0,0]} \tau_{l,m}^{[-1,0,1,-1,0,0]} \Big) \\ &= \langle b_2^2\rangle \Big(\tau_{l,m+1}^{[0,0,0,0,-1,0]} \tau_{l,m-1}^{[0,0,0,-1,1,0]} - \tau_{l+1,m}^{[0,0,0,0,0,-1]} \tau_{l-1,m}^{[0,0,0,-1,0,1]} \Big), \\ &\langle b_3^2\rangle \langle b_1^2\rangle p^{(l-m)/2} \Big((p^{l+m+1} b_4/b_5)^{1/2} \tau_{l,m-1}^{[0,0,0,0,-1,0]} \tau_{l,m+1}^{[0,0,0,0,0,-1]} \\ &\qquad - (p^{l+m+1} b_4/b_5)^{-1/2} \tau_{l+1,m}^{[0,0,0,0,0,-1]} \tau_{l-1,m}^{[0,0,0,0,-1,0]} \Big) \\ &= \langle p^{l+m+1} b_4/b_5\rangle \Big(\tau_{l,m}^{[0,1,-1,1,-1,-1]} \tau_{l,m}^{[0,-1,1,-1,0,0]} - \tau_{l,m}^{[0,-1,0,1,-1,-1]} \tau_{l,m}^{[0,1,0,-1,0,0]} \Big), \end{aligned} \tag{4.19}$$

where $\tau_{l,m}^{[j_0,j_1,j_2,j_3,j_4,j_5]} = \tau_{l,m}(p^{j_0} b_0, p^{j_1} b_1, p^{j_2} b_2, p^{j_3} b_3, p^{j_4} b_4, p^{j_5} b_5; p)$. Therefore our problem is reduced to constructing a family of functions $\tau_{l,m}(b;p)$ $(l,m \in \mathbb{Z})$ that satisfies the bilinear relations (4.18) and (4.19) and possesses the symmetry (4.15),(4.16) and (4.17).

§ 4.4. Construction of the function $\tau_{0,0}(b;p)$

First we construct the functions $\tau_{-1,-1}(b;p)=\tau_{-1,0}(b;p)=\tau_{0,-1}(b;p)=\tau_{0,0}(b;p)$. The bilinear relations (4.18) with $l=m=0$ yield to the equations for $\tau_{0,0}(b;p)$

$$
\begin{aligned}
&\langle b_2\rangle\langle b_2\rangle_+\langle b_0/b_1\rangle\tau_{0,0}^{[0,0,0,0,-1,0]}\tau_{0,0}^{[0,0,0,-1,1,0]}\\
&=\langle b_4\rangle_+\Big(\tau_{0,0}^{[0,-1,1,-1,0,0]}\tau_{0,0}^{[0,1,-1,0,0,0]}-\tau_{0,0}^{[-1,0,1,-1,0,0]}\tau_{0,0}^{[1,0,-1,0,0,0]}\Big),\\
&\langle b_3\rangle\langle b_3\rangle_+\langle b_5/b_4\rangle\tau_{0,0}^{[0,1,-1,0,0,0]}\tau_{0,0}^{[0,-1,0,0,0,0]}\\
&=\langle b_1\rangle_+\Big(\tau_{0,0}^{[0,0,-1,1,-1,0]}\tau_{0,0}^{[0,0,0,-1,1,0]}-\tau_{0,0}^{[0,0,-1,1,0,-1]}\tau_{0,0}^{[0,0,0,-1,0,1]}\Big)
\end{aligned}
\tag{4.20}
$$

due to the symmetry (4.15), where $\langle x\rangle_+=x+x^{-1}$.

Let us consider a pair of non-zero meromorphic functions $(G(x),F(x))$ satisfying the difference equations

$$F(px)=G(x)F(x),\quad G(px)=\langle x\rangle_+G(x),$$

and the reflection formula $F(p^2/x)=F(x)$. A typical choice of such functions is given by

$$
\begin{aligned}
G(x)&=e\left(\pm\tfrac{1}{4}(\xi/d-\tfrac{1}{2})\right)\frac{\left(\pm\sqrt{-1}\,px^{-1};p\right)_\infty}{\left(\pm\sqrt{-1}\,x;p\right)_\infty},\\
F(x)&=e\left(\pm\tfrac{1}{8d^2}(\xi-d)^2\right)\left(\pm\sqrt{-1}\,x,\pm\sqrt{-1}\,p^2x^{-1};p,p\right)_\infty,
\end{aligned}
$$

where $x=e(\frac{1}{4}\xi)$, $p=e(\frac{1}{4}d)$ and $e(\eta):=e^{2\pi\sqrt{-1}\,\eta}$, $(a;p)_\infty=\prod_{i=0}^{\infty}(1-ap^i)$, $(a;p,p)_\infty=\prod_{i,j=0}^{\infty}(1-ap^{i+j})$.

