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Transitions of generalised Bessel kernels related to biorthogonal ensembles

By

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Abstract

Biorthogonal ensembles are generalisations of classical orthogonal ensembles such as the Laguerre or the Hermite ensembles. Local fluctuation of these ensembles at the origin has been studied, and determinantal kernels in the limit are described by the Wright generalised Bessel functions. The limit kernels are one parameter deformations of the Bessel kernel and the sine kernel for the Laguerre weight and the Hermite weight, respectively. We study transitions from these generalised Bessel kernels to the sine kernel under appropriate scaling limits in common with classical kernels.

§ 1. Introduction and main results

§ 1.1. Biorthogonal ensembles

We consider random point fields on an underlying space $I \subset \mathbb{R}$. For $\theta > 0$, we focus on biorothogonal ensembles with N particles, which are described by the following probability density functions on I^N

$$p^N(x_1, \ldots, x_N) = \frac{1}{Z_N} \prod_{1 \le i < j \le N} (x_i - x_j)(x_i^\theta - x_j^\theta) \prod_{k=1}^{N} w(x_k). \tag{1.1}$$

Here Z_N is the normalization constant and $w : I \to (0, \infty)$ is a weight function of a certain class.

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When $\theta = 1$, (1.1) corresponds the classical orthogonal ensembles, which include eigenvalue distributions of Gaussian random matrices with symmetry. The classical ensembles play a fundamental role in random matrix theory, and a lot of results have been established (see for example [3, 9, 14, 18]). Muttalib introduced (1.1) for general θ to describe appropriate models of disordered conductors in the metallic regime [15].

The biorthogonal ensembles have determinantal structure in common with classical ensembles. We define the n-correlation function ρ^n as

$$\rho^n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_I \cdots \int_I p^N(x_1, \dots, x_N)\, dx_{n+1} \dots dx_N.$$

A random point field is called determinantal if there exists a function $K : I \times I \to \mathbb{C}$ such that its correlation functions satisfy

$$\rho^n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n}$$

for each $1 \le n \le N$. Asymptotic behaviour of determinantal random point fields hence boils down to asymptotic analysis for determinantal kernels.

For the biorthogonal ensembles (1.1), determinantal kernels K^N are expressed in terms of biorthogonal polynomials with respect to weight w. Assume that there exist families of polynomials $\{p_i(x)\}_{i \in \{0\} \cup \mathbb{N}}$ and $\{q_i(x)\}_{i \in \{0\} \cup \mathbb{N}}$ such that $\deg(p_i) = \deg(q_i) = i$ and that these satisfy biorthogonal relation

$$\int_I p_i(x) q_j(x^\theta) w(x)\, dx = \delta_{ij}.$$

The polynomials of course depend on θ and w, but we suppress the dependence from the notations. Such families exist uniquely if matrix

$$\left(\int_I x^{i+j\theta} w(x)\, dx \right)_{i,j=0,\dots,n-1}$$

is non-singular for each n. Then a determinantal kernel K^N for (1.1) is explicitly given by

$$K^N(x, y) = \sqrt{w(x)w(y)} \sum_{i=0}^{N-1} p_i(x) q_i(y^\theta). \tag{1.2}$$

In the present paper, we focus on the following two classical weights:

(Biorthogonal Laguerre ensemble) $I = (0, \infty), w(x) = x^\alpha e^{-x}$ for $\alpha > -1$.
(Biorthogonal Hermite ensemble) $I = \mathbb{R}, w(x) = |x|^\alpha e^{-x^2}$ for $\alpha > -1$.

The determinantal kernels (1.2) for the biorthogonal Laguerre and Hermite ensembles are denoted by $K_{\alpha,\theta}^{\mathrm{Lag},N}$ and $K_{\alpha,\theta}^{\mathrm{Her},N}$, respectively.

§ 1.2. Local statistics and generalised Bessel kernels

Local statistics at the origin for the biorthogonal Laguerre and Hermite ensembles have been studied, and limit kernels are given explicitly [4]. Both kernels are described by the Wright generalised Bessel functions. The special function was introduced by Wright [20], which is the entire function given by

$$J_{a,b}(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m!\Gamma(a+bm)},$$

where $a \in \mathbb{C}$ and $b \in (0, \infty)$. Then we set

$$L_{\alpha,\theta}(x,y) = \theta \int_0^1 J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(xt) J_{\alpha+1,\theta}((yt)^\theta) t^\alpha \, dt. \tag{1.3}$$

