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The Painlevé divisors of the autonomous 4-dimensional Painlevé-type equations

By

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Abstract

Following [1], we consider compactification of the Liouville tori for the autonomous 4-dimensional Painlevé-type equations by adjoining the Painlevé divisors. We can see that the genus 2 components of the Painlevé divisors are all isomorphic to the corresponding spectral curve. We illustrate the motivating examples for [21], which asserts that the genus 2 curves are uniquely determined by the Liouville tori of the autonomous 4-dimensional Painlevé-type equations. This may enable us to recover a linear problem from the nonlinear integrable systems.

§ 1. Introduction

One of the important characterizations of integrable systems is that they possess linear problem (Lax pair), even though the systems themselves are nonlinear. If we have a Lax pair, it is straightforward to write down its corresponding nonlinear equation. On the other hand, it is often not easy to find a Lax pair out of nonlinear integrable systems. This paper treats some examples to illustrate how singularity data of a linear equation can be read off from nonlinear equations alone.

There are 40 types of 4-dimensional Painlevé-type equations [28, 11, 8, 10, 9].[1] We study their autonomous/isospectral versions of them as in [18]. In Okounkov-Rains [24], Rains [26], the Painlevé equations are understood as noncommutative generalizations of the Hitchin systems. Our approach is to study the commutative cases, i.e. the Hitchin systems, in order to understand the noncommutative cases, i.e. the Painlevé-type systems.

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[1]See also Rains[25] for general framework for classifying the Hitchin/Painelvé systems.

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The seven examples we consider in this paper are all the cases among these 40 types that are weighted homogeneous polynomial Hamiltonian systems with positive weights for all the canonical variables. This corresponds to considering $P_{\mathrm{I}}(E_8)$, $P_{\mathrm{II}}(E_7)$, $P_{\mathrm{IV}}(E_6)$ and excluding $P_{\mathrm{III}}(D_8)$, $P_{\mathrm{III}}(D_7)$, $P_{\mathrm{III}}(D_6)$, $P_{\mathrm{V}}(D_5)$, $P_{\mathrm{VI}}(D_4)$, for the 2-dimensional Painlevé equations (c.f. [5, 6]). All these systems appeared in the Chiba's list [4] classifying regular weights in Saito's sense [27]. The main part of this paper is the Painlevé divisor. Even though the weights and the Kowalevskaya exponents for these examples have appeared in Chiba [4] already, we will repeat them since they are essential for deriving the Painlevé divisors.

We follow Adler, van Moerbeke, Vanhaecke [1] for computing the Painlevé divisors adjoined in compactification of the Liouville tori. In the 4-dimensional cases, the Painlevé divisors are curves, typically of genus 2. We can check that each genus 2 component of the Painlevé divisor is isomorphic to the corresponding spectral curve of the Lax pair. The result in [21] tells us that these components of the Painlevé divisors are isomorphic to the spectral curves. So, this is saying that family of curves on nonlinear side is the same as family of curves on linear side. Therefore, the generic degeneration of the components of the Painlevé divisors carries information about the singularities of the linear equations modulo interpretation [25]. The explicit construction of Lax pair needs further labor even after knowing the singularity type (spectral type) of the linear equation. Still, the point is that the Painlevé divisors encodes crucial information. Since we already know a Lax pair for each 40 types of the 4-dimensional system [28, 11, 8, 10, 9], just knowing the type of generic degeneration of the Painlevé divisors is enough to recover a linear equation. The present article shows examples with actual derivations of the Painlevé divisors. These computations are the motivation why we wanted to prove the uniqueness of the principal polarization in [21].

The organization of this paper is as follows. After reviewing the preliminaries from [31, 1] in the next section, we study examples of the Garnier systems $H^{\frac{9}{2}}_{\mathrm{Gar}}$, $H^{\frac{7}{2}}_{\mathrm{Gar}}$, H^{5}_{Gar}, H^{4+1}_{Gar} in Section 3. Section 4 treats $H^{A_4}_{\mathrm{NY}}$ and Section 5 treats the Matrix Painlevé equations $H^{\mathrm{Mat}}_{\mathrm{I}}$ and $H^{\mathrm{Mat}}_{\mathrm{II}}$.

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§ 2. Preliminaries

The autonomous Painlevé-type equations are algebraically completely integrable systems (a.c.i.) in the following sense.

Definition 2.1 ([1]). Let $(M, \{\cdot,\cdot\}, \mathbf{H})$ be a complex integrable system, where

$(M^{2n}, \{\cdot,\cdot\})$ is an affine nonsingular Poisson variety of rank $2n$ and $\mathbf{H} = (H_1, \ldots, H_n)$ are n functions in involution. We say $(M, \{\cdot,\cdot\}, \mathbf{H})$ is an algebraically completely integrable if for generic $h \in \mathbb{C}^n$ the fiber $\mathbf{H}_h = \mathbf{H}^{-1}(h)$ is an affine part of an Abelian variety and if the Hamiltonian vector fields X_{H_i} are translation invariant, when restricted to the fibers. The map $\mathbf{H}\colon M \to \mathbb{C}^n$ is called the momentum map.

All the examples we consider in this paper are weight-homogeneous in the following sense.

Definition 2.2 ([31, 1]). Let $\nu = (\nu_1, \ldots, \nu_n)$ be a weight vector, which is a collection of positive integers without a common divisor. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is weight-homogeneous of weight k with respect to $\nu = (\nu_1, \ldots, \nu_n)$ if

$$f(\alpha^{\nu_1} x_1, \ldots, \alpha^{\nu_n} x_n) = \alpha^k f(x_1, \ldots, x_n).$$

A polynomial vector field $\mathcal{V}$ on $\mathbb{C}^n$

$$\begin{aligned} \dot{x}_1 =& f_1(x_1, \ldots, x_n), \\ &\vdots \\ \dot{x}_n =& f_n(x_1, \ldots, x_n), \end{aligned}$$

is called a weight-homogeneous vector field with respect to ν if each of the polynomial $f_1, \ldots, f_n$ is weight-homogeneous with respect to $\nu = (\nu_1, \ldots, \nu_n)$ and if f_i has weighted degree $\nu_i + 1$ for each $i = 1, \cdots n$.

Here is how the formal Laurent series solutions to a weight-homogeneous system is determined.

Proposition 2.3 ([31, 1]). *Suppose that $\mathcal{V}$ is a weight-homogeneous vector field on $\mathbb{C}^n$, given by*

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \ (i = 1, \ldots, n),$$

and suppose that

$$x_i(t) = \sum_{j=0}^{\infty} x_i^{(j)} t^{-\nu_i + j}, \ (i = 1, \ldots, n)$$

is a weight-homogeneous Laurent solution for this vector field. Then the leading coefficients $x_i^{(0)}$ satisfy the non-linear algebraic equations

$$\begin{aligned} \nu_1 x_1^{(0)} + & f_1(x_1^{(0)}, \ldots, x_n^{(0)}) = 0, \\ &\vdots \\ \nu_n x_n^{(0)} + & f_n(x_1^{(0)}, \ldots, x_n^{(0)}) = 0. \end{aligned} \tag{2.1}$$

