RIMS Kôkyûroku Bessatsu
B78 (2020), 053–075

Cluster algebra and q-Painlevé equations: higher order generalization and degeneration structure

By

Takao Suzuki* and Naoto Okubo**

Abstract

In this article we give a birational representation of an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$ with the aid of a cluster mutation. This group provides several higher order generalizations of the q-Painlevé VI equation (q-P_{VI}) as translations. We also discuss a confluence of vertices of a quiver which can be applied to a degeneration structure of the q-Painlevé equations.

§ 1. Introduction

The cluster algebra was introduced by Fomin and Zelevinsky in [2, 3]. It is a variety of commutative ring described in terms of cluster variables and coefficients. Its generating set is defined by an operation called a mutation which transforms a seed consisting of cluster variables, coefficients and a quiver. Then new cluster variables (reps. coefficients) are rational in original cluster variables and coefficients (reps. coefficients). Hence we can obtain various discrete integrable systems from mutation-periodic quivers ([13]) as relations satisfied by cluster variables and coefficients.

In the previous work [15] the q-Painlevé VI equation ([6]) was derived from the mutation-periodic quiver with 8 vertices; see Figure 4 in Section 3. As its extension, we consider the quiver $Q_{mn}(A^{(1)}_{m-1})$*1; see Figure 1. This quiver is invariant under some

Received January 30, 2019. Revised April 22, 2019.
2010 Mathematics Subject Classification(s): 39A13, 13F60, 17B80, 34M55, 37K35

*Department of Mathematics, Kindai University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan
e-mail: `suzuki@math.kindai.ac.jp`
**Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1, Fuchinobe, Chuo-ku, Sagamihara-shi, Kanagawa 252-5258, Japan
e-mail: `okubo@gem.aoyama.ac.jp`

*1This quiver is also investigated in a recent work [12]. We follow the notation of [4] in which the quiver $Q_m(\mathfrak{g})$ is proposed for a Kac-Moody Lie algebra $\mathfrak{g}$. Here the symbol $A^{(1)}_{m-1}$ stands for the affine Lie algebra of type $A^{(1)}_{m-1}$.

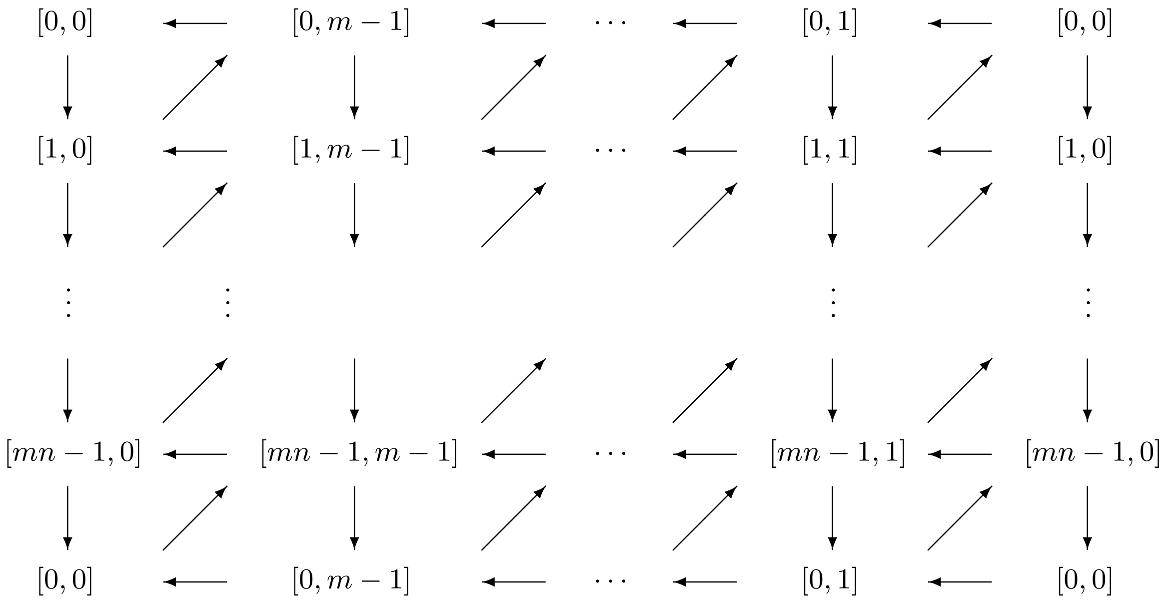


Figure 1. The quiver $Q_{mn}(A^{(1)}_{m-1})$

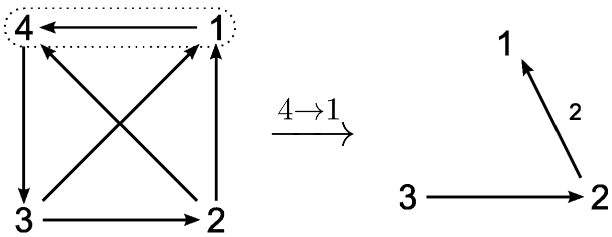


Figure 2. Confluence of quiver (example)

compositions of mutations and permutations of vertices of quivers. Those operations generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)*2}$. And this group provides several generalized q-$P_{\rm VI}$'s as translations. In the previous work [16] we investigated the case $m = 2$ and derived three types of systems. In this article we review it quickly and give an additional explanation about the q-hypergeometric solution of the generalized q-$P_{\rm VI}$; see Section 2.

Our another aim is to propose a confluence of vertices of a quiver; see Figure 2. We define a confluence of vertices of a quiver $i \to j$ by replacing two vertices i, j and an arrow between them with one vertex j. In this article we only consider a confluence of two vertices which are connected by arrows directly. At the level of the corresponding skew-symmetric matrix, the confluence can be interpreted as the following operation.

*2 This fact is first obtained in [12] as an extension of the previous work [9, 10] in which the binational representation of the affine Weyl group of type $(A_{m-1} + A_{n-1})^{(1)}$ is formulated.

1. Add the i-th row to the j-th row.

2. Add the i-th column to the j-th column.

3. Delete the i-th row and the i-th column.

The confluence of skew-symmetric matrix corresponding to the one of Figure 2 is described as follows.

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{4\to 1} \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It can be applied to the degeneration structure of the q-Painlevé equations ([18]), which is described as follows.

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & & & & \dfrac{\underset{|\alpha|^2=8}{A_1^{(1)}}}{A_7^{(1)}} & \\ & & & & & & & & & & & & \nearrow & & \searrow \\ \dfrac{E_8^{(1)}}{A_0^{(1)}} & \to & \dfrac{E_7^{(1)}}{A_1^{(1)}} & \to & \dfrac{E_6^{(1)}}{A_2^{(1)}} & \to & \dfrac{D_5^{(1)}}{A_3^{(1)}} & \to & \dfrac{A_4^{(1)}}{A_4^{(1)}} & \to & \dfrac{E_3^{(1)}}{A_5^{(1)}} & \to & \dfrac{E_2^{(1)}}{A_6^{(1)}} & \to \dfrac{A_1^{(1)}}{A_7^{(1)}} & \dfrac{E_0^{(1)}}{A_8^{(1)}} \end{array}$$

In this table numerators and denominators stand for symmetry and surface types respectively. Following [11], we use the symbols $E_3^{(1)} = (A_2+A_1)^{(1)}$ and $E_2^{(1)} = (A_1+\underset{|\alpha|^2=14}{A_1})^{(1)}$ for the sake of simplicity. The type $D_5^{(1)}/A_3^{(1)}$ corresponds to q-P_{VI}. In the previous work [15] we derived 4 types of q-Painlevé equations containing q-P_{VI} from some mutation-periodic quivers. At that time we gave the quivers by using the operation called a flattening which corresponds to the reduction of the discrete integrable system. Afterward, 9 equations below the one of type $E_6^{(1)}/A_2^{(1)}$ were derived in [1] by using a correspondence between quivers and Newton polygons. We obtain those quivers again with the aid of a confluence of vertices of a quiver; see Section 3.