We see the Laguerre case first. Define the biorthogonal Bessel kernel $K_{\mathrm{Be},\alpha,\theta}$: $(0,\infty) \times (0,\infty) \to \mathbb{R}$ as

$$K_{\mathrm{Be},\alpha,\theta}(x,y) = x^\alpha L_{\alpha,\theta}(x,y). \tag{1.4}$$

Then it was obtained as hard-edge scaling limit in [4] that

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{\theta}}} K_{\alpha,\theta}^{\mathrm{Lag},N}\Big(\frac{x}{N^{\frac{1}{\theta}}}, \frac{y}{N^{\frac{1}{\theta}}}\Big) = K_{\mathrm{Be},\alpha,\theta}(x,y). \tag{1.5}$$

We remark that the limit kernel is one parametrisation of the Bessel kernel $K_{\mathrm{Be},\alpha}$, which is given by

$$K_{\mathrm{Be},\alpha}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)},$$

where J_α is the Bessel function of order α. The Bessel kernel is arising from the hard-edge scaling limit of the Laguerre unitary ensembles [8, 19]. When $\theta = 1$, we see that the biorthogonal Bessel kernel corresponds to the Bessel kernel as expected.

Borodin also showed that the biorthogonal Bessel kernel appeared as hard-edge scaling limit of biorthogonal Jacobi ensemble, that is, the case that the weight is given by $w(x) = x^\alpha$ on $I = (0, 1)$.

To see local statistics for the Hermite case, define the biorthogonal sine kernel $K_{\sin,\alpha,\theta} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$K_{\sin,\alpha,\theta}(x,y) = |x|^\alpha \Big\{ L_{\frac{\alpha-1}{2},\theta}(x^2,y^2) + x^\theta y L_{\frac{\alpha+\theta}{2},\theta}(x^2,y^2) \Big\}.$$

Then it was shown in [4] that

$$\lim_{N\to\infty} \big(\frac{2}{N}\big)^{\frac{1}{2\theta}} K_{\alpha,\theta}^{\mathrm{Her},N}\Big(\big(\frac{2}{N}\big)^{\frac{1}{2\theta}}x, \big(\frac{2}{N}\big)^{\frac{1}{2\theta}}y\Big) = K_{\sin,\alpha,\theta}(x,y).$$

The limit kernel $K_{\sin,\alpha,\theta}$ is a generalisation of the sine kernel

$$K_{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)},$$

which appears as the bulk scaling limit of the Gaussian or Laguerre unitary ensembles. Actually when $\alpha = 0$ and $\theta = 1$, it is not difficult to see

$$K_{\sin,0,1}(x,y) = \frac{2}{\pi} K_{\sin}\Big(\frac{2}{\pi}x, \frac{2}{\pi}y\Big).$$

Recently, more and more studies on the biorthogonal Laguerre ensembles have been done. Random matrix models related to the biorthogonal ensembles were found [6, 11]. They gave specific random matrices whose eigenvalue distributions are given by the biorthogonal Laguerre ensembles.

Furthermore, it has been found intimate relations between the biorthogonal Laguerre ensembles and the products of non-Hermitian random matrices. The Ginibre matrices are square random matrices with no symmetry whose entries are independent complex Gaussian distributions. It was shown that the squared singular values of M products of independent Ginibre matrices were determinantal random point fields, and its kernels were written in terms of Meijer G-functions [2]. This result was extended to the case of products of rectangle Ginibre matrices [1]. For $\theta = M$ or $\theta = 1/M$, these kernels and the limit kernel of hard-edge scaling limit are related to $K_{\alpha,\theta}^{\mathrm{Lag},N}$ and $K_{\mathrm{Be},\alpha,\theta}$, respectively [13, 21].

The global density of the biorthogonal Laguerre ensembles was studied in [7, 10, 11]. The limiting distribution of the empirical distribution of $(\frac{x_1}{N}, \dots, \frac{x_N}{N})$ as $N \to \infty$, where $(x_1, \dots, x_N)$ are particles under the biorthogonal Laguerre ensemble, is given by the associated equilibrium measure.

Furthermore, the global density is specified by the Fuss-Catalan distribution under suitable scaling. Let ρ_θ be the probability density function which is uniquely determined such that the s-th moment is the Fuss-Catalan number

$$\int_0^\infty x^s \rho_\theta(x)\,dx = \frac{1}{\theta s + 1}\binom{\theta s + s}{s}.$$

Then ρ_θ determines a probability measure on $[0, (1+\theta)^{1+\theta}/\theta^\theta]$, which is called the Fuss-Catalan distribution. After change of variables $s_k = (\frac{x_k}{N\theta})^\theta$ for $(x_1, \dots, x_N)$ under the biorthogonal Laguerre ensemble, we get that the empirical distribution of $(s_1, \dots, s_N)$ converges to the Fuss-Catalan distribution as N to infinity.