The subsequent terms $x_i^{(l)}$ *of the Laurent solution satisfy*

$$(l\,\mathrm{Id}_n - \mathcal{K}(x^{(0)}))x^{(l)} = R^{(l)},$$

where

$$x^{(l)} = \begin{pmatrix} x_1^{(l)} \\ \vdots \\ x_n^{(l)} \end{pmatrix}, \quad R^{(l)} = \begin{pmatrix} R_1^{(l)} \\ \vdots \\ R_n^{(l)} \end{pmatrix},$$

and each $R_i^{(l)}$ *is a polynomial of* $x_1^{(j)}, \ldots, x_n^{(j)}$ *with* $1 \leq j \leq l-1$. *The* (i,j)*-th entry of the* $(n \times n)$*-matrix* $\mathcal{K}$ *is the regular function on* $\mathbb{C}^n$ *defined by*

$$\mathcal{K}_{i,j} = \frac{\partial f_i}{\partial x_j} + \nu_i \delta_{i,j}. \tag{2.2}$$

Definition 2.4 ([31, 1])**.** The set of equations (2.1) for $x_i^{(0)}$'s is called the indicial equation of the vector field $\mathcal{V}$. Its solution set, which is an algebraic set in $\mathbb{C}^n$, is called the indicial locus, and is denoted by $\mathcal{I}$. Then matrix $\mathcal{K}$ defined in (2.2) is called the Kovalevskaya matrix. For any $m \in \mathcal{I}$, the eigenvalues of $\mathcal{K}$ evaluated for $x^{(0)} = m$ are called the Kovalevskaya exponents[2] for m.

When a nonnegative integer k is one of the Kovalevskaya exponents, $x^{(k)}$ is not determined by the coefficients of the previous terms and $x^{(k)}$ has at least one new free parameter. The Laurent solutions are organized in families, each family depending on a certain number of independent parameters. Each of these irreducible families of formal Laurent solutions is called a balance.

Proposition 2.5 ([1])**.** *Let* $\mathcal{V}$ *be a weight-homogeneous vector field on* $\mathbb{C}^n$. *The set of all weight-homogeneous Laurent solutions to* $\mathcal{V}$ *is parametrized by a finite number of affine varieties* $\Gamma^{(i)}$. *For any one of these affine varieties* $\Gamma^{(i)}$ *the coefficients that appear in the corresponding weight-homogeneous balance are regular functions on* $\Gamma^{(i)}$.

Definition 2.6 ([1])**.** Let $\mathcal{V}$ be a weight-homogeneous vector field on $\mathbb{C}^n$. The weight-homogeneous balance that corresponds to the affine variety $\Gamma^{(i)}$, as in proposition 2.5, is denoted by $x(t;\Gamma^{(i)})$. For $m \in \mathcal{I}$ the balance specializes to a Laurent series that will be denoted by $x(t;m)$. Each of the affine varieties $\Gamma^{(i)}$ restricted to the fiber of the momentum map is called the Painlevé divisor.

[2]This term is introduced by Yoshida [31] in honor of Kowalevskaya. Our notion deviates from the original [31] in the sense that we also included the eigenvalues corresponding to $m = (0, \cdots, 0)$ for our convenience.

Remark. For the 4-dimensional autonomous Painlevé-type systems, we have 3-dimensional affine varieties $\Gamma^{(i)}$'s corresponding to the principal balances, which are families of the Laurent series solutions with 3 free parameters. When restricted to generic level set of the momentum map, these 3-dimensional $\Gamma^{(i)}$'s give curves, components of the Painlevé divisors of the system. We also have 2-dimensional affine varieties corresponding to the 2-parameter families of Laurent series solutions. These 2-dimensional affine varieties restrict to points on the level set.

§ 3. The Garnier systems

The original Garnier system in 2 variables $H^{1+1+1+1+1}_{\mathrm{Gar},t_i}$ is a generalization of P_{VI} from the perspective of isomonodromic deformation. The systems we consider in this section are the autonomous versions of the degenerated systems of $H^{1+1+1+1+1}_{\mathrm{Gar},t_i}$.

§ 3.1. The Garnier system of type $\frac{9}{2}$

The Garnier system of type $\frac{9}{2}$ is derived from $H^{1+1+1+1+1}_{\mathrm{Gar},t_i}$ by degeneration in [13]. The Garnier system of type $\frac{9}{2}$ is also known in the literature as the second member of the first Painlevé hierarchy[15]. We use the Hamiltonians appeared in [9].[3] The Hamiltonians of the autonomous version are

$$\begin{aligned} H_1 = H^{\frac{9}{2}}_{\mathrm{Gar},s_1} =& p_1q_2^2 - p_1s_1 + p_2s_2 + p_1^4 + 3p_2p_1^2 + p_2^2 - 2q_1q_2, \\ H_2 = H^{\frac{9}{2}}_{\mathrm{Gar},s_2} =& p_1^2q_2^2 - 2p_1q_1q_2 + p_2q_2^2 + p_1^3s_2 + p_1s_2^2 + p_2p_1s_2 + p_2s_1 - p_2p_1^3 - 2p_2^2p_1 \\ & - q_2^2s_2 + q_1^2, \end{aligned}$$

where s_1, s_2 are constants. This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s_1, s_2)$
$(5, 2, 3, 4)$	$(8, 10, 6, 4)$

Table 1. The weights of $H^{\frac{9}{2}}_{\mathrm{Gar}}$

Remark. As observed in Chiba [4], the following holds.

$$\deg(q_i) + \deg(p_i) = \deg H_1 - 1, \quad i = 1, 2.$$

The same equality holds for all the cases in this note.

[3] We have exchanged $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$ with $H^{\frac{9}{2}}_{\mathrm{Gar},s_2}$, and s_1 with s_2 from those in [9] to make $\deg H_1$ smaller than $\deg H_2$.

The Hamiltonian system for $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$ is

$$\begin{aligned}\frac{dq_1}{dt} &= 4p_1^3 + 6p_2p_1 + q_2^2 - s_1, & \frac{dq_2}{dt} &= 3p_1^2 + 2p_2 + s_2,\\ \frac{dp_1}{dt} &= 2q_2, & \frac{dp_2}{dt} &= 2\left(q_1 - p_1q_2\right).\end{aligned}$$

This system has the following three types of balances corresponding to the initial terms $m_1 = (0,0,0,0)$, $m_2 = (-1,1,-1,0)$, $m_3 = (9,3,-3,-9)$.

The balance starting from $m_1 = (0,0,0,0)$ has $2,3,4,5$ as the Kowalevskaya exponents and has 4 free parameters in the Taylor series. When restricted to the fiber, this balance corresponds the affine part of the abelian surface. Details are in [20].

The balance starting from $m_2 = (-1,1,-1,0)$ has $-1,2,3,6$ as the Kowalevskaya exponents, so has 3 free parameters α,β,γ in its Laurent series.

$$\begin{aligned}q_1(t;m_2) =& -\frac{1}{t^5} + \frac{\alpha}{t^3} + \beta + \frac{1}{70}t\left(-35\alpha^3 - 18\alpha s_2 + 10s_1\right) - \frac{15}{2}t^2(\alpha\beta) + \gamma t^3 + O\left(t^4\right),\\ p_1(t;m_2) =& \frac{1}{t^2} + \frac{\alpha}{2} + t^2\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 4\beta t^3 + t^4\left(-\frac{5\alpha^3}{4} - \frac{6\alpha s_2}{7} + \frac{s_1}{7}\right) - 3t^5(\alpha\beta)\\ &+ t^6\left(-\frac{\alpha^4}{8} + \frac{\gamma}{11} + \frac{3\alpha^2 s_2}{385} + \frac{2\alpha s_1}{77} + \frac{18s_2^2}{275}\right) + O\left(t^7\right),\\ q_2(t;m_2) =& -\frac{1}{t^3} + t\left(-\frac{3\alpha^2}{4} - \frac{3s_2}{5}\right) - 6\beta t^2 + t^3\left(-\frac{5\alpha^3}{2} - \frac{12\alpha s_2}{7} + \frac{2s_1}{7}\right) - \frac{15}{2}t^4(\alpha\beta)\\ &+ t^5\left(-\frac{3\alpha^4}{8} + \frac{3\gamma}{11} + \frac{9\alpha^2 s_2}{385} + \frac{6\alpha s_1}{77} + \frac{54s_2^2}{275}\right) + O\left(t^6\right),\\ p_2(t;m_2) =& -\frac{3\alpha}{2t^2} + \left(\frac{3\alpha^2}{2} + s_2\right) + 6\beta t + t^2\left(\frac{9\alpha^3}{8} + \frac{9\alpha s_2}{10}\right)\\ &+ t^4\left(\frac{15\alpha^4}{32} + \frac{9\gamma}{22} - \frac{9\alpha^2 s_2}{308} - \frac{15\alpha s_1}{154} - \frac{27s_2^2}{110}\right) + O\left(t^5\right).\end{aligned}$$

