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An Ultradiscrete Permanent Solution to the Ultradiscrete Two-Dimensional Toda Equation

By

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Abstract

An ultradiscrete permanent solution to the ultradiscrete two-dimensional Toda equation is proposed. The solution is obtained using an ultradiscrete analogue of the Jacobi identity with provided certain elements.

§ 1. Introduction

The two-dimensional Toda equation is known as one of the celebrated equations of integrable system[1]. By discretizing continuous variables in the equation, the discrete two-dimensional Toda equation is obtained[2]. There are some expressions depending on the method of the discretization. For example, one is expressed by

$$\begin{aligned}&\delta\varepsilon\tau(l+1,m,n+1)\tau(l,m+1,n-1)\\=&(1+\delta\varepsilon)\tau(l+1,m+1,n)\tau(l,m,n)-\tau(l+1,m,n)\tau(l,m+1,n),\end{aligned}\tag{1.1}$$

and another is

$$\begin{aligned}&\tau(l,m,n+1)\tau(l+1,m+1,n-1)\\=&\tau(l+1,m+1,n)\tau(l,m,n)-\tau(l+1,m,n)\tau(l,m+1,n),\end{aligned}\tag{1.2}$$

where l, m, n are independent variables and δ, ε are parameters. The former equation is sometimes called the discrete two-dimensional Toda lattice equation and the latter the discrete two-dimensional Toda molecule equation. Both of equations admit the

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determinant solutions and they are verified by using identities of determinants. In this paper we focus on (1.2). The determinant solution is expressed by

$$\tau(l,m,n) = \det[\phi(l+i-1, m+j-1)]_{1\le i,j\le n}$$

$$(1.3) \quad = \begin{vmatrix} \phi(l,m) & \phi(l,m+1) & \cdots & \phi(l,m+n-1) \\ \phi(l+1,m) & \phi(l+1,m+1) & \cdots & \phi(l+1,m+n-1) \\ \cdots & \cdots & \cdots & \cdots \\ \phi(l+n-1,m) & \phi(l+n-1,m+1) & \cdots & \phi(l+n-1,m+n-1) \end{vmatrix},$$

where $\phi(l,m)$ is an arbitrary function and $\tau(l,m,0) = 1$. It is verified by using the Jacobi identity. The ultradiscrete two-dimensional Toda equation can be obtained by ultradiscretizing (1.2) with variable transformation $\tau(l,m,n) = (-1)^{(l+m)(l+m+1)/2} e^{T(l,m,n)/\varepsilon}$[3].

$$(1.4) \quad \begin{aligned} &T(l,m,n+1) + T(l+1,m+1,n-1) \\ &= \max(T(l+1,m+1,n) + T(l,m,n), T(l+1,m,n) + T(l,m+1,n)). \end{aligned}$$

An ultradiscrete solution can be also obtained by ultradiscretizing the solution of the discrete one in general. However the determinant solution (1.3) is not always positive. Thus, ultradiscrete solution to (1.4) cannot be obtained straightforwardly. For this problem, we propose a direct proof in ultradiscrete system and give a solution to (1.4).

This paper consists on below. In section 2, we first introduce the ultradiscrete permanent and an ultradiscrete analogue of the Jacobi identity. In section 3, we impose some conditions for elements of matrix and rewrite the identity. Using the rewritten identity, we give a solution to (1.4) in section 4. Finally, we give concluding remarks in section 5.

§ 2. UP and the ultradiscrete Jacobi Identity

We start from the definition of the ultradiscrete permanent(UP)[4]. Let N be a positive integer. The UP of $N \times N$ matrix $A = [a_{ij}]$ is defined by

$$(2.1) \quad \mathrm{up}[A] := \max_{\pi}(a_{1\pi_1} + a_{2\pi_2} + \cdots + a_{N\pi_N}),$$

where $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ is a set of all possible permutations of $\{1, 2, \ldots, N\}$. There are several identities between UPs[5]. One of them is regarded as an ultradiscrete analogue of the Jacobi identity, which is given by the following theorem[6].