The density function of the Fuss-Catalan distribution is explicitly described by using the following parametrisation. We consider

$$x = \frac{(\sin((1+\theta)\varphi))^{1+\theta}}{\sin\varphi(\sin(\theta\varphi))^\theta} \qquad \text{for } 0 < \varphi < \frac{\pi}{1+\theta},$$

and this gives a one-to-one correspondence between $(0, \pi/(1+\theta))$ and $(0, (1+\theta)^{1+\theta}/\theta^\theta)$. The density function in terms of this parametrisation is given by

$$\rho_\theta(x) = \frac{1}{\pi x}\frac{\sin((1+\theta)\varphi)\sin\varphi}{\sin(\theta\varphi)} = \frac{\sin(\theta\varphi)^{\theta-1}(\sin\varphi)^2}{\pi\sin((\theta+1)\varphi)^\theta} \qquad \text{for } 0<\varphi<\frac{\pi}{1+\theta}.$$

Local statistics of the biorhogonal Laguerre ensembles other than the origin was also shown [21]. For any α and θ, let x_0 be a point in the bulk region $x_0 \in (0, (1+\theta)^{1+\theta}/\theta^\theta)$, then we have

$$\begin{aligned}(1.6)\qquad \lim_{N\to\infty} e^{\pi(\cot\varphi)(x-y)}\frac{x_0^{\frac{1-\theta}{\theta}}}{\rho_\theta(\varphi)}K^{\mathrm{Lag},N}_{\alpha,\theta}\Big(N\theta\Big(x_0+\frac{x}{N\rho_\theta(\varphi)}\Big)^{\frac{1}{\theta}}, N\theta\Big(x_0+\frac{y}{N\rho_\theta(\varphi)}\Big)^{\frac{1}{\theta}}\Big)\\ = K_{\sin}(x,y),\end{aligned}$$

uniformly for x and y in any compact subset in $\mathbb{R}$. At the soft-edge $x_0 = (1+\theta)^{1+\theta}/\theta^\theta$, the Airy kernel was obtained as an appropriate scaling limit [21].

One of the most important topic in random matrix theory is the universality for random matrices in the following sense: limit kernels such as the Bessel, the sine, and the Airy kernels, which are obtained as local fluctuation of classical Gaussian ensembles, are reobtained from eigenvalue distributions of a quite wide class of random matrices, or log-gases with a quite wide class of weight functions. It is reasonable to believe that the biorthogonal Bessel kernel and the birothogonal sine kernel are also universal. In fact, when $\theta = \frac{1}{2}$, the universality of the biorthogonal Bessel kernel was shown for general Laguerre type weight [12].

§ 1.3. Transition of generalised Bessel kernels

For the classical case $\theta = 1$, there exist transition relations between three universal kernels, that is, the Bessel, the sine, and the Airy kernels [5, 8]. We focus on hard-edge to bulk transition in the sense that the distribution at the hard-edge tends to the bulk distribution at large distance from the hard-edge of the system up to scaling. More precisely the scaled Bessel kernel converges to the sine kernel: for any $\alpha > -1$,

$$\lim_{c\to\infty} \pi\sqrt{c}K_{\mathrm{Be},\alpha}(c+\pi\sqrt{c}x, c+\pi\sqrt{c}y) = K_{\sin}(x,y).$$

Local statistics on the bulk (1.6) indicates that hard-edge to bulk transition also holds for general θ, and the main aim of the present paper is to show the transition. Let $\rho^1_{\mathrm{Be},\alpha,\theta}$ be the one-correlation function with respect to $K_{\mathrm{Be},\alpha,\theta}$, that is, $\rho^1_{\mathrm{Be},\alpha,\theta}(x,x) = K_{\mathrm{Be},\alpha,\theta}(x,x)$. Taking into account of asymptotics

$$\rho^1_{\mathrm{Be},\alpha,\theta}(x) \sim \frac{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}{\pi}x^{-\frac{1}{1+\theta}} \qquad \text{as } x\to\infty,$$

we set the scaled biorthogonal Bessel kernel

$$K^c_{\mathrm{Be},\alpha,\theta}(x,y) = \frac{\pi c^{\frac{1}{1+\theta}}}{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})} K_{\mathrm{Be},\alpha,\theta}\Big(c + \frac{\pi c^{\frac{1}{1+\theta}}}{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}x, c + \frac{\pi c^{\frac{1}{1+\theta}}}{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}y\Big). \tag{1.7}$$

Then we have the following hard-edge to bulk transitions for the biorthogonal Bessel kernels.