§ 2. Extended affine Weyl group and Generalized q-P_{VI}

§ 2.1. Cluster mutation

Let $Q = Q_{mn-1}(A_{m-1}^{(1)})$ $(m, n \geq 2)$ be the quiver given in Figure 1. We define a mutation $\mu_{[j,i]}$ at the vertex $[j, i]$ as follows.

1. If there are k_1 arrows from $[j_1, i_1]$ to $[j, i]$ and k_2 arrows from $[j, i]$ to $[j_2, i_2]$, then we add k_1k_2 arrows from $[j_1, i_1]$ to $[j_2, i_2]$.

2. If 2-cycles appear via the operation 1, then we remove all of them.

3. We reverse the direction of all arrows touching $[j,i]$.

Let $I = \{[j,i] \mid 0 \le i \le m-1,\ 0 \le j \le mn-1\}$ be a vertex set. We define a skew-symmetric matrix $\Lambda = \left(\lambda_{[j_1,i_1],[j_2,i_2]}\right)_{[j_1,i_1],[j_2,i_2]\in I}$ corresponding to a quiver Q as follows.

- If there are k arrows from $[j_1,i_1]$ to $[j_2,i_2]$, then we set $\lambda_{[j_1,i_1],[j_2,i_2]} = k$ and $\lambda_{[j_2,i_2],[j_1,i_1]} = -k$.

- If there is no arrow connecting two vertices $[j_1,i_1]$ and $[j_2,i_2]$ directly, then we set $\lambda_{[j_1,i_1],[j_2,i_2]} = \lambda_{[j_2,i_2],[j_1,i_1]} = 0$.

Also let $\boldsymbol{y} = \left(y_{[j,i]}\right)_{[j,i]\in I}$ be a m^2n-tuple of coefficients. Then the mutation $\mu_{[j,i]}$: $(\Lambda, \boldsymbol{y}) \mapsto (\Lambda', \boldsymbol{y}')$ is given by

$$\lambda'_{[j_1,i_1],[j_2,i_2]} = \begin{cases} -\lambda_{[j_1,i_1],[j_2,i_2]} & ([j_1,i_1]=[j,i] \vee [j_2,i_2]=[j,i]) \\ \lambda_{[j_1,i_1],[j_2,i_2]} + \lambda_{[j_1,i_1],[j,i]}\,\lambda_{[j,i],[j_2,i_2]} & (\lambda_{[j_1,i_1],[j,i]} > 0 \wedge \lambda_{[j,i],[j_2,i_2]} > 0) \\ \lambda_{[j_1,i_1],[j_2,i_2]} - \lambda_{[j_1,i_1],[j,i]}\,\lambda_{[j,i],[j_2,i_2]} & (\lambda_{[j_1,i_1],[j,i]} < 0 \wedge \lambda_{[j,i],[j_2,i_2]} < 0) \\ \lambda_{[j_1,i_1],[j_2,i_2]} & (\text{otherwise}) \end{cases},$$

and

$$y'_{[j_1,i_1]} = \begin{cases} y_{[j,i]}^{-1} & ([j_1,i_1]=[j,i]) \\ y_{[j_1,i_1]}\,(1+y_{[j,i]}^{-1})^{\lambda_{[j_1,i_1],[j,i]}} & (\lambda_{[j_1,i_1],[j,i]} < 0) \\ y_{[j_1,i_1]}\,(1+y_{[j,i]})^{\lambda_{[j_1,i_1],[j,i]}} & (\lambda_{[j_1,i_1],[j,i]} > 0) \\ y_{[j_1,i_1]} & (\text{otherwise}) \end{cases}.$$

Note that we don't consider cluster variables in this article. We denote a transposition of vertices $[j_1,i_1]$ and $[j_2,i_2]$ by $([j_1,i_1],[j_2,i_2])$. It act on coefficients as

$$([j_1,i_1],[j_2,i_2])(y_{[j,i]}) = \begin{cases} y_{[j_2,i_2]} & ([j,i]=[j_1,i_1]) \\ y_{[j_1,i_1]} & ([j,i]=[j_2,i_2]) \\ y_{[j,i]} & (\text{otherwise}) \end{cases}.$$

In the following we use a notation of periodicity

$$y_{[j,i]} = y_{[j,i+m]} = y_{[j+mn,i]}.$$

§ 2.2. Birational representation of affine Weyl group

We introduce parameters corresponding to the simple roots of the affine root system as

$$\alpha_j = \prod_{i=0}^{m-1} y_{[j,i]} \quad (j = 0, \dots, mn-1),$$

$$\beta_i = \prod_{j=0}^{mn-1} y_{[j,i]}, \quad \beta'_i = \prod_{j=0}^{mn-1} y_{[j,i+j]} \quad (i = 0, \dots, m-1),$$

with

$$\prod_{j=0}^{mn-1} \alpha_j = \prod_{i=0}^{m-1} \beta_i = \prod_{i=0}^{m-1} \beta'_i = \prod_{i=0}^{m-1} \prod_{j=0}^{mn-1} y_{[j,i]} = q.$$

Note that

$$\alpha_j = \alpha_{j+mn}, \quad \beta_i = \beta_{i+m}, \quad \beta'_i = \beta'_{i+m}.$$

Definition 2.1. We define birational transformations π, π', ρ, called Dynkin diagram automorphisms, by

$$\begin{aligned}
\pi = & ([0,0],[1,1],\dots,[m-1,m-1],[m,0],\dots,[mn-1,m-1]) \\
& \times ([0,1],[1,2],\dots,[m-1,0],[m,1],\dots,[mn-1,0]) \\
& \times \dots \\
& \times ([0,m-1],[1,0],\dots,[m-1,m-2],[m,m-1],\dots,[mn-1,m-2]), \\
\pi' = & ([0,0],[1,0],\dots,[m-1,0],[m,0],\dots,[mn-1,0]) \\
& \times ([0,1],[1,1],\dots,[m-1,1],[m,1],\dots,[mn-1,1]) \\
& \times \dots \\
& \times ([0,m-1],[1,m-1],\dots,[m-1,m-1],[m,m-1],\dots,[mn-1,m-1]), \\
\rho = & ([1,0],[mn-1,m-1])([1,1],[mn-1,0])\dots([1,m-1],[mn-1,m-2]) \\
& \times ([2,0],[mn-2,m-2])([2,1],[mn-2,m-1])\dots([2,m-1],[mn-2,m-3]) \\
& \times \dots \\
& \times ([m-1,0],[mn-m+1,1])([m-1,1],[mn-m+1,2])\dots([m-1,m-1],[mn-m+1,0]) \\
& \times ([m,0],[mn-m,0])([m,1],[mn-m,1])\dots([m,m-1],[mn-m,m-1]) \\
& \times \dots \\
& \times ([N,0],[mn-N,0])([N,1],[mn-N,1])\dots([N,m-1],[mn-N,m-1]),
\end{aligned}$$

where

$$([j_1,i_1],[j_2,i_2],\dots,[j_k,i_k]) = ([j_1,i_1],[j_2,i_2])([j_2,i_2],[j_3,i_3])\dots([j_{k-1},i_{k-1}],[j_k,i_k])$$

stands for a cyclic permutation and $N = \lfloor \frac{mn}{2} \rfloor$. Here we define a composition of two operations μ_1, μ_2 by $(\mu_1\mu_2)(\boldsymbol{y}) = \mu_1(\mu_2(\boldsymbol{y}))$.