Since $H_i(q_1,p_1,q_2,p_2)$ for $i = 1,2$ are constants of motion, and the above Laurent series are the solutions of the Hamiltonian system, $H_i(q_1(t;m_2),p_1(t;m_2),q_2(t;m_2),$ $p_2(t;m_2))$ are t-independent. Therefore, let us express the constants as h_1 and h_2, respectively. Then, $H_i(q_1(t;m_2),p_1(t;m_2),q_2(t;m_2),p_2(t;m_2)) = h_i$ are equivalent to

$$\begin{aligned}h_1 =& \frac{405\alpha^4}{32} + \frac{81\gamma}{22} + \frac{648\alpha^2 s_2}{77} - \frac{150\alpha s_1}{77} - \frac{23s_2^2}{110},\\ h_2 =& \frac{729\alpha^5}{64} + \frac{243\alpha\gamma}{44} + 81\beta^2 + \frac{1539\alpha^3 s_2}{616} - \frac{207\alpha^2 s_1}{308} - \frac{729\alpha s_2^2}{220} + s_1s_2.\end{aligned}$$

This gives the component of the Painlevé divisor. By eliminating γ, the equation becomes

$$-\frac{243\alpha^5}{32} + 81\beta^2 + \frac{3\alpha h_1}{2} - h_2 - \frac{81\alpha^3 s_2}{8} + s_1\left(\frac{9\alpha^2}{4} + s_2\right) - 3\alpha s_2^2 = 0. \tag{3.1}$$

If we take $x = \frac{3}{2}\alpha, y = 9\beta$, the equation becomes

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2. \tag{3.2}$$

When $h_1 = \infty$ and $h_2 = \infty$, the family of genus two curves (3.2) degenerates. Consider a one-parameter subfamily of genus two curves over the line $h_2 = \tilde{a}h_1 + \tilde{b}$, where $\tilde{a}$, $\tilde{b}$ are constants. We study the generic degeneration along the line, by which we mean the degeneration at $h_1 = \infty$ under the assumption $\tilde{a}$ and $\tilde{b}$ are not special.

In this example the generic degeneration is VII* in Namikawa-Ueno's notation [22]. The dual graph and the monodromy matrix are in Appendix. We can find the type of the special fiber of the minimal regular model from their Weierstrass model using Liu's algorithm [17, 16], which is a genus 2 counterpart of Tate algorithm for elliptic curves. See [18] for some more details, where the generic degenerations of the spectral curves for the 40 types of the autonomous 4-dimensional Painlevé-type equations were studied.

Remark. We have already written about the Painlevé divisor for the autonomous Garnier system of type $\frac{9}{2}$ in [20]. However, the degeneration considered in [20] is more special than the generic degeneration. In the light of Rains' work on the generalized Hitchin systems [25], the things to look for is the generic degeneration rather than more special degenerations.

The balance starting from $m_3 = (9, 3, -3, -9)$ has $-3, -1, 8, 10$ as the Kowalevskaya exponents, and has 2 free parameters. When restricted to the fiber of the momentum map, this balance corresponds to a point. The explicit form is in [20].

The situation can be summarized in the following table. Here, we have abbreviated the Kowalevskaya exponents as K-exponents.

K-exponents	# parameters	# balances	generic fiber	degeneration
(2,3,4,5)	4	1	an affine Abelian surface	
(-1,2,5,8)	3	1	a genus two curve	VII*
(-3,-1,8,10)	2	1	a point	

Table 2. The balances of $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$

Remark. Since the system we are considering are weight-homogeneous Hamiltonian (thus divergence-free) system, the trace of the Kowalevskaya matrix is

$$\mathrm{tr}(\mathcal{K}(m)) = \sum_{i=1}^{n} \nu_i,$$

as noted in [1]. Therefore, the sum of Kowalevskaya exponents are always $\sum_{i=1}^{n} \nu_i$, irrespective of the indicial locus $m \in \mathcal{I}$. Indeed, the sums of Kowalevskaya exponents

of each type is always 14 for the Garnier system of type $\frac{9}{2}$. The same equality holds for each case in this paper, which is useful for checking our computation.

Remark. As was shown in Yoshida [31] for more general setting, the weighted degree of the Hamiltonian, which is $\deg H_1 = 8$ in this case, appear as one of the Kowalevskaya exponents. Furthermore, when k is one of the Kowalevskaya exponents, then $\deg H_1 - 1 - k$ is also one of the Kowalevskaya exponents [31]. These facts are also useful for checking our computations.

On the other hand, the Lax pair for $H^{\frac{9}{2}}_{\mathrm{Gar},s_i}$ is as follows.

$$A(x) = A_0x^3 + A_1x^2 + A_2x + A_3,$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2s_1 \\ -p_1 & -q_2 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} q_1 - p_1q_2 & p_1^3 + 2p_1p_2 - q_2^2 + s_1p_1 - s_2 \\ -p_2 + s_1 & -q_1 + p_1q_2 \end{pmatrix}.$$

The spectral curve

$$\det(yI_2 - A(x)) = 0.$$

of the Garnier system of type $\frac{9}{2}$ is expressed as

$$y^2 = x^5 + 3s_2x^3 - s_1x^2 + (2s_2^2 - h_1)x + h_2 - s_1s_2.$$

Equation (3.2) is exactly same as the spectral curve of Garnier 9/2. Therefore, the generic degeneration is also of type VII*. This is one of the results studied in [18].