Proposition 4.3. We find that $\tau_{0,0}(b;p)=\prod_{x\in e^{\Delta_+}}F(px)$ satisfies the equations (4.20), and possesses the symmetry (4.16) and (4.17), where $e^{\Delta_+}:=\{b=e^\beta\mid\beta\in\Delta_+\}$ and Δ_+ is a set of positive roots for the D_5 root system. Further we have $\tau_{-1,-1}(b;p)=\tau_{-1,0}(b;p)=\tau_{0,-1}(b;p)=\tau_{0,0}(b;p)=\prod_{x\in e^{\Delta_+}}F(px)$ from the symmetry (4.15).

§ 4.5. Special polynomials

From the second relation of (4.19), together with the symmetry (4.17), we get the recurrence relation

$$
\begin{aligned}
&\langle b_0^2\rangle\langle b_1^2\rangle\langle b_3^2\rangle\tau_{l+1,m}^{[0,0,0,0,0,-1]}\tau_{l-1,m}^{[0,0,0,0,-1,0]}\\
&=\langle b_0^2\rangle p^{-l}(b_5/pb_4)^{1/2}\left(\tau_{l,m}^{[0,1,-1,1,-1,-1]}\tau_{l,m}^{[0,-1,1,-1,0,0]}-\tau_{l,m}^{[0,-1,0,1,-1,-1]}\tau_{l,m}^{[0,1,0,-1,0,0]}\right)\\
&+\langle b_1^2\rangle p^{l}(pb_4/b_5)^{1/2}\left(\tau_{l,m}^{[1,0,-1,1,-1,-1]}\tau_{l,m}^{[-1,0,1,-1,0,0]}-\tau_{l,m}^{[-1,0,0,1,-1,-1]}\tau_{l,m}^{[1,0,0,-1,0,0]}\right).
\end{aligned}
$$

It is sufficient to consider this recurrence relation for constructing $\tau_{l,m}(b;p)\,(l,m\in\mathbb{Z})$ due to the symmetry (4.17).

Let us introduce a family of functions $U_{l,m}(\varepsilon;p)=U_{l,m}(\varepsilon_1,\dots,\varepsilon_5;p)$ by

$$\tau_{l,m}(b;p)=\prod_{x\in e^{\Delta_+}}F(px)\,U_{l,m}(\varepsilon;p),$$

where the new variables $\varepsilon_i\,(i=1,2,\dots,5)$ are given by

$$b_0=\frac{1}{\varepsilon_1\varepsilon_2},\quad b_i=\frac{\varepsilon_i}{\varepsilon_{i+1}}\,(i=1,2,3,4),\quad b_5=\varepsilon_4\varepsilon_5.$$