Theorem 2.1. *Let N, k, l be positive integers satisfying $1 \le k < l \le N$. Then,*

$$(2.2) \quad \begin{aligned} &\max\left(\mathrm{up}[A] + \mathrm{up}[A^{kl}_{kl}], \mathrm{up}[A^k_k] + \mathrm{up}[A^l_l], \mathrm{up}[A^k_l] + \mathrm{up}[A^l_k]\right) \\ &= \max\left(\mathrm{up}[A] + \mathrm{up}[A^{kl}_{kl}], \mathrm{up}[A^k_k] + \mathrm{up}[A^l_l]\right) \\ &= \max\left(\mathrm{up}[A] + \mathrm{up}[A^{kl}_{kl}], \mathrm{up}[A^k_l] + \mathrm{up}[A^l_k]\right) \\ &= \max\left(\mathrm{up}[A^k_k] + \mathrm{up}[A^l_l], \mathrm{up}[A^k_l] + \mathrm{up}[A^l_k]]\right) \end{aligned}$$

holds. Here A_l^k denotes the $(N-1)\times(N-1)$ matrix obtained by eliminating the kth row and the lth column from A.

Due to the symmetry derived from the definition of UP, (2.2) cannot be applied to the solution to the ultradiscrete two-dimensional Toda equation directly. In the next section, we impose some conditions for the elements of UP in (2.2).

§ 3. Specialization of the ultradiscrete Jacobi Identity

In this section, we show the following theorem.

Theorem 3.1. *Suppose $a_{ij} = |x_i + jy_i|$ for $i, j = 1, 2, \ldots, N$, where x_i, y_i are arbitrary parameters satisfying $0 \le y_1 \le y_2 \le \cdots \le y_N$, and $|\cdot|$ denotes absolute value. Then*

$$\mathrm{up}[A] + \mathrm{up}[A^{1N}_{1N}] = \max\left(\mathrm{up}[A^1_1] + \mathrm{up}[A^N_N], \mathrm{up}[A^1_N] + \mathrm{up}[A^N_1]\right) \tag{3.1}$$

holds.

Proof. Equation (3.1) holds if we show the inequality

$$\mathrm{up}[A] + \mathrm{up}[A^{1N}_{1N}] \ge \mathrm{up}[A^1_1] + \mathrm{up}[A^N_N] \tag{3.2}$$

for (2.2). Each UP in (3.2) can be expanded by using the following formula[4]

$$\begin{aligned}&\mathrm{up}\begin{bmatrix} |x_1+y_1| & |x_1+2y_1| & \cdots & |x_1+Ny_1| \\ |x_2+y_2| & |x_2+2y_2| & \cdots & |x_2+Ny_2| \\ \vdots & \vdots & \ddots & \vdots \\ |x_N+y_N| & |x_N+2y_N| & \cdots & |x_N+Ny_N| \end{bmatrix} \\ &= \max_{\rho_i\in\{-1,1\}}\left(\sum_{i=1}^N \rho_i x_i + \frac{1}{2}\sum_{i=1}^N\left((1+\rho_i)i + \rho_i\sum_{j=i}^N(1-\rho_j)\right)y_i\right),\end{aligned} \tag{3.3}$$

where x_i is arbitrary and $0 \le y_1 \le y_2 \le \cdots \le y_N$. Moreover, by introducing a transformation $\rho_i = 2\mu_i - 1$, $\mathrm{up}[A]$ is reduced to

$$\begin{aligned}\mathrm{up}[A] = \max_{\mu_i\in\{0,1\}}&\left(\sum_{i=1}^N \mu_i(2x_i + y_i(2N-i+1)) + \sum_{1\le j<i\le N}\mu_i y_j - 2\sum_{1\le i<j\le N}\mu_i\mu_j y_i\right) \\ &-\sum_{i=1}^N((N-i+1)y_i + x_i).\end{aligned} \tag{3.4}$$

Similar procedure, we obtain

$$\begin{aligned}
\mathrm{up}[A_{1N}^{1N}] &= \max_{\mu_i\in\{0,1\}} \left(\sum_{i=2}^{N-1} \mu_i(2x_i + y_i(2N-i)) + \sum_{2\le j<i\le N-1} \mu_i y_j - 2 \sum_{2\le i<j\le N-1} \mu_i\mu_j y_i \right) \\
&\quad - \sum_{i=2}^{N-1} ((N-i+1)y_i + x_i) \\
\mathrm{up}[A_1^1] &= \max_{\mu_i\in\{0,1\}} \left(\sum_{i=2}^{N} \mu_i(2x_i + y_i(2N-i+2)) + \sum_{2\le j<i\le N} \mu_i y_j - 2 \sum_{2\le i<j\le N} \mu_i\mu_j y_i \right) \\
&\quad - \sum_{i=2}^{N} ((N-i+2)y_i + x_i) \\
\mathrm{up}[A_N^N] &= \max_{\mu_i\in\{0,1\}} \left(\sum_{i=1}^{N-1} \mu_i(2x_i + y_i(2N-i-1)) + \sum_{1\le j<i\le N-1} \mu_i y_j - 2 \sum_{1\le i<j\le N-1} \mu_i\mu_j y_i \right) \\
&\quad - \sum_{i=1}^{N-1} ((N-i)y_i + x_i).
\end{aligned}$$