These transformations act on the coefficients and the parameters as

$$\begin{aligned}
&\pi(y_{[j,i]}) = y_{[j+1,i+1]},\\
&\pi'(y_{[j,i]}) = y_{[j+1,i]},\\
&\rho(y_{[j_2m+j_1,i]}) = y_{[mn-j_2m-j_1,i-j_1]} \quad (j_1 = 0,\dots,m-1;\ j_2 = 0,\dots,n-1).
\end{aligned}$$

and

$$\begin{aligned}
&\pi(\alpha_j) = \alpha_{j+1}, \quad \pi(\beta_i) = \beta_{i+1}, \quad \pi(\beta'_i) = \beta'_i,\\
&\pi'(\alpha_j) = \alpha_{j+1}, \quad \pi'(\beta_i) = \beta_i, \quad \pi'(\beta'_i) = \beta'_{i-1},\\
&\rho(\alpha_j) = \alpha_{mn-j}, \quad \rho(\beta_i) = \beta'_i, \quad \rho(\beta'_i) = \beta_i,
\end{aligned}$$

for $i = 0,\dots,m-1$ and $j = 0,\dots,mn-1$.

Definition 2.2. We define birational transformations r_0, called a simple reflection, by

$$r_0 = \mu_{[0,0]}\,\mu_{[0,1]}\dots\mu_{[0,m-2]}\,([0,m-2],[0,m-1])\,\mu_{[0,m-2]}\dots\mu_{[0,1]}\,\mu_{[0,0]}.$$

We also define birational transformations $r_1,\dots,r_{mn-1}$ by using π, r_0 as

$$r_j = \pi^{-1} r_{j-1}\,\pi \quad (j = 1,\dots,mn-1).$$

The transformation r_0 acts on the coefficients and the parameters as

$$\begin{aligned}
&r_0(y_{[0,i]}) = \frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,i+k_2]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,i+k_2+1]}}, \quad r_0(y_{[1,i]}) = \frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,i+k_2]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,i+k_2]}}\, y_{[1,i]},\\
&r_0(y_{[mn-1,i]}) = \frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,i+k_2+1]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,i+k_2+1]}}\, y_{[mn-1,i]}, \quad r_0(y_{[j,i]}) = y_{[j,i]} \quad (j \neq 0, 1, mn-1),
\end{aligned}$$

for $i = 0,\dots,m-1$ and

$$\begin{aligned}
&r_0(\alpha_0) = \frac{1}{\alpha_0},\ \ r_0(\alpha_1) = \alpha_0\,\alpha_1,\ \ r_0(\alpha_{mn-1}) = \alpha_0\,\alpha_{mn-1},\ \ r_0(\alpha_k) = \alpha_k\ \ (k \neq 0, 1, mn-1),\\
&r_0(\beta_i) = \beta_i, \quad r_0(\beta'_i) = \beta'_i \quad (i = 0,\dots,m-1),
\end{aligned}$$

Definition 2.3. We define birational transformations s_0, called a simple reflection, by

$$s_0 = \mu_{[0,0]}\,\mu_{[1,0]}\dots\mu_{[mn-2,0]}\,([mn-2,0],[mn-1,0])\,\mu_{[mn-2,0]}\dots\mu_{[1,0]}\,\mu_{[0,0]}.$$

We also define birational transformations $s_1, \dots, s_{m-1}$ and $s'_0, \dots, s'_{m-1}$ by using π, ρ, s_0 as

$$s_i = \pi^{-1} s_{i-1}\, \pi \quad (i = 1, \dots, m-1),$$

and

$$s'_i = \rho\, s_i\, \rho \quad (i = 0, \dots, m-1).$$

In the case $m = 2$, the transformation r_0 acts on the coefficients and the parameters as

$$s_0(y_{[j,0]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}, \quad s_0(y_{[j,1]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}\, y_{[j,0]}\, y_{[j,1]},$$

for $j = 0, \dots, mn-1$ and

$$\begin{aligned} &s_0(\alpha_j) = \alpha_j \quad (j = 0, \dots, mn-1), \\ &s_0(\beta_0) = \frac{1}{\beta_0}, \quad s_0(\beta_1) = \beta_0^2\, \beta_1, \quad s_0(\beta'_i) = \beta'_i \quad (i = 0, \dots, m-1). \end{aligned}$$

In the case $m \geq 3$, the action of r_0 is described as

$$\begin{aligned} &s_0(y_{[j,0]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}, \quad s_0(y_{[j,1]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2,0]}}\, y_{[j,1]}, \\ &s_0(y_{[j,m-1]}) = \frac{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1} y_{[j+k_2+1,0]}}{\sum_{k_1=0}^{mn-1} \prod_{k_2=0}^{k_1-1} y_{[j+k_2+1,0]}}\, y_{[j,m-1]}, \quad s_0(y_{[j,i]}) = y_{[j,i]} \quad (i \neq 0, 1, m-1), \end{aligned}$$

for $j = 0, \dots, mn-1$ and

$$\begin{aligned} &s_0(\alpha_j) = \alpha_j \quad (j = 0, \dots, mn-1), \\ &s_0(\beta_0) = \frac{1}{\beta_0}, \quad s_0(\beta_1) = \beta_0\, \beta_1, \quad s_0(\beta_{m-1}) = \beta_0\, \beta_{m-1}, \quad s_0(\beta_k) = \beta_k \quad (k \neq 0, 1, m-1), \\ &s_0(\beta'_i) = \beta'_i \quad (i = 0, \dots, m-1). \end{aligned}$$

Fact 2.4 ([5, 12, 16]). The birational transformations defined in the above satisfy the fundamental relations of the extended affine Weyl group of type $(A_{mn-1} + A_{m-1} + A_{m-1})^{(1)}$

$$\begin{aligned} &r_j^2 = s_i^2 = (s'_i)^2 = 1, \\ &(r_j\, r_{j+1})^3 = (s_i\, s_{i+1})^3 = (s'_i\, s'_{i+1})^3 = 1, \\ &(r_{j_1} r_{j_2})^2 = (s_{i_1} s_{i_2})^2 = (s'_{i_1} s'_{i_2})^2 = 1 \quad (i_1 \neq i_2, i_2 \pm 1;\ j_1 \neq j_2, j_2 \pm 1), \\ &(r_j\, s_i)^2 = (r_j\, s'_i)^2 = (s_{i_1} s'_{i_2})^2 = 1, \end{aligned}$$

and

$$\begin{aligned}
&\pi^{mn} = 1, \quad (\pi')^{mn} = 1, \quad \rho^2 = 1, \quad \pi\,\pi' = \pi'\pi, \quad \pi^m = (\pi')^m, \quad \pi'\rho = \rho\,\pi^{-1},\\
&\pi\, r_j = r_{j-1}\,\pi, \quad \pi' r_j = r_{j-1}\,\pi', \quad \rho\, r_j = r_{mn-j}\,\rho,\\
&\pi\, s_i = s_{i-1}\,\pi, \quad \pi\, s'_i = s'_i\,\pi, \quad \pi' s_i = s_i\,\pi', \quad \pi' s'_i = s'_{i+1}\,\pi', \quad \rho\, s_i = s'_i\,\rho,
\end{aligned}$$

for $i, i_1, i_2 = 0, \dots, m-1$ and $j = 0, \dots, mn-1$, where

$$r_j = r_{j+mn}, \quad s_i = s_{i+m}, \quad s'_i = s'_{i+m}.$$

§ 2.3. Example (case $m = 2$)

This case has already been considered in our previous work.

Fact 2.5 ([16]). Let

$$\begin{aligned}
T_1 &= s'_1\, s_1\, \pi'\pi^{-1},\\
T_2 &= (r_0\, r_1 \dots r_{n-2}\, r_n\, r_{n+1} \dots r_{2n-2}\, \pi')^2,\\
T_3 &= r_1\, r_2 \dots r_{2n-1}\, s'_1\, \pi',\\
T_4 &= (r_0\, r_2 \dots r_{2n-2}\, \pi')^2.
\end{aligned}$$

Then they provides three types of q-Painlevé systems as follows[*3].