§ 3.2. The Garnier system of type $\frac{7}{2} + 1$

The Garnier system of type $\frac{7}{2}+1$ appeared in the context of degeneration in [12]. We use the Hamiltonians from [9]. Its restriction to the second member of the thirty-fourth Painlevé hierarchy is discussed in [14]. The autonomous versions of the Hamiltonians are

$$\begin{aligned} H_1 = H^{\frac{7}{2}+1}_{\mathrm{Gar},s_1} =& H_{\mathrm{I}}(s_1; q_1, p_1) + p_2(2q_1 - q_2^2 - s_2) + \theta^0 q_2, \\ H_2 = H^{\frac{7}{2}+1}_{\mathrm{Gar},s_2} =& p_2^2 - s_2p_2q_2^2 - s_2^2p_2 + \theta^0 s_2q_2 + 2p_1p_2q_2 - q_1q_2(p_2q_2 - \theta^0) \\ & - p_2q_1(q_1 - s_2) - s_1p_2 - \theta^0 p_1, \end{aligned}$$

where H_{I} is the Hamiltonian of the 2-dimensional P_{I}

$$H_{\mathrm{I}}(t;q,p) = p^2 - q^3 - tq.$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1,p_1,q_2,p_2)$	$\deg(H_1,H_2,s_1,s_2,\theta^0)$
$(2,3,1,4)$	$(6,8,4,2,5)$

Table 3. The weights of $H_{\mathrm{Gar}}^{\frac{7}{2}+1}$

The Hamiltonian system for $H_{\mathrm{Gar},s_1}^{\frac{7}{2}+1}$ is

$$\frac{dq_1}{dt} = 2p_1, \qquad \frac{dq_2}{dt} = 2q_1 - q_2^2 - s_2,$$
$$\frac{dp_1}{dt} = -2p_2 + 3q_1^2 + s_1, \qquad \frac{dp_2}{dt} = 2p_2q_2 - \theta^0.$$

The types of balance are summarized as follows.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,2,3,4)	4	1	an affine Abelian surface	
(-1,2,3,6)	3	2	a genus two curve	$\mathrm{II}^* - \mathrm{II}_0^*$
(-3,-1,6,8)	2	2	a point	

Table 4. The balances of $H_{\mathrm{Gar},s_1}^{\frac{7}{2}+1}$

The balance starting from $m_2 = (1,-1,-1,0)$ has $-1,2,3,6$ as the Kowalevskaya exponents, so has 3 free parameters.

$$\begin{aligned}
q_1(t;m_2) =& \frac{1}{t^2} + \alpha + t^2\left(-3\alpha^2 + \frac{6\alpha s_2}{5} - \frac{s_1}{5}\right) + t^3\left(\frac{4\alpha\beta}{3} - \frac{2\theta^0}{9}\right) + \gamma t^4 + O\left(t^5\right),\\
p_1(t;m_2) =& -\frac{1}{t^3} + t\left(-3\alpha^2 + \frac{6\alpha s_2}{5} - \frac{s_1}{5}\right) + t^2\left(2\alpha\beta - \frac{\theta_0}{3}\right) + 2\gamma t^3 + O\left(t^4\right),\\
q_2(t;m_2) =& -\frac{1}{t} + t\left(s_2 - 2\alpha\right) + \beta t^2 + t^3\left(-10\alpha^2 + \frac{32\alpha s_2}{5} - s_2^2 - \frac{2s_1}{5}\right)\\
&+ t^4\left(\frac{10\alpha\beta}{3} - \frac{2\theta^0}{9} - \beta s_2\right) + \frac{1}{15}t^5\left(-200\alpha^3 - 5\beta^2 + 10\gamma + 228\alpha^2 s_2\right.\\
&\left.-84\alpha s_2^2 - 8\alpha s_1 + 10s_2^3 + 4s_1s_2\right) + O\left(t^6\right),\\
p_2(t;m_2) =& \frac{3\alpha}{t^2} + \left(3\alpha s_2 - 6\alpha^2\right) + t\left(2\alpha\beta - \frac{\theta^0}{3}\right) - \frac{3}{5}t^2\left(15\alpha^3 - 6\alpha^2 s_2 + \alpha s_1\right) + O\left(t^3\right).
\end{aligned}$$

We find the component of the Painlevé divisor at (h_1, h_2) by substituting the Laurent series in the equation $H_i(q_1(t;m_2), p_1(t;m_2), q_2(t;m_2), p_2(t;m_2)) = h_i, i = 1, 2$.

$$\begin{aligned} h_1 =& -70\alpha^3 - 7\gamma + 42\alpha^2 s_2 - 6\alpha s_2^2 - 4\alpha s_1, \\ h_2 =& -129\alpha^4 - 3\alpha\beta^2 - 21\alpha\gamma + \beta\theta^0 + 45\alpha^3 s_2 + 9\alpha^2 s_2^2 - 3\alpha^2 s_1 - 3\alpha s_2^3 - 3\alpha s_1 s_2. \end{aligned}$$

By eliminating γ, the equation becomes

$$3\alpha\beta^2 - \beta\theta^0 - 81\alpha^4 + 81\alpha^3 s_2 - 9\alpha^2(s_1 + 3s_2^2) + 3\alpha(s_1 s_2 + s_2^3 - h_1) + h_2 = 0. \tag{3.3}$$

If we take $x = 3\alpha, y = \beta$, the equation becomes

$$xy^2 - \theta^0 y - x^4 + 3s_2 x^3 - (s_1 + 3s_2^2)x^2 + (s_1 s_2 + s_2^3 - h_1)x + h_2 = 0. \tag{3.4}$$

The balance starting from $m_3 = (0, 0, 1, 0)$ also has $-1, 2, 3, 6$ as the Kowalevskaya exponents, so has 3 free parameters.

$$\begin{aligned} q_1(t;m_3) =& \alpha + \beta t + t^2\left(3\alpha^2 + s_1\right) + t^3\left(2\alpha\beta - \frac{2\theta^0}{3}\right) + \gamma t^4 + O\left(t^5\right), \\ p_1(t;m_3) =& \frac{\beta}{2} + t\left(3\alpha^2 + s_1\right) + t^2\left(3\alpha\beta - \theta^0\right) + 2\gamma t^3 + O\left(t^4\right), \\ q_2(t;m_3) =& \frac{1}{t} + t\left(\frac{2\alpha}{3} - \frac{s_2}{3}\right) + \frac{\beta t^2}{2} + \frac{1}{45}t^3\left(50\alpha^2 + 4\alpha s_2 - s_2^2 + 18s_1\right) \\ &+ \frac{1}{18}t^4\left(10\alpha\beta - 4\theta^0 + \beta s_2\right) \\ &+ t^5\left(\frac{2\gamma}{7} - \frac{40\alpha^3}{189} - \frac{\beta^2}{28} + \frac{4\alpha^2 s_2}{45} + \frac{4\alpha s_2^2}{315} - \frac{8\alpha s_1}{105} - \frac{2s_2^3}{945} + \frac{4s_1 s_2}{105}\right) + O\left(t^6\right) \\ p_2(t;m_3) =& \theta^0 t + t^2\left(9\alpha^3 + \frac{3\beta^2}{2} - 3\gamma + 3\alpha s_1\right) + O\left(t^3\right). \end{aligned}$$

The component of the Painlevé divisor corresponding to this Laurent series is defined by the following intersection.

$$\begin{aligned} h_1 =& -10\alpha^3 - \frac{5\beta^2}{4} + 3\gamma - 4\alpha s_1, \\ h_2 =& \frac{1}{2}\left(-18\alpha^4 - 3\alpha\beta^2 + 6\alpha\gamma + \beta\theta^0 - 18\alpha^3 s_2 - 6\alpha^2 s_1 - 6\alpha s_1 s_2 - 3\beta^2 s_2 + 6\gamma s_2\right). \end{aligned}$$

Eliminating γ gives exactly the same equation as equation (3.3). Therefore, the Painlevé divisor consists of two isomorphic curves of genus 2.

On the other hand, the matrix giving its Lax pair is

$$A(x) = \frac{A_3}{x} + A_2 + A_1 x + A_0 x^2$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & q_1 - 2s_2 \\ 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} -p_1 & -p_2 + q_1^2 - s_2 q_1 + s_1 + s_2^2 \\ -q_1 - s_2 & p_1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} q_2 \\ 1 \end{pmatrix} \begin{pmatrix} p_2 & -p_2 q_2 + \theta^0 \end{pmatrix}.$$

The spectral curve of this Lax pair is

$$y^2 - \frac{\theta^0 y}{x} - x^3 + 3s_2 x^2 - (s_1 + 3s_2^2)x - H_1 + s_2^3 + s_1 s_2 + \frac{H_2}{x} = 0.$$

The equation is the same as the Painlevé divisor (3.4) if we multiply by x.