Thus (3.2) is reduced into
(3.5)

$$\begin{aligned}
&\max_{\mu_i\in\{0,1\}} \left(\sum_{i=1}^{N} \mu_i(2x_i + y_i(2N-i+1)) + \sum_{1\le j<i\le N} \mu_i y_j - 2 \sum_{1\le i<j\le N} \mu_i\mu_j y_i \right) \\
&+ \max_{\nu_i\in\{0,1\}} \left(\sum_{i=2}^{N-1} \nu_i(2x_i + y_i(2N-i)) + \sum_{2\le j<i\le N-1} \nu_i y_j - 2 \sum_{2\le i<j\le N-1} \nu_i\nu_j y_i \right) + y_N - y_1 \\
&\ge \max_{\mu_i\in\{0,1\}} \left(\sum_{i=2}^{N} \mu_i(2x_i + y_i(2N-i+2)) + \sum_{2\le j<i\le N} \mu_i y_j - 2 \sum_{2\le i<j\le N} \mu_i\mu_j y_i \right) \\
&+ \max_{\nu_i\in\{0,1\}} \left(\sum_{i=1}^{N-1} \nu_i(2x_i + y_i(2N-i-1)) + \sum_{1\le j<i\le N-1} \nu_i y_j - 2 \sum_{1\le i<j\le N-1} \nu_i\nu_j y_i \right).
\end{aligned}$$

Introducing transformations $\lambda_i = \mu_i + \nu_i$, $\sigma_i = \mu_i - \nu_i$, we rewrite LHS of (3.5) as

$$\begin{aligned}
&\max_{(\lambda_i,\sigma_i)} \left(\sum_{i=1}^{N} \lambda_i(2x_i + y_i(2N-i)) + \frac{1}{2}\sum_{i=1}^{N} \lambda_i(y_i - y_1) + \sum_{1\le j<i\le N} \lambda_i y_j - \sum_{1\le i<j\le N} \lambda_i\lambda_j y_i \right. \\
&\quad \left. + \frac{1}{2}\sum_{i=1}^{N} \sigma_i(y_i + y_1) - \sum_{1\le i<j\le N} \sigma_i\sigma_j y_i \right) + y_N - y_1,
\end{aligned}$$

where $\max_{(\lambda_i,\sigma_i)} F(\lambda_1,\ldots,\lambda_N,\sigma_1,\ldots,\sigma_N)$ denotes the maximum value of F among 2^{2N-2} cases of $(\lambda_1,\ldots,\lambda_N,\sigma_1,\ldots,\sigma_N)$ replacing by

$$(\lambda_i,\sigma_i)\in\begin{cases}\{(1,1),(0,0)\} & (i=1)\\ \{(2,0),(1,1),(1,-1),(0,0)\} & (i=2,3,\ldots,N-1)\\ \{(1,1),(0,0)\} & (i=N)\end{cases}. \tag{3.6}$$

Note σ_i is determined as 0 when $\lambda_i\neq 1$. Thus we have

(3.7)

$$\begin{aligned}&\mathrm{up}[A]+\mathrm{up}[A^{1N}_{1N}]\\ &=\max_{\substack{\lambda_i\in\{0,1,2\}\\ \lambda_1,\lambda_N\in\{0,1\}}}\left(\sum_{i=1}^{N}\lambda_i(2x_i+y_i(2N-i))+\frac12\sum_{i=1}^{N}\lambda_i(y_i-y_1)+\sum_{1\le j<i\le N}\lambda_i y_j-\sum_{1\le i<j\le N}\lambda_i\lambda_j y_i\right.\\ &\qquad\left.+\max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_1=\sigma_N=1}}\left(\frac12\sum_{i=1}^{n}\sigma_{k_i}(y_{k_i}+y_1)-\sum_{1\le i<j\le n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right)\right)+y_N-y_1,\end{aligned}$$