Then we obtain the reccurence relation for $U_{l,m}(\varepsilon;p)$

$$\begin{aligned}
&-\langle\varepsilon_1\varepsilon_2\rangle\langle\varepsilon_1/\varepsilon_2\rangle\langle\varepsilon_3/\varepsilon_4\rangle U_{l+1,m}^{[0,0,0,0,0]}U_{l-1,m}^{[0,0,0,0,2]}\\
&=-\langle\varepsilon_1\varepsilon_2\rangle p^{-l}\varepsilon_5\\
&\quad\times\Big[\langle\varepsilon_2/\varepsilon_3\rangle_+\langle\varepsilon_1\varepsilon_3\rangle_+\langle\varepsilon_1/\varepsilon_4\rangle_+\langle\varepsilon_2\varepsilon_4\rangle_+U_{l,m}^{[1,-1,1,-1,1]}U_{l,m}^{[-1,1,-1,1,1]}\\
&\qquad-\langle\varepsilon_1/\varepsilon_3\rangle_+\langle\varepsilon_2/\varepsilon_4\rangle_+\langle\varepsilon_2\varepsilon_3\rangle_+\langle\varepsilon_1\varepsilon_4\rangle_+U_{l,m}^{[-1,1,1,-1,1]}U_{l,m}^{[1,-1,-1,1,1]}\Big]\\
&+\langle\varepsilon_1/\varepsilon_2\rangle p^{l}\varepsilon_5^{-1}\\
&\quad\times\Big[\langle\varepsilon_2/\varepsilon_3\rangle_+\langle\varepsilon_1/\varepsilon_3\rangle_+\langle\varepsilon_1\varepsilon_4\rangle_+\langle\varepsilon_2\varepsilon_4\rangle_+U_{l,m}^{[-1,-1,1,-1,1]}U_{l,m}^{[1,1,-1,1,1]}\\
&\qquad-\langle\varepsilon_1\varepsilon_3\rangle_+\langle\varepsilon_2/\varepsilon_4\rangle_+\langle\varepsilon_2\varepsilon_3\rangle_+\langle\varepsilon_1/\varepsilon_4\rangle_+U_{l,m}^{[1,1,1,-1,1]}U_{l,m}^{[-1,-1,-1,1,1]}\Big],
\end{aligned}\tag{4.21}$$

where $U_{l,m}^{[i_1,i_2,i_3,i_4,i_5]}=U_{l,m}(p^{i_1/2}\varepsilon_1,p^{i_2/2}\varepsilon_2,p^{i_3/2}\varepsilon_3,p^{i_4/2}\varepsilon_4,p^{i_5/2}\varepsilon_5;p)$, and the initial conditions

$$U_{-1,-1}=U_{-1,0}=U_{0,-1}=U_{0,0}=1.\tag{4.22}$$

Let us consider the symmetry of the family of functions $U_{l,m}(\varepsilon;p)$. First we get

$$U_{-l-1,-m-1}(\varepsilon;p)=U_{-l-1,m}(\varepsilon;p)=U_{l,-m-1}(\varepsilon;p)=U_{l,m}(\varepsilon;p)$$

from (4.15). In order to describe the symmetry that originates from (4.16) and (4.17), we introduce a group isomorphic to the Weyl group of D_5 and its counterpart.

Let $\mathfrak{S}_{2n}$ be a symmetric group on the set $\{\pm1,\pm2,\dots,\pm n\}$. We see that

$$\mathfrak{S}_n^{\pm}=\left\{\sigma\in\mathfrak{S}_{2n}\,\middle|\,\sigma(-i)=-\sigma(i),\ i=1,2,\dots,n\right\}$$

is a subgroup of order $2^n n!$ of $\mathfrak{S}_{2n}$, and that any element $\sigma\in\mathfrak{S}_n^{\pm}$ can be denoted by

$$\sigma=\begin{pmatrix}1 & 2 & \cdots & n\\ \epsilon_1k_1 & \epsilon_2k_2 & \cdots & \epsilon_nk_n\end{pmatrix},\qquad \epsilon_i=\pm1,\quad k_i=|\sigma(i)|.$$

We also see that $\mathfrak{S}_n^{+}=\left\{\sigma\in\mathfrak{S}_n^{\pm}\,\middle|\,\epsilon_1\epsilon_2\cdots\epsilon_n=+1\right\}$ is a subgroup of order $2^{n-1}n!$ of $\mathfrak{S}_n^{\pm}$, and $\mathfrak{S}_n^{+}\cong W(D_n)$.

Theorem 4.4. Define the action of $\sigma \in \mathfrak{S}_5^{\pm}$ on the variables $\varepsilon_1, \ldots, \varepsilon_5$ by $\sigma(\varepsilon_i) = \varepsilon_{k_i}^{\epsilon_i}$ $(i = 1, 2, \ldots, 5)$. Then we find that the family of functions $U_{l,m}(\varepsilon; p)$ satisfies

$$U_{l,m}(\sigma(\varepsilon); p) = U_{l,m}(\varepsilon; p), \sigma \in \mathfrak{S}_5^{+},$$
$$U_{m,l}(\sigma(\varepsilon); p) = U_{l,m}(\varepsilon; p), \sigma \in \mathfrak{S}_5^{-},$$

where $\mathfrak{S}_5^{-} = \mathfrak{S}_5^{\pm} - \mathfrak{S}_5^{+}$.