§ 3.3. The Garnier system of type 5

The Garnier system of type 5 is derived in [13] by degeneration. Its restriction to the second member of the P_{II} is discussed in [14]. We use the Hamiltonians from [11].

$$H_1 = H^5_{\mathrm{Gar},s_1} = -q_1(q_1 p_1 - \theta_1^\infty) + q_2(q_1(p_2 + q_2) - 2p_1 + s_1) + p_1(p_2 - 2s_2),$$
$$H_2 = H^5_{\mathrm{Gar},s_2} = H_{\mathrm{IV}}(0, \theta_1^\infty; 2s_2; q_2, p_2) + q_1 q_2(q_1 q_2 - 2p_1 + s_1) + p_1(p_1 - p_2 q_1 - s_1),$$

where H_{IV} is the Hamiltonian of the 2-dimensional P_{IV}

$$H_{\mathrm{IV}}(\alpha, \beta; t; q, p) = pq(p - q - t) + \beta p + \alpha q. \tag{3.5}$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s_1, s_2, \theta_1^\infty)$
$(1, 3, 2, 2)$	$(5, 6, 3, 2, 4)$

Table 5. The weights of H^5_{Gar}

The Hamiltonian system is

$$\frac{dq_1}{dt} = p_2 - 2(q_2 + s_2) - q_1^2, \qquad \frac{dq_2}{dt} = p_1 + q_1 q_2,$$
$$\frac{dp_1}{dt} = 2p_1 q_1 - q_2(p_2 + q_2) - \theta_1^\infty, \quad \frac{dp_2}{dt} = -p_2 q_1 + 2p_1 - 2q_1 q_2 - s_1.$$

The types of balances are summarized in the following table.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,2,2,3)	4	1	an affine Abelian surface	
(-1,1,3,5)	3	2	a genus two curve	$\mathrm{IX}-3$
(-2,-1,5,6)	2	3	a point	

Table 6. The balances of H^5_{Gar,s_1}

For example, the component of the Painlevé divisor corresponding to the balance starting from $(-1,1,-1,0)$ has following equation.

$$\begin{aligned}&y^2+4(x^3+2s_2x-s_1)y-5x^6-20s_2x^4+10s_1x^3-(20s_2^2+36\theta_1^\infty)x^2\\&+(20s_1s_2+36h_1)x-36h_2-5s_1^2=0.\end{aligned}$$

The following change of variables

$$x=\tilde{x}, y=3\tilde{y}+2\tilde{x}^3+4s_2\tilde{x}+2s_1,$$

turns the equation into the Weierstrass form

$$\tilde{y}^2=\tilde{x}^6+4s_2\tilde{x}^4+2s_1\tilde{x}^3+4(\theta_1^\infty+s_2^2)\tilde{x}^2 4(h_1+s_1s_2)\tilde{x}+4h_2+s_1^2 \tag{3.6}$$

On the other hand, the Lax pair for H^5_{Gar} in [11] is given by the following matrix

$$A(x)=A_4x^3+A_3x^2+A_2x+A_1,$$

where

$$\begin{aligned}&A_4=\begin{pmatrix}0&0\\0&1\end{pmatrix},\ A_3=\begin{pmatrix}0&q_2\\-1&0\end{pmatrix},\ A_2=\begin{pmatrix}-q_2&-p_1\\-q_1&q_2+2s_2\end{pmatrix},\\&A_1=\begin{pmatrix}p_1-q_1q_2&p_1q_1-(p_2-q_2-2s_2)q_2-\theta_1^\infty\\-p_2&-p_1+q_1q_2+s_1\end{pmatrix}.\end{aligned}$$

This gives the following spectral curve.

$$y^2-(x^3+2s_2x+s_1)y-\theta_1^\infty x^2-h_1x-h_2=0.$$

The following change of variables

$$x=\tilde{x},\ y=\frac{1}{2}\left(\tilde{y}+\tilde{x}^3+2s_2\tilde{x}+s_1\right)$$

gives exactly the same equation as (3.6). Therefore, these curves are isomorphic.

§ 3.4. The Garnier system of type $4+1$

The Garnier system of type $4+1$ is derived in [13] by degeneration. Its restriction to the second member of the fourth Painlevé hierarchy is discussed in [14]. We use the Hamiltonians from [11].

$$\begin{aligned}H_1=H^{4+1}_{\mathrm{Gar},s_1}=&H_{\mathrm{II}}\left(-\theta_1^\infty;s_1;q_1,p_1\right)+p_2q_2(q_1-q_2+s_2)+p_1p_2+\theta^0q_2,\\H_2=H^{4+1}_{\mathrm{Gar},s_2}=&-{p_2}^2q_2-s_2p_2{q_2}^2+{s_2}^2p_2q_2+\theta^0s_2q_2-\theta_1^\infty p_2+p_1p_2(q_1-2q_2+s_2)\\&+q_1q_2(p_2q_2-\theta^0)+\theta^0p_1+s_1p_2q_2,\end{aligned}$$

where, H_{II} is the Hamiltonian of the 2-dimensional P_{II}

$$H_{\mathrm{II}}\left(\alpha;t;q,p\right)=\ p^2-(q^2+t)p-\alpha q,$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1,p_1,q_2,p_2)$	$\deg(H_1,H_2,s_1,s_2,\theta^0,\theta_1^\infty)$
$(1,2,1,2)$	$(4,5,2,1,3,3)$

Table 7. The weights of H^{4+1}_{Gar}

The Hamiltonian system for $H^{4+1}_{\mathrm{Gar},s_1}$ is

$$\begin{aligned}\frac{dq_1}{dt}&=2p_1+p_2-q_1^2-s_1, & \frac{dq_2}{dt}&=p_1+q_2\left(q_1-q_2+s_2\right),\\ \frac{dp_1}{dt}&=2p_1q_1-p_2q_2-\theta_1^\infty, & \frac{dp_2}{dt}&=\ -p_2\left(q_1-2q_2+s_2\right)-\theta^0.\end{aligned}$$

We will skip the detail, but the situation is summarized in the following table. From direct computation, we can check that 3 genus 2 curves corresponding to the balances with 3 free parameters is isomorphic to the spectral curve.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,1,2,2)	4	1	an affine Abelian surface	
(-1,1,2,4)	3	3	a genus two curve	$\mathrm{III}^*-\mathrm{II}_0^*$
(-2,-1,4,5)	2	5	a point	

Table 8. The balances of $H^{4+1}_{\mathrm{Gar},s_1}$

For instance, the balance starting from $(-1,1,-1,0)$ has $-1,1,2,4$ as the Kowalevskaya exponents and the corresponding component of the Painlevé divisor has the following form.

$$\begin{aligned}&2xy^2+12x^3y+16s_2x^2y+(2s_1+16s_2^2)xy+3\theta^0y-54x^5-96s_2x^4-(30s_1+58s_2^2)x^3\\&-(28s_1s_2+12s_2^3-27\theta^0-36\theta_1^\infty)x^2+(30\theta^0s_2+18\theta_1^\infty s_2-4s_1^2-6s_1s_2^2-18h_1)x\\&-9h_2+9\theta^0s_2^2+6\theta^0s_1=0.\end{aligned}$$

On the other hand, the Lax pair in [11] gives the following spectral curve for $H^{4+1}_{\mathrm{Gar},s_1}$.

$$y^2+\left(x^2+2s_2x+s_1+s_2^2-\frac{\theta^0}{x}\right)y+\theta_1^\infty x-h_1+\theta_1^\infty s_2-\frac{h_2}{x}=0.$$

We can check that each of the following ratios of the Igusa invariants

$$\frac{J_4}{J_2^2},\ \frac{J_6}{J_2^3},\ \frac{J_{10}}{J_2^5},$$

for the components of the Painlevé divisor and the spectral curve is the same, where J_{2i}'s are the Igusa invariants [7]. Therefore, these curves are all isomorphic genus two curves.