where n is the total number of i such that $\lambda_i=1$ and k_i is a rearranged number satisfying

$$1\le k_1<k_2<\cdots<k_n\le N,\qquad \lambda_{k_i}=1. \tag{3.8}$$

Similarly, RHS of (3.5) is expressed by

(3.9)

$$\begin{aligned}&\mathrm{up}[A^{1}_{1}]+\mathrm{up}[A^{N}_{N}]\\ &=\max_{\substack{\lambda_i\in\{0,1,2\}\\ \lambda_1,\lambda_N\in\{0,1\}}}\left(\sum_{i=1}^{N}\lambda_i(2x_i+y_i(2N-i))+\frac12\sum_{i=1}^{N}\lambda_i(y_i-y_1)+\sum_{1\le j<i\le N}\lambda_i y_j-\sum_{1\le i<j\le N}\lambda_i\lambda_j y_i\right.\\ &\qquad\left.+\max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_1=-1,\sigma_N=1}}\left(\frac12\sum_{i=1}^{n}\sigma_{k_i}(3y_{k_i}-y_1)-\sum_{1\le i<j\le n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right)\right).\end{aligned}$$

Comparing (3.7) and (3.9), one can check the terms depending on λ are the same expressions. Thus (3.2) holds if we show

$$\begin{aligned}&\max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_1=\sigma_N=1}}\left(\frac12\sum_{i=1}^{n}\sigma_{k_i}(y_{k_i}+y_1)-\sum_{1\le i<j\le n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right)+y_N-y_1\\ &\ge\max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_1=-1,\sigma_N=1}}\left(\frac12\sum_{i=1}^{n}\sigma_{k_i}(3y_{k_i}-y_1)-\sum_{1\le i<j\le n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right).\end{aligned}\tag{3.10}$$

We denote LHS and RHS of (3.10) as L and R hereafter. From proposition A.1 in the appendix, L and R can be given as below. When $k_1 \neq 1$, L is reduced to

$$L = \frac{1}{2}\sum_{i=1}^{n}(-1)^{n-i}(y_{k_i} + y_1) - \sum_{1\leq i<j\leq n}(-1)^{i+j}y_{k_i} + y_N - y_1 \tag{3.11}$$

by setting $\sigma_{k_i} = (-1)^{n-i}$, and

$$R = \frac{1}{2}\left(\sum_{i=1}^{n-1}(-1)^{n-i+1}(y_{k_i} - y_1) + 3y_{k_n} - y_1\right) - \sum_{1\leq i<j\leq n-1}(-1)^{i+j}y_{k_i} \tag{3.12}$$

by $\sigma_{k_i} = (-1)^{n-i+1}$ for $i = 1, 2, \ldots, n-1$ and $\sigma_{k_n} = 1$. Hence, in this case we have

$$L - R = y_N - y_{k_n} \geq 0. \tag{3.13}$$

When $k_1 = 1$, L and R are rewritten as

$$\begin{aligned} L &= \max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_N=1}}\left(y_1 + \frac{1}{2}\sum_{i=2}^{n}\sigma_{k_i}(y_{k_i} - y_1) - \sum_{2\leq i<j\leq n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right) + y_N - y_1, \\ R &= \max_{\substack{\sigma_{k_i}\in\{-1,1\}\\ \sigma_N=1}}\left(-y_1 + \frac{1}{2}\sum_{i=2}^{n}\sigma_{k_i}(3y_{k_i} + y_1) - \sum_{2\leq i<j\leq n}\sigma_{k_i}\sigma_{k_j}y_{k_i}\right). \end{aligned} \tag{3.14}$$

In this case, L is reduced to

$$L = \frac{1}{2}\sum_{i=2}^{n}(-1)^{n-i}(y_{k_i} - y_1) - \sum_{2\leq i<j\leq n}(-1)^{i+j}y_{k_i} + y_N \tag{3.15}$$

by setting $\sigma_{k_1} = 1$, $\sigma_{k_i} = (-1)^{n-i}$ for $i = 2, 3, \ldots, n$, and

$$R = -y_1 + \frac{1}{2}\left(\sum_{i=2}^{n-1}(-1)^{n-i+1}(y_{k_i} + y_1) + 3y_{k_n} + y_1\right) - \sum_{2\leq i<j\leq n-1}(-1)^{i+j}y_{k_i} \tag{3.16}$$

by $\sigma_{k_1} = -1$, $\sigma_{k_i} = (-1)^{n-i+1}$ for $i = 2, 3, \ldots, n-1$, $\sigma_{k_n} = 1$. Hence, we have

$$L - R = y_N - y_{k_n} \geq 0. \tag{3.17}$$

Therefore this completes the proof. □

§ 4. UP solution to the ultradiscrete two-dimensional Toda equation

From theorem 3.1, we obtain an UP solution to the ultradiscrete two-dimensional Toda equation, namely, we obtain the following theorem.