Theorem 1.1. *For any $\alpha > -1$ and $\theta > 0$, we have*

$$\lim_{c\to\infty} e^{\pi\cot(\frac{\pi}{1+\theta})(x-y)} K^c_{\mathrm{Be},\alpha,\theta}(x,y) = K_{\sin}(x,y). \tag{1.8}$$

uniformly for x and y in compact subsets.

Remark. The factor $e^{\pi\cot(\frac{\pi}{1+\theta})(x-y)}$ in the left hand side of (1.8) is not essential, because $K(x,y)$ and $\frac{f(x)}{f(y)}K(x,y)$ define the same determinantal kernel for a non-vanishing function f. We choose the factor in Theorem 1.1 such that the limit form is $K_{\sin}$.

Theorem 1.1 is lifted to the convergence of associated random point fields. Determinantal random point fields associated with Hermitian kernels are studied in [16, 17]. Although $K_{\mathrm{Be},\alpha,\theta}$ is non-Hermitian, there exists determinantal random point field $\mu_{\mathrm{Be},\alpha,\theta}$ associated with $K_{\mathrm{Be},\alpha,\theta}$, because (1.5) holds also in compact uniform sense [11, 21]. Furthermore, compact uniform convergence of kernels implies weak convergence of the associated determinantal random point fields. (See e.g. [16, Proposition 3.11] for Hermitian kernels. The same claim holds for the non-Hermitian case.) We then conclude the following as a corollary of Theorem 1.1.

Corollary 1.2. *Let $\mu^c_{\mathrm{Be},\alpha,\theta}$ be the determinantal random point field associated with $K^c_{\mathrm{Be},\alpha,\theta}$. Let $\mu_{\sin}$ be the sine random point field, that is, the determinantal random point field associated with $K_{\sin}$. Then we have*

$$\lim_{c\to\infty} \mu^c_{\mathrm{Be},\alpha,\theta} = \mu_{\sin} \qquad \textit{weakly.}$$

In contrast to the biorthogonal Laguerre ensembles, studies on the biorthogonal Hermite ensembles has not made much progress. To our best knowledge, the global density for the biorthogonal Hermite ensembles is not known precisely, and especially, bulk region has not been found. Accordingly, local statistics on bulk except for the origin has not been shown. On the other hand, computation for Theorem 1.1 also yields a transition of the biorthogonal sine kernels, that is, the biorthogonal sine kernels are approximately the sine kernel at large distance from the origin. This fact supports the intuition that local statistics on bulk except for the origin is given by the sine kernel. Remark that

$$\rho^1_{\sin,\alpha,\theta}(x) \sim \frac{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}{\pi} x^{\frac{-1+\theta}{1+\theta}} \qquad \text{as } |x| \to \infty,$$

where $\rho^1_{\sin,\alpha,\theta}$ is the one-correlation function with respect to $K_{\sin,\alpha,\theta}$, that is, $\rho^1_{\sin,\alpha,\theta}(x) = K_{\sin,\alpha,\theta}(x,x)$. We then set

$$K^c_{\sin,\alpha,\theta}(x,y) = \frac{\pi c^{\frac{1-\theta}{1+\theta}}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})} K_{\sin,\alpha,\theta}\Big(c + \frac{\pi c^{\frac{1-\theta}{1+\theta}}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})} x, c + \frac{\pi c^{\frac{1-\theta}{1+\theta}}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})} y\Big).$$

Theorem 1.3. *For any $\alpha > -1$ and $\theta > 0$, we have*

$$\lim_{c\to\infty} e^{\pi\cot(\frac{\pi}{1+\theta})(x-y)} K^c_{\sin,\alpha,\theta}(x,y) = K_{\sin}(x,y),$$

uniformly for x and y in compact subsets.

Remark. If the determinantal random point field associated with $K_{\sin,\alpha,\theta}$ exists, then we immediately see that Theorem 1.3 derives convergence of the associated random point fields same as Corollary 1.2. However, we cannot show the existence using the results in [16, 17] directly, since $K_{\sin,\alpha,\theta}$ is not Hermitian kernel. Compact uniform convergence of (1.6) yields the existence, whilst (1.6) has been proven only pointwise convergence currently unlike the case of the biorthogonal Bessel kernel.

§ 2. Proof of the main results

§ 2.1. Asymptotics of the Wright generalised Bessel functions

In this subsection we prepare asymptotics of the Wright generalised Bessel functions. Remarking that our definition of the Wright generalised Bessel functions is different to that in [20], we quote the following asymptotic result.