§ 4. The Noumi-Yamada system of type $A_4^{(1)}$

The autonomous Noumi-Yamada system [23] of type $A_4^{(1)}$ is

$$\frac{df_i}{dt} = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \quad (i \in \mathbb{Z}/5\mathbb{Z}), \tag{4.1}$$

with the autonomous condition $\sum_{i=0}^{5} \alpha_i = 0$. These systems can be expressed as the Hamiltonian systems with the Hamiltonians

$$\begin{aligned}
H_1 =& H_{\mathrm{NY}}^{A_4} = H_{\mathrm{IV}}\left(\theta_3^\infty, \theta_1^\infty - \theta_3^\infty; s; q_1, p_1\right) + H_{\mathrm{IV}}\left(\theta_1^0 + \theta_2^\infty, \theta_1^\infty - \theta_2^\infty - \theta_3^\infty; s; q_2, p_2\right) \\
&+ 2q_1p_1p_2, \\
H_2 =& \left(q_1 - q_2\right)\left(p_1 + p_2 - q_2 - s\right)\left(p_2\left(\theta_1 + \theta_2 + \theta_1^\infty + \theta_2^\infty\right) + p_1p_2q_1\right) \\
&+ p_1\left(q_1 - q_2\right)\left(\theta_1^\infty p_2 + \theta_1 q_1\right) + \theta_2^\infty p_1 q_1\left(p_1 + p_2 - q_2 - s\right) \\
&+ \theta_2^\infty\left(\theta_1^\infty p_1 + \left(\theta_1 + \theta_2 + \theta_1^\infty + \theta_2^\infty\right)\left(p_1 + p_2 - s\right)\right) \\
&+ \left(\theta_1 + \theta_2 + \theta_1^\infty + \theta_2^\infty\right)\left(\theta_1 q_1 - q_2\left(\theta_1 + \theta_2^\infty\right)\right).
\end{aligned}$$

where H_{IV} is the Hamiltonian (3.5) of the 2-dimensional fourth Painlevé equation and s is a constant. The correspondence of the canonical variables and the originals is as follows:

$$p_1 = f_2,\ q_1 = -f_1,\ p_2 = f_4,\ q_2 = -f_1 - f_3.$$

The Hamiltonian system is

$$\begin{aligned}
\frac{dq_1}{dt} &= q_1\left(-q_1 + 2p_1 + 2p_2 - s\right) + \theta_1 + \theta_2 + 2\theta_1^\infty + \theta_2^\infty, \\
\frac{dp_1}{dt} &= p_1\left(2q_1 - p_1 - 2p_2 + s\right) + \theta_1 + \theta_2 + \theta_1^\infty + \theta_2^\infty, \\
\frac{dq_2}{dt} &= q_2\left(-q_2 + 2p_2 - s\right) + 2p_1q_1 + \theta_1 + \theta_2 + 2\theta_1^\infty, \\
\frac{dp_2}{dt} &= p_2\left(2q_2 - p_2 + s\right) - \theta_1 - \theta_2^\infty.
\end{aligned}$$

The earlier appearance of these equations (4.1) is in Adler [2] and Veselov-Shabat [30] In the autonomous situation, we have another functionally independent function in involution, which we denote by H_2. H_2 can be explicitly computed from one of the coefficients of the characteristic polynomial of matrix $A(x)$ giving its Lax pair. This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s, \theta_1^0, \theta_1^\infty, \theta_2^\infty, \theta_3^\infty)$
$(1, 1, 1, 1)$	$(3, 5, 1, 2, 2, 2, 2)$

Table 9. The weights of $H_{\mathrm{NY}}^{A_4}$

The balances for $H_{\mathrm{NY}}^{A_4}$ can be summarized as follows.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,1,1,1)	4	1	an affine Abelian surface	
(-1,1,1,3)	3	5	a genus two curve	$\mathrm{VI}^* - \mathrm{II}_3$
(-3,-1,3,5)	2	5	a point	
(-1,-1,3,3)	2	5	a point	

Table 10. The balances of $H_{\mathrm{NY}}^{A_4}$

Remark. A space which parametrizes all the meromorphic solutions of $H_{\mathrm{NY}}^{A_4}$ is constructed by Tahara [29]. The relations between each formal solution and Bäcklund transformation are also discussed there.

The balance starting from $m_2 = (-1, 0, -1, -1)$, $m_3 = (0, 1, 0, 0)$, $m_4 = (0, -1, 0, 1)$, $m_5 = (1, 0, 0, 0)$, $m_6 = (0, 0, 1, 0)$ all have $-1, 1, 1, 3$ as the Kowalevskaya exponents, so has 3 free parameters. The equations of these components of the Painlevé divisor is different from that of spectral curve. For instance, the equation for the component of the Painlevé divisor corresponding to the balance starting from $m_2 = (-1, 0, -1, -1)$ is as follows.

$$\begin{aligned}&(x+s)^2y^3 + (x+s)(x^2 - s^2 + 4(\theta_1 - \theta_2^\infty))y^2 + 4(\theta_1 + \theta_2 + \theta_2^\infty)(x^2 - s^2)y \\ &+ 8(h_1 - (\theta_1^\infty - \theta_2^\infty)s)(x+s)y - 16\theta_1\theta_2^\infty y + 16\theta_1^\infty\theta_3^\infty(x+s) - 32h_2 = 0.\end{aligned}$$

On the other hand, the Lax pair in [11] gives the following spectral curve for $H_{\mathrm{NY}}^{A_4}$.

$$\begin{aligned}&\frac{1}{x^4}(x^4y^3 + xy^2\left(sx + (-\theta_1 - \theta_2)\,x^2 - 1\right) + y\left(x\left(h_1 - s\left(\theta_1 + \theta_2 + \theta^\infty\right)\right)\right. \\ &\quad \left. +\theta_1 + \left(\theta_1^\infty\theta_2^\infty + \theta_3^\infty\theta_2^\infty + \theta_1^\infty\theta_3^\infty\right)x^2\right) + h_2 - \theta_1^\infty\theta_2^\infty x\left(\theta_1 + \theta_2 + \theta_1^\infty + \theta_2^\infty\right))\end{aligned}$$

As in the case of $H^{4+1}_{\mathrm{Gar},s_1}$, we can check that each ratio of the Igusa invariants for the components of the Painlevé divisor and the spectral curve is the same. Therefore, these curves are all isomorphic genus two curves.

§ 5. The Matrix Painlevé systems

The 4-dimensional sixth matrix Painlevé system $H^{\mathrm{Mat}}_{\mathrm{VI}}$ is derived from isomonodromic deformation of the Fuchsian equation in [28]. It is also known from [3]. The systems we consider can be obtained by degeneration.