- T_1 provides the q-Painlevé system q-$P_{(n,n)}$ arising from the q-DS hierarchy given in [20, 21].
- T_2 provides the q-Garnier system given in [19].
- T_4 provides the q-Painlevé system arising from the q-LUC hierarchy given in §3.4 of [23].

In this section we focus on the translation T_1 and investigate a particular solution in terms of the basic hypergeometric function ${}_n\phi_{n-1}$. The actions of T_1 on the parameters are described as

$$T_1(\alpha_j) = \alpha_j \quad (j = 0, \dots, 2n-1), \quad T_1(\beta_0) = q\,\beta_0, \quad T_1(\beta_1) = \frac{\beta_1}{q},$$
$$T_1(\beta'_0) = q\,\beta'_0, \quad T_1(\beta'_1) = \frac{\beta'_1}{q}.$$

[*3] We conjecture that T_3 provides the variation of the q-Garnier system $T^{-1}_{a_{N+1}} T^{-1}_{c_1}$ given in §3.2.4 of [14].

Recall that

$$\alpha_j = y_{[j,0]}\, y_{[j,1]} \quad (j = 0, \ldots, 2n-1),$$
$$\beta_i = \prod_{j=0}^{2n-1} y_{[j,i]}, \quad \beta'_i = \prod_{j=0}^{n-1} y_{[2j,i]}\, y_{[2j+1,i+1]} \quad (i = 0, 1),$$

and

$$y_{[j,i]} = y_{[j,i+2]} = y_{[j+2n,i]}, \quad \alpha_j = \alpha_{j+2n}, \quad \beta_i = \beta_{i+2}, \quad \beta'_i = \beta'_{i+2}.$$

The actions of T_1 on the coefficients $y_{[0,0]}, \ldots, y_{[2n-1,0]}$ are described as

$$\begin{aligned} T_1(y_{[2j,0]}) &= \alpha_{2j}\, \alpha_{2j+1}\, y_{[2j,0]} \frac{S'_j\, S_{j+1}}{S_j\, S'_{j+1}}, \\ T_1(y_{[2j+1,0]}) &= \frac{1 + y_{[2j+2,0]}}{1 + y_{[2j,0]}}\, y_{[2j+3,0]} \frac{S_j\, S'_{j+1} + \alpha_{2j+1}\, y_{[2j,0]}\, S'_j\, S_{j+1}}{S_{j+1}\, S'_{j+2} + \alpha_{2j+3}\, y_{[2j+2,0]}\, S'_{j+1}\, S_{j+2}}, \end{aligned} \tag{2.1}$$

for $j = 0, \ldots, n-1$, where

$$\begin{aligned} S_j &= \sum_{k=j}^{n-1} (1 + y_{[2k,0]}) \prod_{l=j}^{k-1} y_{[2l,0]}\, y_{[2l+1,0]} + \sum_{k=0}^{j-1} (1 + y_{[2k,0]}) \prod_{l=0}^{k+n-1} y_{[2l,0]}\, y_{[2l+1,0]}, \\ S'_j &= \sum_{k=j}^{n-1} \left(1 + y^{-1}_{[2k,0]}\right) \prod_{l=j}^{k-1} y^{-1}_{[2l,0]}\, y^{-1}_{[2l+1,1]} + \sum_{k=0}^{j-1} \left(1 + y^{-1}_{[2k,0]}\right) \prod_{l=0}^{k+n-1} y^{-1}_{[2l,0]}\, y^{-1}_{[2l+1,1]}. \end{aligned} \tag{2.2}$$

Lemma 2.6. If, in system (2.1), we assume that

$$y_{[2j+1,1]} = -1 \quad (j = 0, \ldots, n-1),$$

then the coefficients $y_{[0,0]}, y_{[2,0]}, \ldots, y_{[2n-2,0]}$ satisfy

$$T_1(y_{[2j,0]}) = \alpha_{2j}\, \alpha_{2j+1}\, y_{[2j,0]} \frac{S_{j+1}}{S_j} \quad (j = 0, \ldots, n-1), \tag{2.3}$$

where

$$S_j = 1 + \sum_{k=j}^{j+n-2} (-1)^{k-j} (1 - \alpha_{2k+1}) \prod_{l=j}^{k-1} \alpha_{2l+1} \prod_{l=j}^{k} y_{[2l,0]} + (-1)^{n-1} \frac{\prod_{k=0}^{n-1} \alpha_{2k+1}}{\alpha_{2j-1}} \prod_{k=0}^{n-1} y_{[2k,0]}.$$

Proof. Substituting

$$y_{[2j+1,0]} = -\alpha_{2j+1}, \quad y_{[2j+1,1]} = -1 \quad (j = 0, \ldots, n-1),$$

into (2.2), we obtain

$$S_j = 1 + \sum_{k=j}^{j+n-2} (-1)^{k-j}(1-\alpha_{2k+1}) \prod_{l=j}^{k-1} \alpha_{2l+1} \prod_{l=j}^{k} y_{[2l,0]} + (-1)^{n-1} \frac{\prod_{k=0}^{n-1} \alpha_{2k+1}}{\alpha_{2j-1}} \prod_{k=0}^{n-1} y_{[2k,0]},$$

$$S'_j = 1 + \frac{(-1)^{n-1}}{\prod_{k=0}^{n-1} y_{[2k,0]}},$$

and

$$S_j + \alpha_{2j+1}\, y_{[2j,0]}\, S_{j+1} = \big(1 + y_{[2j,0]}\big) \left(1 + (-1)^{n-1} \prod_{k=0}^{n-1} \alpha_{2k+1}\, y_{[2k,0]}\right),$$

for $j = 0, \ldots, n-1$. Then system (2.1) implies (2.3). We also obtain

$$T_1(y_{[2j+1,0]}) = y_{[2j+3,0]} \quad (j = 0, \ldots, n-1),$$

namely the assumption of this lemma is consistent with system (2.1). □

Thanks to this lemma, we can show the following theorem easily by a direct calculation.

Theorem 2.7. Let n-tuple $(x_0, \ldots, x_{n-1})$ be a solution of a system of linear q-difference equations

(2.4)

$$\begin{aligned} T_1(x_j) = {} & \left(\prod_{l=0}^{2j-1} \alpha_l\right) x_j + \sum_{k=j+1}^{n-1} \left((-1)^{k-j-1}(1-\alpha_{2k-1}) \prod_{l=0}^{2k-2} \alpha_l\right) x_k \\ & + \sum_{k=0}^{j-1} \left((-1)^{k-j-1+n}(1-\alpha_{2k-1}) \prod_{l=0}^{2k-2} \alpha_l\right) q\,\beta'_0\, x_k + \left((-1)^{n-1} \prod_{l=0}^{2j-2} \alpha_l\right) q\,\beta'_0\, x_j, \end{aligned}$$

for $j = 0, \ldots, n-1$. We also set

$$y_{[2j,0]} = \alpha_{2j}\, \frac{x_{j+1}}{x_j} \quad (j = 0, \ldots, n-2), \quad y_{[2n-2,0]} = \alpha_{2n-2}\, \beta'_0\, \frac{x_0}{x_{n-1}}.$$

Then $y_{[0,0]}, y_{[2,0]}, \ldots, y_{[2n-2,0]}$ satisfy system (2.3).

System (2.4) is equivalent to the one given in [20] whose solution is described in terms of the basic hypergeometric function ${}_n\phi_{n-1}$. Here the parameter β'_0 plays the role of the independent variable.