Theorem 4.1. *Define* $T(l,m,n) = \mathrm{up}[|p_{l+i-1}+(m+j-1)q_{l+i-1}|]_{1\le i,j\le n}$, *where* p_i, q_i *are arbitrary parameters satisfying*

$$0 \le q_1 \le q_2 \le \cdots \le q_N \tag{4.1}$$

and $T(l,m,0) = 0$. *Then* $T(l,m,n)$ *satisfies the ultradiscrete two-dimensional Toda equation*

$$\begin{aligned} &T(l,m,n) + T(l+1,m+1,n-2) \\ &= \max(T(l+1,m+1,n-1)+T(l,m,n-1), T(l+1,m,n-1)+T(l,m+1,n-1)). \end{aligned} \tag{4.2}$$

Finally, we comment on the relation between the obtained UP and its discrete analogue. The obtained UP is expressed by $\mathrm{up}[|x_i+jy_i|]_{1\le i,j\le n}$ as well as some ultradiscrete soliton solutions[4, 8]. For this type of UP, we have the following theorem[9].

Theorem 4.2. *Define*

$$D = [X_iY_i^j + (-1)^{i+1}(X_iY_i^j)^{-1}]_{1\le i,j\le N}, \tag{4.3}$$

where $X_i > 0$, $1 < Y_1 < Y_2 < \cdots < Y_N$. *Then* $\det[D]$ *can be ultradiscretized as the UP, that is,*

$$\lim_{\varepsilon\to+0} \varepsilon \log\det[D] = \mathrm{up}[|x_i + jy_i|]_{1\le i,j\le N} \tag{4.4}$$

holds under the transformations $X_i = e^{x_i/\varepsilon}$, $Y_i = e^{y_i/\varepsilon}$.

Theorem 4.2 is proved by expanding (4.3) and using (3.3). From this theorem, we can confirm that some discrete soliton solutions in determinant form take positive values and can be ultradiscretized as ultradiscrete ones in UP form(See [9]). However, the theorem cannot apply to the determinant solution to (1.2) even if we set $\phi(l,m)$ in (1.3) as (4.3). In fact, $\det[D_1^1]$ may take a negative value. This fact shows there is a gap between UP soliton solutions and the UP solution to (4.2).

§ 5. Concluding Remarks

We have given an UP solution to the ultradiscrete two-dimensional Toda equation. It is confirmed by using an ultradiscrete analogue of the Jacobi identity. In other words, we have given a direct proof in ultradiscrete system. This approach enable us avoiding for considering positivity and magnitude relations among parameters in discrete systems. However, instead of that, the problem remains that the discrete analogue of our obtained solution is not cleared. Clarifying the relations is one of future problems. In addition, we note that a solution to the ultradiscrete two-dimensional Toda equation is discussed in [7] from the view point of biorthogonal polynomials. Its solution is derived from the determinant solution with a finite boundary condition while our solution semi-infinite condition. Investigating the relation between them is also a future problem.

§ A. Maximum values of (3.10)

In this appendix, we give the following proposition[8].

Proposition A.1. *Suppose m, n are positive integers and p_i, q_i satisfy*

$$\begin{array}{ll} & 0 = p_{n+1} \leq p_n \leq p_{n-1} \leq \cdots \leq p_1 \\ \text{(A.1)} & 0 = q_{n+1} \leq q_n \leq q_{n-1} \leq \cdots \leq q_1 \\ & (m-1)(p_i - p_j) \leq q_i - q_j \leq m(p_i - p_j) \qquad (1 \leq i < j \leq n+1) \end{array}.$$