§ 5.1. The first matrix Painlevé system

The first matrix Painlevé system is derived as a degeneration of $H^{\mathrm{Mat}}_{\mathrm{VI}}$ in [8]. In the autonomous situation, we also have another functionally independent function in involution, which we denote by H_2.

$$\begin{aligned}
H_1 =& H^{\mathrm{Mat}}_{\mathrm{I}} = \frac{p_1^2}{2} - 2p_2\left(p_2q_2 - \theta_2^\infty\right) - 2q_1 s - 2q_1\left(q_1^2 - q_2\right) + 4q_1q_2,\\
\tilde{H}_2 =& q_2\left(p_1p_2 + 3q_1^2 - q_2 + s\right)^2 - \theta_2^\infty p_1\left(p_1p_2 + 3q_1^2 - q_2 + s\right) - 2(\theta_2^\infty)^2 q_1,\\
H_2 =& \tilde{H}_2 + \frac{1}{4}H_1^2.
\end{aligned}$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s, \theta_2^\infty)$
$(2, 3, 4, 1)$	$(6, 12, 4, 5)$

Table 11. The weights of $H^{\mathrm{Mat}}_{\mathrm{I}}$

The Hamiltonian system for $H^{\mathrm{Mat}}_{\mathrm{I}}$ is given as follows

$$\begin{aligned}
\frac{dq_1}{dt} &= p_1, & \frac{dq_2}{dt} &= -4p_2q_2 + 2\theta_2^\infty,\\
\frac{dp_1}{dt} &= 2\left(3q_1^2 - 3q_2 + s\right), & \frac{dp_2}{dt} &= 2(p_2^2 - 3q_1).
\end{aligned}$$

The balances are summarized as follows.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,2,3,4)	4	1	an affine Abelian surface	
(-1,2,3,6)	3	1	a genus 2 curve	$\mathrm{I}_0 - \mathrm{II}^* - 1$
(-1,2,3,6)	3	1	a genus 4 curve	
(-2,-1,6,7)	2	1	a point	
(-7,-1,6,12)	2	1	a point	

Table 12. The balances of $H_{\mathrm{I}}^{\mathrm{Mat}}$

The balance starting from $m_2 = \left(\frac{1}{2}, -1, -\frac{1}{4}, 1\right)$ has $-1, 2, 3, 6$ as the Kovalevskaya exponents, so has 3 free parameters α, β, γ

$$\begin{aligned}
q_1(t;m_2) =& \frac{1}{2t^2} + \alpha + \beta t + t^2\left(6\alpha^2 + \frac{2s}{5}\right) + 4\alpha\beta t^3 + \gamma t^4 + O\left(t^5\right),\\
p_1(t;m_2) =& -\frac{1}{t^3} + \beta + t\left(12\alpha^2 + \frac{4s}{5}\right) + 12\alpha\beta t^2 + 4\gamma t^3 + O\left(t^4\right),\\
q_2(t;m_2) =& -\frac{1}{4t^4} + \frac{\alpha}{t^2} + \frac{\beta}{t} + \left(5\alpha^2 + \frac{3s}{5}\right) + 2\alpha\beta t + t^2\left(12\alpha^3 + \beta^2 - \gamma + \frac{4\alpha s}{5}\right)\\
&+ O\left(t^3\right),\\
p_2(t;m_2) =& \frac{1}{t} + 2\alpha t + 3\beta t^2 + t^3\left(28\alpha^2 + \frac{12s}{5}\right) + t^4\left(30\alpha\beta - 2\theta_2^\infty\right)\\
&+ t^5\left(224\alpha^3 + 18\beta^2 - 6\gamma + \frac{96\alpha s}{5}\right) + O\left(t^6\right).
\end{aligned}$$

We find the component of the Painlevé divisor at (h_1, h_2) by substituting the Laurent series in the equation $H_i(q_1(t;m_2), p_1(t;m_2), q_2(t;m_2), p_2(t;m_2)) = h_i, i = 1, 2$.

$$\begin{aligned}
h_1 =& 160\alpha^3 + 15\beta^2 - 14\gamma + 12\alpha s,\\
h_2 =& 2\alpha(\theta_2^\infty)^2 - \frac{1}{4}\left(176\alpha^3 + 13\beta^2 - 14\gamma + 16\alpha s\right)^2 + \frac{1}{4}h_1^2.
\end{aligned}$$

By eliminating γ, the equation becomes

$$2\alpha(\theta_2^\infty)^2 - \frac{1}{4}\left(16\alpha^3 - 2\beta^2 + h_1 + 4\alpha s\right)^2 + \frac{1}{4}h_1^2 - h_2 = 0.$$

If we express the curve in $x = 2\alpha, y = \beta$, we have

$$\left(y^2 - x^3 - sx - \frac{h_1}{2}\right)^2 - (\theta_2^\infty)^2 x - \frac{1}{4}h_1^2 + h_2 = 0. \tag{5.1}$$

This is a curve of genus 2.

The balance starting from $m_3 = \left(0,0,0,-\frac{1}{2}\right)$ also has $-1,2,3,6$ as Kovalevskaya exponents and has 3 parameters.

$$\begin{aligned}
q_1(t;m_3) =&\alpha + \beta t + t^2\left(3\alpha^2+s\right) + 2t^3\left(\alpha\beta+\theta_2^\infty\right) + \gamma t^4 + O\left(t^5\right),\\
p_1(t;m_3) =&\beta + t\left(6\alpha^2+2s\right) + 6t^2\left(\alpha\beta+\theta_2^\infty\right) + 4\gamma t^3 + O\left(t^4\right),\\
q_2(t;m_3) =&-2\theta_2^\infty t + t^2\left(6\alpha^3+\beta^2-2\gamma+2\alpha s\right) + O\left(t^3\right),\\
p_2(t;m_3) =&-\frac{1}{2t} - 2\alpha t - \frac{3\beta t^2}{2} + t^3\left(-2\alpha^2-\frac{6s}{5}\right) - 2\theta_2^\infty t^4\\
&+\frac{1}{70}t^5\left(160\alpha^3+45\beta^2-60\gamma+96\alpha s\right) + O\left(t^6\right).
\end{aligned}$$

The component of the Painlevé divisor corresponding to this balance is

$$\begin{aligned}
h_1 =&-5\alpha^3+\gamma-3\alpha s,\\
h_2 =&\frac{1}{4}\beta^2\left(6\alpha^3+\beta^2-2\gamma+2\alpha s\right) + \theta_2^\infty(3\alpha^2\beta+\beta s-2\alpha\theta_2^\infty) + \frac{1}{4}h_1^2.
\end{aligned}$$

In terms of $x=\alpha, y=\beta$, the equation is

$$y^4 - 4\left(x^3+sx-\frac{1}{2}h_1\right)y^2 + 4\theta_2^\infty\left(3x^2+s\right)y - 8(\theta_2^\infty)^2x + h_1^2 - 4h_2 = 0.$$

This is a curve of genus 4.

The balance starting from $m_4 = (1,-2,0,-2)$ and $m_5 = (1,-2,0,\frac{3}{2})$ has only 2 nonnegative integers among the Kovalevskaya exponents, so have only 2 free parameters. Each of them corresponds to a point in the level set $\cap_{i=1,2}\{H_i = h_i\}$.