§ 3. Degeneration structure of q-Painlevé equations

In this section we start with the quiver $Q_8 = Q_4(A_1^{(1)})$ and consider confluences of vertices of quivers. These procedures give the degeneration scheme of q-Painlevé

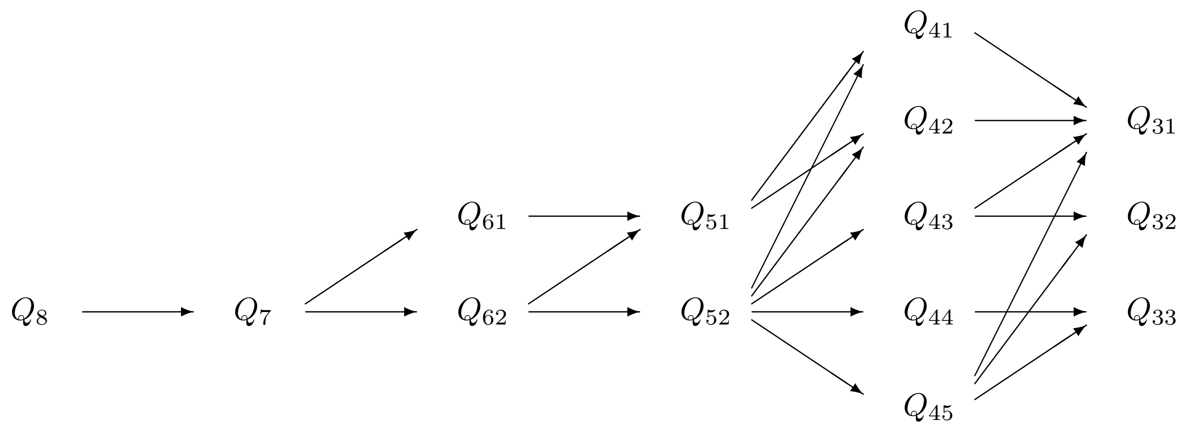


Figure 3. Confluence of the quiver $Q_4(A_1^{(1)})$

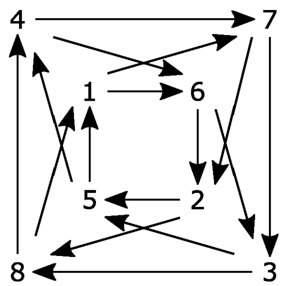


Figure 4. Q_8

equations below the one of type $D_5^{(1)}/A_3^{(1)}$. We list a correspondence between the quivers in Figure 3 and the q-Painlevé equations below[*4].

Q_8	Q_7	Q_{62}	Q_{52}	Q_{44}	Q_{45}	Q_{33}
$D_5^{(1)}/A_3^{(1)}$	$A_4^{(1)}/A_4^{(1)}$	$E_3^{(1)}/A_5^{(1)}$	$E_2^{(1)}/A_6^{(1)}$	$A_1^{(1)}/A_7^{(1)}$	$\underset{\lvert\alpha\rvert^2=8}{A_1^{(1)}}/A_7^{(1)}$	$E_0^{(1)}/A_8^{(1)}$

§ 3.1. Quiver Q_8

For the sake of simplicity, we rename the vertices of $Q_4(A_1^{(1)})$ as

$$[0,0]=1, \quad [0,1]=2, \quad [1,0]=6, \quad [1,1]=5,$$
$$[2,0]=3, \quad [2,1]=4, \quad [3,0]=8, \quad [3,1]=7.$$

[*4] Among the other 7 quivers, Q_{41} and Q_{31} are ones of finite type. We expect that the rest 5 quivers correspond to the q-hypergeometric functions for the following reasons. The assumption of Lemma 2.6 turns into $y_5 = y_7 = -1$ in the quiver Q_8. On the other hand, if we remove two vertices $5, 7$ and all arrows touching $5, 7$ from the quiver Q_8, then we obtain the one Q_{61}. Besides, the degeneration scheme below Q_{61} is similar to the one of q-hypergeometric functions.

Then we obtain the quiver Q_8; see Figure 4. The skew-symmetric matrix Λ_8 corresponding to Q_8 is given by

$$\Lambda_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the following we denote a mutation at the vertex i by μ_i and a transposition of vertices i_1, i_2 by (i_1, i_2).

The quiver Q_8 is invariant under compositions of mutations and permutations of vertices of quivers

$$r_0 = (1,4), \quad r_1 = (2,3), \quad r_2 = \mu_1\,(1,2)\,\mu_1, \quad r_3 = \mu_5\,(5,6)\,\mu_5, \quad r_4 = (5,8), \quad r_5 = (6,7),$$
$$\pi_1 = (1,5,2,6)(4,8,3,7), \quad \pi_2 = (1,2)(3,4)(5,6)(7,8).$$

Their actions on the coefficients $\boldsymbol{y} = (y_1, \ldots, y_8)$ generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $D_5^{(1)}$[*5]. The parameters corresponding to the simple roots of the affine root system are given by

$$\alpha_0 = \frac{y_4}{y_1}, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_1\,y_2, \quad \alpha_3 = y_5\,y_6, \quad \alpha_4 = \frac{y_8}{y_5}, \quad \alpha_5 = \frac{y_7}{y_6}.$$

The transformations π_1, π_2 act on the parameters as

$$\pi_1((\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) = (\alpha_4, \alpha_5, \alpha_3, \alpha_2, \alpha_1, \alpha_0),$$
$$\pi_2((\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_5, \alpha_4).$$

A translation of this Weyl group provides q-P_{VI}; see [24].

§ 3.2. Quiver Q_7

Thanks to a symmetry of the quiver Q_8, it is enough to investigate the following confluences.

$$8 \to 1, \quad 8 \to 2,$$

[*5]The formulations of the extended affine Weyl groups in this section were given systematically in the previous work [1].

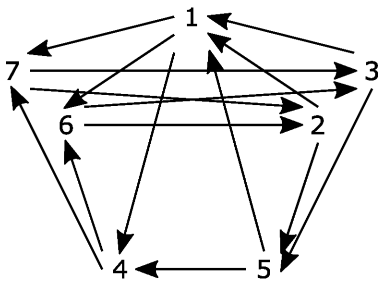


Figure 5. Q_7

from which we obtain the quiver Q_7. To be precise, we have to take a permutation $(1,2,6)(3,7,4,5)$ after the confluence $8 \to 2$. In the following we omit permutations after confluence procedures.

The quiver Q_7 is invariant under compositions of mutations and permutations

$$r_0 = \mu_1\,\mu_2\,(2,6)\,\mu_2\,\mu_1, \quad r_1 = (2,3), \quad r_2 = \mu_2\,(2,4)\,\mu_2, \quad r_3 = \mu_5\,(5,6)\,\mu_5, \quad r_4 = (6,7),$$
$$\pi_1 = (1,5,3,7,4)(2,6)\,\mu_2.$$

Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $A_4^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1\,y_2\,y_6, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_2\,y_4, \quad \alpha_3 = y_5\,y_6, \quad \alpha_4 = \frac{y_7}{y_6}.$$

The transformation π_1 acts on the parameters as

$$\pi_1((\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)) = (\alpha_3,\alpha_4,\alpha_0,\alpha_1,\alpha_2).$$

A translation of this Weyl group provides q-Painlevé V equation; see [17, 22].