Define $g(\sigma_1, \sigma_2, \ldots, \sigma_n)$ by

$$\text{(A.2)} \qquad g(\sigma_1, \sigma_2, \ldots, \sigma_n) := \sum_{i=1}^{n} \sigma_i q_i - \sum_{1 \leq i < j \leq n} \sigma_i \sigma_j p_j,$$

where σ_i is 1 or -1. Then,

$$\text{(A.3)} \qquad g(\sigma_1, \sigma_2, \ldots, \bar{\sigma}_{n-k}, \ldots, \bar{\sigma}_n) \geq g(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

holds for $0 \leq k \leq n-1$. Here $\bar{\sigma}_i$ is defined by

$$\text{(A.4)} \qquad \bar{\sigma}_i = \begin{cases} 1 & \left(\sum_{l=1}^{l=n-k-1} \sigma_l + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \leq m-1 \right) \\ -1 & \left(\sum_{l=1}^{l=n-k-1} \sigma_l + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \geq m \right) \end{cases}.$$

Proof. We prove by mathematical induction on k. For $k = 0$, (A.3) holds since

$$\text{(A.5)} \qquad g(\sigma_1, \sigma_2, \ldots, \bar{\sigma}_n) - g(\sigma_1, \sigma_2, \ldots, -\bar{\sigma}_n) = 2\bar{\sigma}_n \left(q_n - p_n \sum_{i=1}^{n-1} \sigma_i \right) \geq 0.$$

We assume

$$\text{(A.6)} \qquad g(\sigma_1, \sigma_2, \ldots, \sigma_{n-k-1}, \bar{\sigma}_{n-k}, \ldots, \bar{\sigma}_n) \geq g(\sigma_1, \sigma_2, \ldots, \sigma_{n-k-1}, \sigma_{n-k}, \ldots, \sigma_n)$$

holds for a certain k. Let us prove the inequality

$$\text{(A.7)} \qquad g(\sigma_1, \sigma_2, \ldots, \bar{\sigma}_{n-k-1}, \bar{\sigma}_{n-k}, \ldots, \bar{\sigma}_n) \geq g(\sigma_1, \sigma_2, \ldots, -\bar{\sigma}_{n-k-1}, \hat{\sigma}_{n-k}, \ldots, \hat{\sigma}_n),$$

where $\bar{\sigma}_i$ and $\hat{\sigma}_i$ are defined by

(A.8)
$$\bar{\sigma}_{n-k-1} = \begin{cases} 1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l \leq m-1\right) \\ -1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l \geq m\right) \end{cases},$$
$$\bar{\sigma}_i = \begin{cases} 1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l + \bar{\sigma}_{n-k-1} + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \leq m-1\right) \\ -1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l + \bar{\sigma}_{n-k-1} + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \geq m\right) \end{cases},$$
$$\hat{\sigma}_i = \begin{cases} 1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l - \bar{\sigma}_{n-k-1} + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \leq m-1\right) \\ -1 & \left(\sum_{l=1}^{l=n-k-2} \sigma_l - \bar{\sigma}_{n-k-1} + \sum_{l=n-k}^{i-1} \bar{\sigma}_l \geq m\right) \end{cases}.$$

Then, from (A.8), we have

(A.9)
$$\begin{aligned} & g(\sigma_1, \sigma_2, \ldots, \bar{\sigma}_{n-k-1}, \bar{\sigma}_{n-k}, \ldots, \bar{\sigma}_n) - g(\sigma_1, \sigma_2, \ldots, -\bar{\sigma}_{n-k-1}, \hat{\sigma}_{n-k}, \ldots, \hat{\sigma}_n) \\ =& 2\bar{\sigma}_{n-k-1} q_{n-k-1} + \sum_{i=n-k}^{n} (\bar{\sigma}_i - \hat{\sigma}_i) q_i - 2 \sum_{i=1}^{n-k-2} \sigma_i \bar{\sigma}_{n-k-1} p_{n-k-1} \\ & - \sum_{i=1}^{n-k-2} \sum_{j=n-k}^{n} \sigma_i (\bar{\sigma}_j - \hat{\sigma}_j) p_j - \sum_{j=n-k}^{n} \bar{\sigma}_{n-k-1} (\bar{\sigma}_j + \hat{\sigma}_j) p_j - \sum_{i=n-k}^{n} \sum_{j=i+1}^{n} (\bar{\sigma}_i \bar{\sigma}_j - \hat{\sigma}_i \hat{\sigma}_j) p_j. \end{aligned}$$

We denote $\sum_{i=1}^{n-k-2} \sigma_i$ by S and consider two cases of $S \geq m$ or $S \leq m-1$.