On the other hand, a linear problem for the Matrix P_{I} is as follows [8].

$$A(x) = A_0x^2 + A_1x + A_2,$$

where

$$\begin{aligned}
&A_0 = \begin{pmatrix} O & I\\ O & O\end{pmatrix},\ A_1 = \begin{pmatrix} O & Q\\ I & O\end{pmatrix},\ A_2 = \begin{pmatrix} -P & Q^2+s\\ -Q & P\end{pmatrix},\\
&Q = \begin{pmatrix} q_1 & u\\ -q_2/u & q_1\end{pmatrix},\ P = \begin{pmatrix} p_1/2 & -p_2u\\ (p_2q_2-\theta_2^\infty)/u & p_1/2\end{pmatrix}.
\end{aligned}$$

The (affine) spectral curve defined by $\det(yI - A(x)) = 0$ is

$$\begin{aligned}
y^4 - \left(2x^3+2sx+H_1\right)y^2 + x^6 + 2sx^4 + H_1x^3\\
+ s^2x^2 + \left(H_1s - (\theta_2^\infty)^2\right)x + H_2 = 0.
\end{aligned} \tag{5.2}$$

This is exactly the same as the component of the Painlevé divisor (5.1) under the identification $H_1 = h_1, H_2 = h_2$.

§ 5.2. The second matrix Painlevé system

We use the Hamiltonians from [11].

$$\begin{aligned} H_1 =& H_{\mathrm{II}}^{\mathrm{Mat}} = \operatorname{tr}\left(P^2 - (Q^2+s)P - \theta_1^\infty Q\right) \\ =& -2(q_1+p_2)\left(p_2q_2 - \theta_1^\infty - \theta_2^\infty\right) - p_1\left(q_1^2 - q_2 + s\right) - 2p_2q_1q_2 + \frac{p_1^2}{2} + 2\theta_1^\infty q_1, \\ H_2 =& q_2\left(p_1p_2 + 3q_1^2 - q_2 + s\right)^2 - \theta_2^\infty\left(p_1\left(p_1p_2 + 3q_1^2 - q_2 + s\right) + 2\theta_2^\infty q_1\right), \end{aligned}$$

where the matrices Q and P are

$$Q = \begin{pmatrix} q_1 & 1 \\ -q_2 & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1/2 & -p_2 \\ p_2q_2 - \theta_1^\infty - \theta_2^\infty & p_1/2 \end{pmatrix}.$$

The Hamiltonian system for $H_{\mathrm{II}}^{\mathrm{Mat}}$ is

$$\begin{aligned} \frac{dq_1}{dt} &= p_1 - q_1^2 + q_2 - s, & \frac{dq_2}{dt} &= 2\left(-2p_2q_2 - 2q_1q_2 + \theta_1^\infty + \theta_2^\infty\right), \\ \frac{dp_1}{dt} &= 2p_1q_1 + 4p_2q_2 - 2\left(2\theta_1^\infty + \theta_2^\infty\right), & \frac{dp_2}{dt} &= 2p_2\left(p_2 + 2q_1\right) - p_1. \end{aligned}$$

This is a weight-homogeneous Hamiltonian system with the following weights.

$\deg(q_1, p_1, q_2, p_2)$	$\deg(H_1, H_2, s, \theta_1^\infty, \theta_2^\infty)$
$(1, 2, 2, 1)$	$(4, 8, 2, 3, 3)$

Table 13. The weights of $H_{\mathrm{II}}^{\mathrm{Mat}}$

The balances for this system are as follows.

K-exponents	# parameters	# balances	generic fiber	degeneration
(1,1,2,2)	4	1	an affine Abelian surface	
(-1,1,2,4)	3	1	a genus 5 curve	
(-1,1,2,4)	3	2	a genus 2 curve	$\mathrm{I}_0 - \mathrm{III}^* - 1$
(-1,-1,4,4)	2	1	two points	
(-2,-1,4,5)	2	2	a point	
(-5,-1,4,8)	2	2	a point	

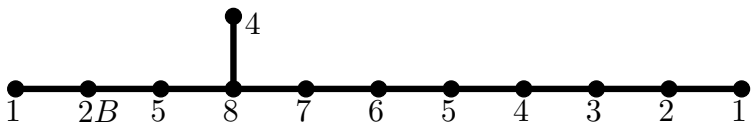
Table 14. The balances of $H_{\mathrm{II}}^{\mathrm{Mat}}$

The equations for the components of the Painlevé divisor is complicated so we will not write down. However, we can check that genus 2 components are isomorphic to the spectral curve, by comparing the ratios of the Igusa invariants.

Remark. The Bäcklund transformations induced by the Katz operations for the linear equation is studied for the matrix Painlevé equations in [19]. The situation having two isomorphic genus 2 curves and a genus 5 curve, as above, might suggest there is no extra Bäcklund transformations for $H_{\mathrm{II}}^{\mathrm{Mat}}$ other than those $A_1^{(1)}$ symmetry for 2-dimensional H_{II}. This needs further clarification.

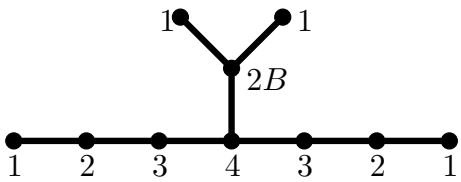
§ Appendix A. Generic degenerations

We list the dual graphs and the monodromy matrix [22] of the special fibers of genus two curves appeared in the examples. For the spectral curves, these special fibers have information of the spectral type modulo suitable interpretation [25]. The generic degenerations of the spectral curves for the 40 types of the autonomous 4-dimensional Painlevé-type equations were studied in [18].



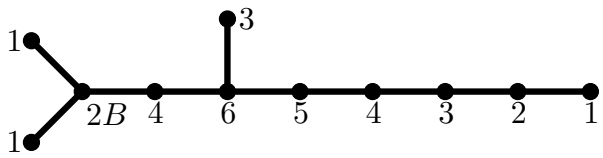
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1. $H^{\frac{9}{2}}_{\mathrm{Gar},s_1}$ VII^*



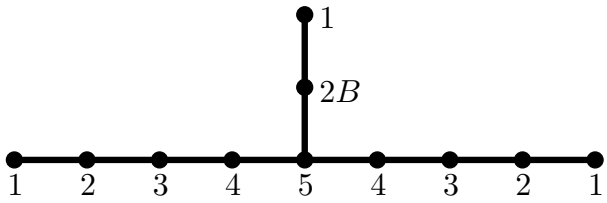
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Figure 2. $H^{4+1}_{\mathrm{Gar},s_1}$ $\mathrm{III}^* - \mathrm{II}_0^*$



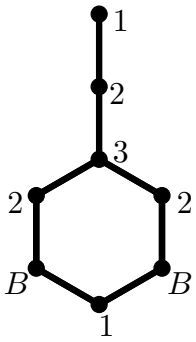
$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Figure 3. $H^{\frac{7}{2}+1}_{\mathrm{Gar},s_1}$ $\mathrm{II}^* - \mathrm{II}_0^*$



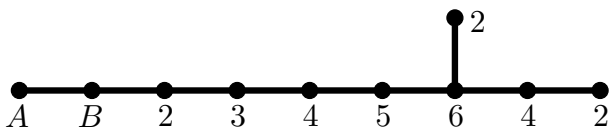
$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 4. H^5_{Gar,s_1} $\mathrm{IX}-3$



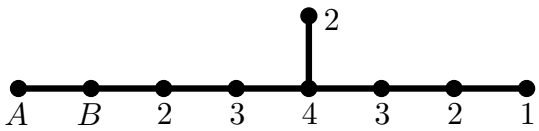
$$\begin{pmatrix} -1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 5. $H^{A_4}_{\mathrm{NY}}$ $\mathrm{VI}^* - \mathrm{II}_3$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 6. $H^{\mathrm{Mat}}_{\mathrm{I}}$ $\mathrm{I}_0 - \mathrm{II}^*_0$



$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 7. $H^{\mathrm{Mat}}_{\mathrm{II}}$ $\mathrm{I}_0 - \mathrm{III}^*_0$

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