In the confluence $8 \to 1$ a degeneration of the coefficients is given by a replacement $y_1 \to y_1/\varepsilon$, $y_8 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_8	$r_2\,r_4\,r_3\,r_4\,r_2 = \mu_1\,\mu_8\,\mu_2\,(2,6)\,\mu_2\,\mu_8\,\mu_1$	r_1	$r_0\,r_2\,r_0$	r_3	r_5
Q_7	r_0	r_1	r_2	r_3	r_4

For example, the action $r_2\,r_4\,r_3\,r_4\,r_2(y_1\,y_8)$ in Q_8 is reduced to the one $r_0(y_1)$ in Q_7 as

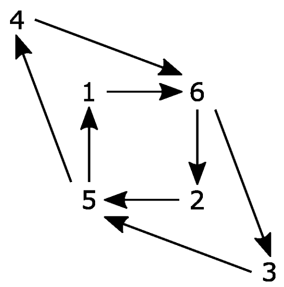


Figure 6. Q_{61}

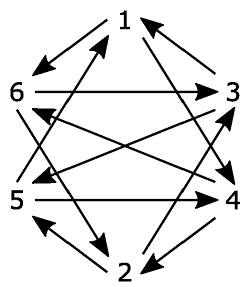


Figure 7. Q_{62}

follows.

$$\begin{aligned} r_2\, r_4\, r_3\, r_4\, r_2(y_1\, y_8) &= \frac{(1+y_1+y_1\,y_6+y_1\,y_6\,y_2)(1+y_8+y_8\,y_1+y_8\,y_1\,y_6)}{y_1\,y_6\,(1+y_2+y_2\,y_8+y_2\,y_8\,y_1)(1+y_6+y_6\,y_2+y_6\,y_2\,y_8)} \\ &\xrightarrow[y_8\to\varepsilon]{y_1\to y_1/\varepsilon} \frac{(\varepsilon+y_1+y_1\,y_6+y_1\,y_6\,y_2)(1+\varepsilon+y_1+y_1\,y_6)}{y_1\,y_6\,(1+y_2+y_2\,\varepsilon+y_2\,y_1)(1+y_6+y_6\,y_2+y_6\,y_2\,\varepsilon)} \\ &\xrightarrow{\varepsilon\to 0} \frac{1+y_1+y_1\,y_6}{y_6\,(1+y_2+y_2\,y_1)} \\ &= r_0(y_1). \end{aligned}$$

Note that, throughout this section, we haven't clarified degenerations of mutations or transformations denoted by π_1, π_2 yet. It is a future problem.

§ 3.3. Quiver with 6 vertices

For the quiver Q_7, it is enough to investigate the following confluences.

$$7 \to 1, \quad 7 \to 2, \quad 7 \to 4, \quad 5 \to 1, \quad 5 \to 2, \quad 5 \to 4, \quad 4 \to 1, \quad 3 \to 1.$$

Then we obtain the quivers Q_{61} and Q_{62}.

3.3.1. Q_{61}

The quiver Q_{61} is obtained via the following confluences.

$$Q_7 \to Q_{61}: \quad 7 \to 1, \quad 5 \to 4, \quad 3 \to 1.$$

To be precise, the quiver obtained after the confluence $5 \to 4$ is different from the one Q_{61}. We have to take a mutation μ_1 after the confluence procedure in order to obtain Q_{61}.

The quiver Q_{61} is invariant under compositions of mutations and permutations

$$\begin{aligned} &r_1 = \mu_5\,(5,6)\,\mu_5, \quad r_2 = \mu_1\,(1,2)\,\mu_1, \quad r_3 = (1,4), \quad r_4 = (2,3), \\ &\pi_1 = (1,5,4,6)\,\mu_1\,\mu_5, \quad \pi_2 = (1,2)(3,4)(5,6). \end{aligned}$$

The actions of simple reflections $r_1, \ldots, r_4$ on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type D_4. The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_5\, y_6, \quad \alpha_2 = y_1\, y_2, \quad \alpha_3 = \frac{y_4}{y_1}, \quad \alpha_4 = \frac{y_3}{y_2}.$$

The transformations π_1, π_2 act on the parameters as

$$\pi_1((\alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (\alpha_3, \alpha_2, \alpha_1, \alpha_4), \quad \pi_2((\alpha_1, \alpha_2, \alpha_3, \alpha_4)) = (\alpha_1, \alpha_2, \alpha_4, \alpha_3).$$

Note that π_1 is not an involution unlike π_2 because $\pi_1^2(y_i) \neq y_i$ for any i.

In this case the simple reflections aren't obtained via a limiting procedure. They are derived from the simple reflections of the quiver Q_8. A set of the coefficients $\{y_1, \ldots, y_6\}$ is closed under actions of $r_0, \ldots, r_3$ in Q_8. Moreover, the quiver Q_{61} is obtained by removing two vertices $7, 8$ and all arrows touching $7, 8$ from Q_8. These facts induce the following degeneration.

Q_8	r_3	r_2	r_0	r_1
Q_{61}	r_1	r_2	r_3	r_4

3.3.2. Q_{62}

The quiver Q_{62} is obtained via the following confluences.

$$Q_7 \to Q_{62}: \quad 7 \to 2, \quad 7 \to 4, \quad 5 \to 1, \quad 5 \to 2, \quad 4 \to 1.$$

Similarly as Q_{61}, we have to take mutations μ_2 and μ_7 after the confluences $7 \to 4$ and $5 \to 2$ respectively. We also have to take μ_2 and μ_7 in this order after $5 \to 1$ or $4 \to 1$.

The quiver Q_{62} is invariant under compositions of mutations and permutations

$$r_0 = \mu_1\,(1,2)\,\mu_1, \quad r_1 = \mu_3\,(3,4)\,\mu_3, \quad r_2 = \mu_5\,(5,6)\,\mu_5, \quad \pi_1 = (1,3,6)(2,4,5),$$
$$s_0 = \mu_1\,\mu_3\,(3,6)\,\mu_3\,\mu_1, \quad s_1 = \mu_2\,\mu_4\,(4,5)\,\mu_4\,\mu_2, \quad \pi_2 = (1,2)(3,4)(5,6).$$

Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $(A_2 + A_1)^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1\, y_2, \quad \alpha_1 = y_3\, y_4, \quad \alpha_2 = y_5\, y_6, \quad \beta_0 = y_1\, y_3\, y_6, \quad \beta_1 = y_2\, y_4\, y_5.$$

The transformations π_1, π_2 act on the parameters as

$$\pi_1((\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)) = (\alpha_1, \alpha_2, \alpha_0, \beta_0, \beta_1), \quad \pi_2((\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)) = (\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_0).$$

Translations of this Weyl group provide q-Painlevé III and IV equations; see [7, 8, 22].

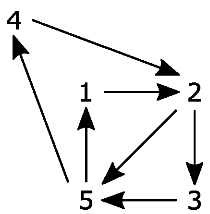


Figure 8. Q_{51}

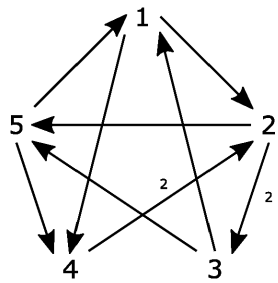


Figure 9. Q_{52}

In the confluence $7 \to 2$ a degeneration of the coefficients is given by a replacement $y_2 \to y_2/\varepsilon$, $y_7 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_7	$r_4\,r_0\,r_4 = \mu_7\,\mu_2\,(1,2)\,\mu_2\,\mu_7$	$r_1\,r_2\,r_1$	r_3	$r_1\,r_0\,r_1$	$r_2\,r_4\,r_3\,r_4\,r_2 = \mu_2\,\mu_7\,\mu_4\,(4,5)\,\mu_4\,\mu_7\,\mu_2$
Q_{62}	r_0	r_1	r_2	s_0	s_1

§ 3.4. Quiver with 5 vertices

For the quivers Q_{61} and Q_{62}, it is enough to investigate the following confluences.

$$Q_{61}: \quad 6 \to 1, \quad 6 \to 2.$$
$$Q_{62}: \quad 6 \to 1, \quad 6 \to 2, \quad 6 \to 3, \quad 6 \to 4, \quad 4 \to 1, \quad 4 \to 2.$$

Then we obtain the quivers Q_{51} and Q_{52}.