In the case of $S \geq m$, $\bar{\sigma}_i$ and $\hat{\sigma}_i$ are given by

(A.10)
$$\bar{\sigma}_i = \begin{cases} -1 & (i = n-k-1, \ldots, S^*-1) \\ (-1)^{S^*+i} & (i = S^*, \ldots, n) \end{cases}, \qquad \hat{\sigma}_i = \begin{cases} -\bar{\sigma}_i & (i = S^*) \\ \bar{\sigma}_i & (i \neq S^*) \end{cases}$$

from (A.8). The symbol S^* denotes $n-k+S-m$. Thus we have

(A.11)
$$\begin{aligned}
&g(\sigma_1,\sigma_2,\dots,\bar\sigma_{n-k-1},\bar\sigma_{n-k},\dots,\bar\sigma_n)-g(\sigma_1,\sigma_2,\dots,-\bar\sigma_{n-k-1},\hat\sigma_{n-k},\dots,\hat\sigma_n)\\
=&2\left(-q_{n-k-1}+q_{S^*}+S(p_{n-k-1}-p_{S^*})-\sum_{i=n-k}^{S^*-1}p_j+(S^*-n+k)p_{S^*}\right)\\
\geq&2\left(-q_{n-k-1}+q_{S^*}+S(p_{n-k-1}-p_{S^*})-(S^*-n+k)p_{n-k}+(S^*-n+k)p_{S^*}\right)\\
=&2\left(-q_{n-k-1}+q_{S^*}+S(p_{n-k-1}-p_{n-k})+m(p_{n-k}-p_{S^*})\right)\\
\geq&2\left(-q_{n-k-1}+q_{S^*}+m(p_{n-k-1}-p_{n-k})+m(p_{n-k}-p_{S^*})\right)\\
=&2\left(m(p_{n-k-1}-p_{S^*})-(q_{n-k-1}-q_{S^*})\right)\\
\geq&0.
\end{aligned}$$

In the case of $S\leq m-1$, $\bar\sigma_i$ and $\hat\sigma_i$ are given by

(A.12)
$$\bar\sigma_i=\begin{cases}1 & (i=n-k-1,\dots,S_*-1)\\ (-1)^{S_*+i+1} & (i=S_*,\dots,n)\end{cases},\qquad \hat\sigma_i=\begin{cases}-\bar\sigma_i & (i=S_*)\\ \bar\sigma_i & (i\neq S_*)\end{cases},$$

where $S_*=n-k-S+m$. Thus we have

$$\begin{aligned}
&g(\sigma_1,\sigma_2,\dots,\bar\sigma_{n-k-1},\bar\sigma_{n-k},\dots,\bar\sigma_n)-g(\sigma_1,\sigma_2,\dots,-\bar\sigma_{n-k-1},\hat\sigma_{n-k},\dots,\hat\sigma_n)\\
=&2\left(q_{n-k-1}-q_{S^*}-S(p_{n-k-1}-p_{S_*})-\sum_{i=n-k}^{S^*-1}p_j+(S^*-n+k)p_{S_*}\right)\\
\geq&2\left(q_{n-k-1}-q_{S^*}-S(p_{n-k-1}-p_{S^*})-(S_*-n+k)p_{n-k}+(S_*-n+k)p_{S^*}\right)\\
=&2\left(q_{n-k-1}-q_{S^*}-S(p_{n-k-1}-p_{n-k})-(m-1)(p_{n-k}-p_{S_*})\right)\\
\geq&2\left(q_{n-k-1}-q_{S^*}-(m-1)(p_{n-k-1}-p_{n-k})-(m-1)(p_{n-k}-p_{S_*})\right)\\
=&2\left(q_{n-k-1}-q_{S^*}-(m-1)(p_{n-k-1}-p_{S_*})\right)\\
\geq&0.
\end{aligned}\tag{A.13}$$

Therefore, (A.7) holds and we completed the proof. □

Proposition leads that

$$\max_{\substack{\sigma_i\in\{-1,1\}\\1\leq i\leq n}} g(\sigma_1,\sigma_2,\dots,\sigma_n)=g(\bar\sigma_1,\bar\sigma_2,\dots,\bar\sigma_n),\tag{A.14}$$

where

$$\bar\sigma_i=\begin{cases}1 & (i=1,2,\dots,m)\\ (-1)^{m+i} & (i=m+1,m+2,\dots,n)\end{cases}.\tag{A.15}$$

Using this result, we can determine σ_i for L and R in (3.10).

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