Lemma 2.1 ([20]). *Assume $\alpha > -1$ and $\theta > 0$. We assume $\arg(z) = \xi$, $|\xi| \le \pi$, and set*

$$Z_1 = (\theta|z|)^{\frac{1}{1+\theta}} e^{i\frac{\xi+\pi}{1+\theta}}, \quad Z_2 = (\theta|z|)^{\frac{1}{1+\theta}} e^{i\frac{\xi-\pi}{1+\theta}}.$$

Then we have

$$J_{\alpha,\theta}(z) = H(Z_1) + H(Z_2),$$

where $H(z)$ satisfies

$$H(z) = z^{\frac{1}{2}-\alpha} e^{(1+\theta^{-1})z} \Big\{ \sum_{m=0}^{M} \frac{(-1)^m a_m}{z^m} + \mathcal{O}(|z|^{-M-1}) \Big\}, \qquad \text{as } |z| \to \infty$$

for any M. Here $\{a_k\}_{k\in\mathbb{N}}$ are constants which depend on α and θ.

We note that $\{a_k\}_{k\in\mathbb{N}}$ are explicitly given in [20], and especially $a_0 = (2\pi(1+\theta))^{-\frac{1}{2}}$. For simplifying notations we put

$$\hat{\theta} = \frac{1+\theta}{\theta^{\frac{\theta}{1+\theta}}}. \tag{2.1}$$

Then Lemma 2.1 yields the following key estimates.

Lemma 2.2.

(1) For $x \ge 0$ we have

$$J_{\alpha,\theta}(x) = \tilde{J}_{\alpha,\theta}(x)(1+\mathcal{O}(x^{-\frac{1}{1+\theta}})), \tag{2.2}$$

where

$$\tilde{J}_{\alpha,\theta}(x) = \sqrt{\frac{2}{\pi(1+\theta)}}(\theta x)^{(\frac{1}{2}-\alpha)\frac{1}{1+\theta}} \exp\Big\{(1+\theta^{-1})(\theta x)^{\frac{1}{1+\theta}}\cos\big(\frac{\pi}{1+\theta}\big)\Big\} \times \cos\Big(\Big(\frac{1}{2}-\alpha\Big)\frac{\pi}{1+\theta} + (1+\theta^{-1})(\theta x)^{\frac{1}{1+\theta}}\sin\big(\frac{\pi}{1+\theta}\big)\Big). \tag{2.3}$$

(2) We have

$$\begin{aligned}\tilde{J}_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(x)\tilde{J}_{\alpha+1,\theta}(y^{\theta}) =& \frac{\theta^{\frac{1}{1+\theta}}}{\pi(1+\theta)} x^{\frac{\theta-2\alpha-2}{2(1+\theta)}} y^{-\frac{\theta(1+2\alpha)}{2(1+\theta)}} \exp\Big\{\hat{\theta}\cos\big(\frac{\pi}{1+\theta}\big)(-x^{\frac{\theta}{1+\theta}} + y^{\frac{\theta}{1+\theta}})\Big\} \\ &\times \Big\{ \cos\Big(\frac{\theta-4\alpha-3}{2(1+\theta)}\pi + \hat{\theta}\sin\big(\frac{\pi}{1+\theta}\big)(x^{\frac{\theta}{1+\theta}} + y^{\frac{\theta}{1+\theta}})\Big) \\ &+ \cos\Big(\frac{\theta-1}{2(1+\theta)}\pi + \hat{\theta}\sin\big(\frac{\pi}{1+\theta}\big)(x^{\frac{\theta}{1+\theta}} - y^{\frac{\theta}{1+\theta}})\Big)\Big\}.\end{aligned}$$

Proof. We use Lemma 2.1 for $M = 0$. Then we see (2.2) from direct computation. It is not difficult to see (2) from (2.3) and trigonometric formulae. □

§ 2.2. Proof of Theorem 1.1

We give a proof of Theorem 1.1 in this subsection using the asymptotic results in the previous subsection. To simplify notations we use

$$x_c = c + \frac{\pi c^{\frac{1}{1+\theta}}}{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}x, \quad y_c = c + \frac{\pi c^{\frac{1}{1+\theta}}}{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}y. \tag{2.4}$$