3.4.1. Q_{51}

The quiver Q_{51} is obtained via the following confluences.

$$Q_{61} \to Q_{51}: \quad 6 \to 1, \quad 6 \to 2.$$
$$Q_{62} \to Q_{51}: \quad 6 \to 1, \quad 6 \to 3, \quad 4 \to 2.$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1\,\mu_2\,(2,5)\,\mu_2\,\mu_1, \quad r_2 = (1,4), \quad r_3 = \mu_1\,(1,3)\,\mu_1, \quad \pi_1 = (2,3,5)\,\mu_2.$$

The actions of simple reflections r_1, r_2, r_3 on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type A_3. The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_1\,y_2\,y_5, \quad \alpha_2 = \frac{y_4}{y_1}, \quad \alpha_3 = y_1\,y_3.$$

Note that the transformation π_1 is not an involution due to the same reason as Q_{61}, although it acts on the parameters as

$$\pi_1((\alpha_1,\alpha_2,\alpha_3)) = (\alpha_3,\alpha_2,\alpha_1).$$

In the confluence $6 \to 2$ ($Q_{61} \to Q_{51}$) a degeneration of the coefficients is given by a replacement $y_2 \to y_2/\varepsilon$, $y_6 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{61}	$r_2\, r_1\, r_2 = \mu_1\, \mu_6\, \mu_2\, (2,5)\, \mu_2\, \mu_6\, \mu_1$	r_3	$r_4\, r_2\, r_4$
Q_{51}	r_1	r_2	r_3

3.4.2. Q_{52}

The quiver Q_{52} is obtained via the following confluences.

$$Q_{62} \to Q_{52}: \quad 6 \to 2, \quad 6 \to 4, \quad 4 \to 1.$$

It is invariant under compositions of mutations and permutations

$$r_0 = \mu_1\, \mu_2\, (2,5)\, \mu_2\, \mu_1, \quad r_1 = \mu_3\, (3,4)\, \mu_3, \quad \pi_1 = (1,3,2,4,5)\, \mu_1.$$

The actions of simple reflections r_0, r_1 on the coefficients generate a group of birational transformations which is isomorphic to the affine Weyl group of type $A_1^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1\, y_2\, y_5, \quad \alpha_1 = y_3\, y_4.$$

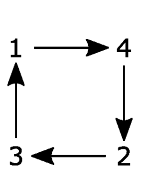
The transformation π_1 is not an involution due to the same reason as Q_{61}, although it acts on the parameters as

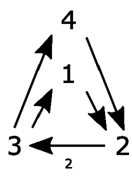
$$\pi_1((\alpha_0,\alpha_1)) = (\alpha_1,\alpha_0).$$

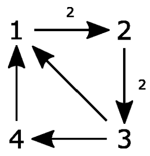
Therefore the group $\langle r_0, r_1, \pi_1 \rangle$ is not an extended affine Weyl group of type $A_1^{(1)}$. Nevertheless those transformations provide the q-Painlevé II equation and another q-Painlevé equation; see [1, 15].

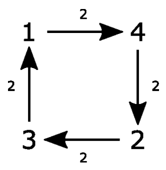
In the confluence $6 \to 2$ a degeneration of the coefficients is given by a replacement $y_2 \to y_2/\varepsilon$, $y_6 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{62}	$r_0\, r_2\, r_0 = \mu_1\, \mu_6\, \mu_2\, (2,5)\, \mu_2\, \mu_6\, \mu_1$	r_1
Q_{52}	r_0	r_1









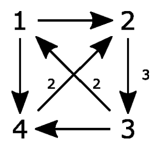


Figure 10. Q_{41} Figure 11. Q_{42} Figure 12. Q_{43} Figure 13. Q_{44} Figure 14. Q_{45}

§ 3.5. Quiver with 4 vertices

For the quivers Q_{51} and Q_{52}, it is enough to investigate the following confluences.

Q_{51} : $5 \to 1,\quad 5 \to 3,\quad 4 \to 2,\quad 3 \to 2.$

Q_{52} : $5 \to 1,\quad 5 \to 2,\quad 5 \to 3,\quad 5 \to 4,\quad 4 \to 1,\quad 4 \to 2,\quad 3 \to 1,\quad 3 \to 2,\quad 2 \to 1.$

Then we obtain the quivers Q_{41}, Q_{42}, Q_{43}, Q_{44} and Q_{45}. Note that, in the quiver Q_{51}, all of arrows are removed via the confluence $5 \to 2$.

3.5.1. Q_{41}

The quiver Q_{41} is obtained via the following confluences.

$$Q_{51} \to Q_{41} : \quad 5 \to 1, \quad 4 \to 2.$$
$$Q_{52} \to Q_{41} : \quad 5 \to 2, \quad 2 \to 1.$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1\,(1,2)\,\mu_1, \quad r_2 = \mu_3\,(3,4)\,\mu_3, \quad \pi_1 = (1,4,2,3).$$

Since the fundamental relation $(r_1\,r_2)^3 = 1$ is satisfied, the actions of simple reflections r_1, r_2 on the coefficients generate a group of birational transformations which is isomorphic to the Weyl group of type A_2. The parameters corresponding to the simple roots are given by

$$\alpha_1 = y_1\,y_2, \quad \alpha_2 = y_3\,y_4.$$

The transformation π_1 acts on the parameters as

$$\pi_1((\alpha_1, \alpha_2)) = (\alpha_2, \alpha_1).$$

In the confluence $5 \to 1$ ($Q_{51} \to Q_{41}$) a degeneration of the coefficients is given by a replacement $y_1 \to y_1/\varepsilon$, $y_5 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{51}	$r_1 = \mu_5\,\mu_1\,(1,2)\,\mu_1\,\mu_5$	$r_2\,r_3\,r_2$
Q_{41}	r_1	r_2

3.5.2. Q_{42}

The quiver Q_{42} is obtained via the following confluences.

$$Q_{51} \to Q_{42}: \quad 5 \to 3, \quad 3 \to 2.$$
$$Q_{52} \to Q_{42}: \quad 5 \to 4, \quad 3 \to 1.$$

It is invariant under compositions of mutations and permutations

$$r_1 = (1,4), \quad \pi_1 = (2,3)\,\mu_2.$$

The action of a simple reflection r_1 on the coefficients generates a group of birational transformations which is isomorphic to the Weyl group of type A_1. The parameter corresponding to the simple root is given by

$$\alpha_1 = \frac{y_4}{y_1},$$

which is invariant under the action of π_1.

In the confluence $5 \to 3$ ($Q_{51} \to Q_{42}$) a degeneration of the coefficients is given by a replacement $y_3 \to y_3/\varepsilon$, $y_5 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{51}	r_2
Q_{42}	r_1

3.5.3. Q_{43}

The quiver Q_{43} is obtained via the following confluences.

$$Q_{52} \to Q_{43}: \quad 4 \to 2, \quad 3 \to 2.$$

It is invariant under a composition of a mutation and a permutation

$$\pi_1 = (1,4,3)\,\mu_1.$$

3.5.4. Q_{44}

The quiver Q_{44} is obtained via the following confluence.

$$Q_{52} \to Q_{44}: \quad 5 \to 1.$$

It is invariant under compositions of mutations and permutations

$$r_0 = \mu_1\,(1,2)\,\mu_1, \quad r_1 = \mu_3\,(3,4)\,\mu_3, \quad \pi_1 = (1,4,2,3).$$

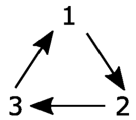


Figure 15. Q_{31}

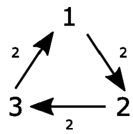


Figure 16. Q_{32}

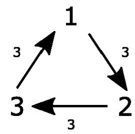


Figure 17. Q_{33}

Their actions on the coefficients generate a group of birational transformations which is isomorphic to an extended affine Weyl group of type $A_1^{(1)}$. The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1\, y_2, \quad \alpha_1 = y_3\, y_4.$$

The transformation π_1 acts on the parameters as

$$\pi_1((\alpha_0, \alpha_1)) = (\alpha_1, \alpha_0).$$

A translation of this Weyl group provides a q-Painlevé equation; see [1].