Further we observe that $U_{l,m}(\varepsilon; p)$ possesses the symmetry $U_{l,m}(\varepsilon; p^{-1}) = U_{l,m}(\varepsilon; p)$, which does not originate from the condition (4.14) but from (4.13).

It is obvious that $U_{l,m}(\varepsilon; p) \in \mathbb{Q}(\varepsilon_1, \ldots, \varepsilon_5, p)$ from the recurrence relation (4.21) and the initial conditions (4.22). Further we obtain from Theorem 3.3 in [11] the following.

Theorem 4.5. We have $U_{l,m}(\varepsilon; p) \in \mathbb{Z}[\varepsilon_1^{\pm 1}, \ldots, \varepsilon_5^{\pm 1}, p^{\pm 1}]$ and all the coefficients are non-negative integers.

Let $\chi_{\pi_\lambda}(t)$ be a character of the irreducible representation π_λ of $SO(10)$ associated with the highest weight λ, which are given by

$$\chi_{\pi_\lambda}(t) = \frac{\displaystyle\sum_{\sigma \in \mathfrak{S}_5} \sum_{\epsilon \in E,\, \epsilon_1 \cdots \epsilon_5 = 1} \operatorname{sgn}(\sigma)\, t_{\sigma(1)}^{\epsilon_1(\lambda_1+4)} t_{\sigma(2)}^{\epsilon_2(\lambda_2+3)} t_{\sigma(3)}^{\epsilon_3(\lambda_3+2)} t_{\sigma(4)}^{\epsilon_4(\lambda_4+1)} t_{\sigma(5)}^{\epsilon_5\lambda_5}}{\displaystyle\prod_{1 \le i < j \le 5} \left[\left(t_i + \frac{1}{t_i} \right) - \left(t_j + \frac{1}{t_j} \right) \right]},$$

where $\lambda \in \mathbb{Z}^5$ or $\lambda \in \mathbb{Z}^5 + \frac{1}{2}(1,1,1,1,1)$, and $E = \{\epsilon = (\epsilon_1, \ldots, \epsilon_5) \,|\, \epsilon_j = \pm 1\, (1 \le j \le 5)\}$. Introduce the notation $\chi_{2\lambda}(\varepsilon) := \chi_{\pi_\lambda}(\varepsilon^2)$. Then some members of the family of polynomials $U_{l,m}(\varepsilon; p)$ are expressed by

$$U_{1,0}(\varepsilon; p) = \chi_{(1,1,1,1,1)}(\varepsilon), \quad U_{0,1}(\varepsilon; p) = \chi_{(1,1,1,1,-1)}(\varepsilon),$$

$$U_{2,0}(\varepsilon; p) = (p^{-4} + p^4)\chi_{(3,1,1,1,1)}(\varepsilon) + (p^{-2} + p^2)\left[\chi_{(3,3,3,1,1)}(\varepsilon) + \chi_{(1,1,1,1,-1)}(\varepsilon)\right]$$
$$+ \left[\chi_{(3,3,3,3,3)}(\varepsilon) + \chi_{(3,3,1,1,-1)}(\varepsilon) + \chi_{(3,1,1,1,1)}(\varepsilon)\right],$$

$$U_{1,1}(\varepsilon; p) = \frac{p^{-5} + p^5}{p^{-1} + p} + \frac{p^{-3} + p^3}{p^{-1} + p}\chi_{(2,2,0,0,0)}(\varepsilon) + \frac{p^{-1} + p}{p^{-1} + p}\chi_{(2,2,2,2,0)}(\varepsilon).$$

Unfortunately we have not obtained an explicit formula for $U_{l,m}(\varepsilon; p)$. As we stated above the family of polynomials $U_{l,m}(\varepsilon; p)$ $(l, m \in \mathbb{Z})$ possesses the external symmetry described by the affine Weyl group of type $(A_1 + A_1)^{(1)}$. This suggests that the polynomials $U_{l,m}(\varepsilon; p)$ can be indicated by a pair of partitions $\lambda = (l, l-1, \ldots, 2, 1)$ and $\mu = (m, m-1, \ldots, 2, 1)$. Construction of an explicit formula is still an open problem.

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