Then from (1.3), (1.4), and (1.7) we have

$$\begin{aligned}K^c_{\mathrm{Be},\alpha,\theta}(x,y) &= \frac{\theta^{\frac{\theta}{1+\theta}}\pi}{\sin(\frac{\pi}{1+\theta})} c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_0^1 J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(x_c t) J_{\alpha+1,\theta}((y_c t)^{\theta}) t^{\alpha}\, dt \\ &= \frac{\theta^{\frac{\theta}{1+\theta}}\pi}{\sin(\frac{\pi}{1+\theta})}(I_1 + I_2),\end{aligned} \tag{2.5}$$

where we set

$$I_1(x,y) = c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_{c^{-1}\log c}^{1} J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(x_c t) J_{\alpha+1,\theta}((y_c t)^{\theta}) t^{\alpha}\, dt,$$
$$I_2(x,y) = c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_{0}^{c^{-1}\log c} J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(x_c t) J_{\alpha+1,\theta}((y_c t)^{\theta}) t^{\alpha}\, dt.$$

Let us analyse I_1 first, which turns out to be a main term. Because $\mathcal{O}((x_c t)^{-\frac{1}{1+\theta}}) = \mathcal{O}((\log c)^{-\frac{1}{1+\theta}})$ and $\mathcal{O}(((y_c t)^{\theta})^{-\frac{1}{1+\theta}}) = \mathcal{O}((\log c)^{-\frac{\theta}{1+\theta}})$ as c to infinity uniformly for $t \geq c^{-1}\log c$ and x, y in compact subsets, we obtain from Lemma 2.2 (1) that

$$I_1(x,y) = c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_{c^{-1}\log c}^{1} \tilde{J}_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(x_c t) \tilde{J}_{\alpha+1,\theta}((y_c t)^{\theta}) t^{\alpha}\, dt \big(1 + \mathcal{O}\big((\log c)^{-\min\{\frac{1}{1+\theta},\frac{\theta}{1+\theta}\}}\big)\big),$$

where the error term is uniform for x, y in compact subsets. Then Lemma 2.2 (2) yields

$$I_1(x,y) = \frac{\theta^{\frac{1}{1+\theta}}}{\pi(1+\theta)} (\tilde{I}_1(x,y) + \tilde{I}_2(x,y)) \big(1 + \mathcal{O}\big((\log c)^{-\min\{\frac{1}{1+\theta},\frac{\theta}{1+\theta}\}}\big)\big), \tag{2.6}$$

where

$$\tilde{I}_1(x,y) = c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_{c^{-1}\log c}^{1} x_c^{\frac{\theta-2\alpha-2}{2(1+\theta)}} y_c^{-\frac{\theta(1+2\alpha)}{2(1+\theta)}} t^{-\frac{1}{1+\theta}} \exp\big\{\hat{\theta}\cos\big(\frac{\pi}{1+\theta}\big)(-x_c^{\frac{\theta}{1+\theta}} + y_c^{\frac{\theta}{1+\theta}}) t^{\frac{\theta}{1+\theta}}\big\}$$
$$\times \cos\Big(\frac{\theta-4\alpha-3}{2(1+\theta)}\pi + \hat{\theta}\sin\big(\frac{\pi}{1+\theta}\big)(x_c^{\frac{\theta}{1+\theta}} + y_c^{\frac{\theta}{1+\theta}}) t^{\frac{\theta}{1+\theta}}\Big)\, dt,$$
$$\tilde{I}_2(x,y) = c^{\frac{1}{1+\theta}} x_c^{\alpha} \int_{c^{-1}\log c}^{1} x_c^{\frac{\theta-2\alpha-2}{2(1+\theta)}} y_c^{-\frac{\theta(1+2\alpha)}{2(1+\theta)}} t^{-\frac{1}{1+\theta}} \exp\big\{\hat{\theta}\cos\big(\frac{\pi}{1+\theta}\big)(-x_c^{\frac{\theta}{1+\theta}} + y_c^{\frac{\theta}{1+\theta}}) t^{\frac{\theta}{1+\theta}}\big\}$$
$$\times \cos\Big(\frac{\theta-1}{2(1+\theta)}\pi + \hat{\theta}\sin\big(\frac{\pi}{1+\theta}\big)(x_c^{\frac{\theta}{1+\theta}} - y_c^{\frac{\theta}{1+\theta}}) t^{\frac{\theta}{1+\theta}}\Big)\, dt.$$

Here $\hat{\theta}$ is given by (2.1) as before.