In the confluence $5 \to 1$ a degeneration of the coefficients is given by a replacement $y_1 \to y_1/\varepsilon$, $y_5 \to \varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{52}	$r_0 = \mu_5\, \mu_1\, (1, 2)\, \mu_1\, \mu_5$	r_1
Q_{44}	r_0	r_1

3.5.5. Q_{45}

The quiver Q_{45} is obtained via the following confluences.

$$Q_{52} \to Q_{45} : \quad 5 \to 3, \quad 4 \to 1.$$

It is invariant under a composition of a mutation and a permutation

$$\pi_1 = (1, 2, 3, 4)\, \mu_1,$$

which provides the q-Painlevé I equation; see [15].

§ 3.6. Quiver with 3 vertices

For the quiver Q_{41}, Q_{42}, Q_{43}, Q_{44} and Q_{45}, it is enough to investigate the following confluences.

$$
\begin{aligned}
Q_{41} :&\quad 4 \to 1.\\
Q_{42} :&\quad 4 \to 2, \quad 4 \to 3.\\
Q_{43} :&\quad 4 \to 1, \quad 4 \to 3, \quad 3 \to 2, \quad 2 \to 1.\\
Q_{44} :&\quad 4 \to 1.\\
Q_{45} :&\quad 4 \to 1, \quad 4 \to 2, \quad 4 \to 3, \quad 3 \to 1, \quad 3 \to 2, \quad 2 \to 1.
\end{aligned}
$$

Then we obtain the quivers Q_{31}, Q_{32} and Q_{33}. Note that, in the quiver Q_{42} and Q_{43}, all of arrows are removed via the confluence $3 \to 2$ and $3 \to 1$ respectively.

3.6.1. Q_{31}

The quiver Q_{31} is obtained via the following confluences.

$$
\begin{aligned}
Q_{41} \to Q_{31} :&\quad 4 \to 1.\\
Q_{42} \to Q_{31} :&\quad 4 \to 2, \quad 4 \to 3.\\
Q_{43} \to Q_{31} :&\quad 3 \to 2, \quad 2 \to 1.\\
Q_{45} \to Q_{31} :&\quad 4 \to 3, \quad 3 \to 2, \quad 2 \to 1.
\end{aligned}
$$

It is invariant under compositions of mutations and permutations

$$r_1 = \mu_1\,\mu_2\,(2,3)\,\mu_2\,\mu_1, \quad \pi_1 = (1,2,3).$$

The action of r_1 on the coefficients generates a group of birational transformations which is isomorphic to the Weyl group of type A_1. The parameter corresponding to the simple root is given by

$$\alpha_1 = y_1\,y_2\,y_3.$$

which is invariant under the action of π_1.

In the confluence $4 \to 1$ ($Q_{41} \to Q_{31}$) a degeneration of the coefficients is given by a replacement $y_1 \to \varepsilon$, $y_4 \to y_1/\varepsilon$ and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows.

Q_{41}	$r_1\,r_2\,r_1 = \mu_1\,\mu_4\,\mu_2\,(2,3)\,\mu_2\,\mu_4\,\mu_1$
Q_{31}	r_1

3.6.2. Q_{32}

The quiver Q_{32} is obtained via the following confluences.

$$\begin{aligned} Q_{43} \to Q_{32}: &\quad 4 \to 1, \quad 4 \to 3. \\ Q_{44} \to Q_{32}: &\quad 4 \to 1. \\ Q_{45} \to Q_{32}: &\quad 4 \to 2, \quad 3 \to 1. \end{aligned}$$

It is invariant under compositions of mutations and permutations

$$\pi_1 = (2,3)\,\mu_1, \quad \pi_2 = (1,2,3).$$

3.6.3. Q_{33}

The quiver Q_{33} is obtained via the following confluence.

$$Q_{45} \to Q_{33}: \quad 4 \to 1.$$

It is invariant under a permutation

$$\pi_1 = (1,2,3).$$

Acknowledgement

This work was supported by JSPS KAKENHI Grant Number 15K04911.

References

[1] M. Bershtein, P. Gavrylenko and A. Marshakov, *Cluster integrable systems, q-Painlevé equations and their quantization*, J. High Energ. Phys. (2018) 077.
[2] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002) 497–529.
[3] S. Fomin and A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007) 112–164.
[4] R. Inoue, T. Ishibashi and H. Oya, *Cluster realizations of Weyl groups and higher Teichmüller theory*, arXiv:1902.02716.
[5] R. Inoue, T. Lam and P. Pylyavskyy, *On the cluster nature and quantization of geometric R-matrices*, to appear in Publ. RIMS (arXiv:1607.00722).
[6] M. Jimbo and H. Sakai, *A q-analog of the sixth Painlevé equation*, Let. Math. Phys. **38** (1996) 145–154.
[7] K. Kajiwara and K. Kimura, *On a q-difference Painlevé III equation. I. Derivation, symmetry and Riccati type solutions*, J. Nonlin. Math. Phys. **10** (2003) 86–102.
[8] K. Kajiwara, M. Noumi and Y. Yamada, *A study on the fourth q-Painlevé equation*, J. Phys. A: Math. Gen. **34** (2001) 8563–8581.
[9] K. Kajiwara, M. Noumi and Y. Yamada, *Discrete dynamical systems with* $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ *symmetry*, Lett. Math. Phys. **60** (2002) 211–219.

[10] K. Kajiwara, M. Noumi and Y. Yamada, q-Painlevé systems arising from q-KP hierarchy, Lett. Math. Phys. **62** (2003) 259–268.

[11] K. Kajiwara, M. Noumi and Y. Yamada, *Geometric aspects of Painlevé equations*, J. Phys. A: Math. Theor. **50** (2017) Number 7.

[12] T. Masuda, N. Okubo and T. Tsuda, *in preparation.*

[13] T. Nakanishi, *Periodicities in cluster algebras and dilogarithm identities*, EMS Series of Congress Reports (2011) 407–444.

[14] H. Nagao and Y. Yamada, *Variations of q-Garnier system*, J. Phys. A: Math. and Theor. **51** (2018) 135204.

[15] N. Okubo, *Bilinear equations and q-discrete Painlevé equations satisfied by variables and coefficients in cluster algebras*, J. Phys. A: Math. Theor. **48** (2015) 355201.

[16] N. Okubo and T. Suzuki, *Generalized q-Painlevé VI systems of type* $(A_{2n+1}+A_1+A_1)^{(1)}$ *arising from cluster algebra*, arXiv:1810.03252.

[17] A. Ramani, B. Grammaticos and Y. Ohta, *The q-Painlevé V equation and its geometrical description*, J. Phys. A: Math. Gen. **34** (2001) 2505–2513.

[18] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001) 165–229.

[19] H. Sakai, *A q-analog of the Garnier system*, Funkcial. Ekvac. **48** (2005) 237–297.

[20] T. Suzuki, *A q-analogue of the Drinfeld-Sokolov hierarchy of type A and q-Painlevé system*, AMS Contemp. Math. **651** (2015) 25–38.

[21] T. Suzuki, *A reformulation of the generalized q-Painlevé VI system with* $W(A^{(1)}_{2n+1})$ *symmetry*, J. Integrable Syst. **2** (2017) xyw017.

[22] T. Tsuda, *Tau functions of q-Painlevé III and IV equations*, Lett. Math. Phys. **75** (2006) 39–47.

[23] T. Tsuda, *On an integrable system of q-difference equations satisfied by the universal characters: its Lax formalism and an application to q-Painlevé equations*, Comm. Math. Phys. **293** (2010) 347–359.

[24] T. Tsuda and T. Masuda, *q-Painlevé VI equation arising from q-UC hierarchy*, Comm. Math. Phys. **262** (2006) 595–609.