Lemma 2.3. *We have the following compact uniform convergence for x and y:*

$$\lim_{c\to\infty} \tilde{I}_1(x,y) = 0, \qquad \lim_{c\to\infty} \tilde{I}_2(x,y) = e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)} \frac{(1+\theta)\sin(\frac{\pi}{1+\theta})}{\theta} K_{\sin}(x,y), \tag{2.7}$$

and in particular,

$$\lim_{c\to\infty} I_1(x,y) = e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)} \frac{\sin(\frac{\pi}{1+\theta})}{\theta^{\frac{\theta}{1+\theta}}\pi} K_{\sin}(x,y). \tag{2.8}$$

Proof. Note that $\lim_{c\to\infty} c^{\frac{1}{1+\theta}} x_c^{\alpha} x_c^{\frac{\theta-2\alpha-2}{2(1+\theta)}} y_c^{-\frac{\theta(1+2\alpha)}{2(1+\theta)}} = 1$ since x_c and y_c are given as (2.4). Furthermore, we see

$$x_c^{\frac{\theta}{1+\theta}} + y_c^{\frac{\theta}{1+\theta}} = 2c^{\frac{\theta}{1+\theta}} + \mathcal{O}(1), \tag{2.9}$$
$$x_c^{\frac{\theta}{1+\theta}} - y_c^{\frac{\theta}{1+\theta}} = \frac{\pi}{\hat{\theta}\sin(\frac{\pi}{1+\theta})}(x-y) + \mathcal{O}(c^{-\frac{\theta}{1+\theta}}), \tag{2.10}$$

as c to infinity, where the error terms are uniform for x, y in compact subsets. Because of (2.9) and $\hat{\theta}\sin(\frac{\pi}{1+\theta}) > 0$, cosine in the integrand of $\tilde{I}_1$ oscillates. Therefore integration by parts yields that

$$\begin{aligned}\lim_{c\to\infty} \tilde{I}_1(x,y) &= \lim_{c\to\infty} \int_{c^{-1}\log c}^{1} t^{-\frac{1}{1+\theta}} e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)t^{\frac{\theta}{1+\theta}}} \\ &\qquad \times \cos\Big(\frac{\theta-4\alpha-3}{2(1+\theta)}\pi + \hat{\theta}\sin\big(\frac{\pi}{1+\theta}\big)(2c^{\frac{\theta}{1+\theta}} + \mathcal{O}(1))t^{\frac{\theta}{1+\theta}}\Big)\,dt \\ &= 0.\end{aligned}$$

On the other hand, (2.10) and straightforward calculation yield

$$\begin{aligned}\lim_{c\to\infty} \tilde{I}_2(x,y) &= \int_0^1 t^{-\frac{1}{1+\theta}} e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)t^{\frac{\theta}{1+\theta}}} \cos\Big(\frac{\theta-1}{2(1+\theta)}\pi + \pi(x-y)t^{\frac{\theta}{1+\theta}}\Big)\,dt \\ &= \frac{1+\theta}{\theta}\int_0^1 e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)t} \sin\Big(\frac{\pi}{1+\theta} - \pi(x-y)t\Big)\,dt \\ &= e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)} \frac{(1+\theta)\sin(\frac{\pi}{1+\theta})}{\theta}\frac{\sin\pi(x-y)}{\pi(x-y)}.\end{aligned}$$

Finally (2.8) follows from (2.6) and (2.7). □

Lemma 2.4. *We have*

$$\lim_{c\to\infty} I_2(x,y) = 0,$$

uniformly for x and y in compact subsets.

Proof. By change of variables we get

$$I_2(x,y) = c^{-\frac{\theta}{1+\theta}}\Big(\frac{x_c}{c}\Big)^{\alpha}\int_0^{\log c} J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(c^{-1}x_c t)J_{\alpha+1,\theta}((c^{-1}y_c t)^{\theta})t^{\alpha}\,dt.$$

Fix a compact set $K \subset [0,\infty)$. Then from Lemma 2.2 (1), there exist positive constants m and c_1 which are independent of t, x, y such that for any $t \geq m$ and $x, y \in K$,

$$|J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(c^{-1}x_c t)J_{\alpha+1,\theta}((c^{-1}y_c t)^{\theta})t^{\alpha}| \leq c_1 \exp\Big\{\hat{\theta}\cos\big(\frac{\pi}{1+\theta}\big)\big(\frac{t}{c}\big)^{\frac{\theta}{1+\theta}}(-x_c^{\frac{\theta}{1+\theta}} + y_c^{\frac{\theta}{1+\theta}})\Big\}.$$

Combining this with (2.10) we have

$$\lim_{c\to\infty}\sup_{x,y\in K}\Big|c^{-\frac{\theta}{1+\theta}}\Big(\frac{x_c}{c}\Big)^{\alpha}\int_m^{\log c} J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(c^{-1}x_c t)J_{\alpha+1,\theta}((c^{-1}y_c t)^{\theta})t^{\alpha}\,dt\Big| = 0. \tag{2.11}$$

Since the Wright generalised Bessel functions are analytic, then there exists a positive constant c_2 such that

$$\sup_{c\in\mathbb{N}, t\in[0,m],\, x,y\in K}\Big|\Big(\frac{x_c}{c}\Big)^{\alpha} J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(c^{-1}x_c t)J_{\alpha+1,\theta}((c^{-1}y_c t)^{\theta})t^{\alpha}\Big| \leq c_2,$$

which yields

$$\begin{aligned}(2.12)\qquad \lim_{c\to\infty}\sup_{x,y\in K}&\left|c^{-\frac{\theta}{1+\theta}}\left(\frac{x_c}{c}\right)^{\alpha}\int_0^m J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(c^{-1}x_c t)J_{\alpha+1,\theta}((c^{-1}y_c t)^{\theta})t^{\alpha}\,dt\right|\\ &\le \lim_{c\to\infty} mc_2 c^{-\frac{\theta}{1+\theta}}=0.\end{aligned}$$

We then conclude the lemma from (2.11) and (2.12). □

Proof of Theorem 1.1

Theorem 1.1 immediately follows from (2.5), Lemma 2.3 and Lemma 2.4. □

§ 2.3. Proof of Theorem 1.3

We can prove Theorem 1.3 by similar calculation as in the proof of Theorem 1.1. For simplicity we set

$$\tilde{x}_c = c+\frac{\pi c^{\frac{1-\theta}{1+\theta}}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}x,\quad \tilde{y}_c = c+\frac{\pi c^{\frac{1-\theta}{1+\theta}}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}y.$$

Then

$$(2.13)\qquad K^c_{\sin,\alpha,\theta}(x,y)=\frac{\pi c^{\frac{1-\theta}{1+\theta}}|\tilde{x}_c|^{\alpha}}{2\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}\Big\{L_{\frac{\alpha-1}{2},\theta}(\tilde{x}_c^2,\tilde{y}_c^2)+\tilde{x}_c^{\theta}\tilde{y}_c L_{\frac{\alpha+\theta}{2},\theta}(\tilde{x}_c^2,\tilde{y}_c^2)\Big\}.$$

Lemma 2.5. *We have*

$$\begin{aligned}(2.14)\qquad \lim_{c\to\infty}c^{\frac{1-\theta}{1+\theta}}|\tilde{x}_c|^{\alpha}L_{\frac{\alpha-1}{2},\theta}(\tilde{x}_c^2,\tilde{y}_c^2)&=\lim_{c\to\infty}c^{\frac{1-\theta}{1+\theta}}|\tilde{x}_c|^{\alpha}\tilde{x}_c^{\theta}\tilde{y}_c L_{\frac{\alpha+\theta}{2},\theta}(\tilde{x}_c^2,\tilde{y}_c^2)\\ &=\frac{\theta^{\frac{1}{1+\theta}}\sin(\frac{\pi}{1+\theta})}{\pi}e^{\pi\cot(\frac{\pi}{1+\theta})(-x+y)}K_{\sin}(x,y),\end{aligned}$$

uniformly for x and y in compact subsets.

Proof. Following the proof in Section 2.2, we divide the integral into two parts as follows:

$$\begin{aligned}(2.15)\qquad & c^{\frac{1-\theta}{1+\theta}}|\tilde{x}_c|^{\alpha}L_{\frac{\alpha-1}{2},\theta}(\tilde{x}_c^2,\tilde{y}_c^2)\\ &=\theta c^{\frac{1-\theta}{1+\theta}}|\tilde{x}_c|^{\alpha}\Big\{\int_0^{c^{-2}\log c}+\int_{c^{-2}\log c}^1\Big\}J_{\frac{\alpha+1}{2\theta},\frac{1}{\theta}}(\tilde{x}_c^2 t)J_{\frac{\alpha+1}{2},\theta}((\tilde{y}_c^2 t)^{\theta})t^{\frac{\alpha-1}{2}}\,dt.\end{aligned}$$

Using Lemma 2.2 and the fact that

$$\tilde{x}_c^{\frac{2\theta}{1+\theta}}-\tilde{y}_c^{\frac{2\theta}{1+\theta}}=\frac{\pi}{\hat{\theta}\sin(\frac{\pi}{1+\theta})}(x-y)+\mathcal{O}(c^{-\frac{2\theta}{1+\theta}}),$$

we get that the second integral of the right hand side of (2.15) converges to the most right hand side of (2.14), and the first integral vanishes. The calculations are the same manner as in Lemma 2.3 and Lemma 2.4 respectively, we then omit it.

Furthermore, we can obtain the second equality in (2.14) by the same way. □

Proof of Theorem 1.3

Theorem 1.3 immediately follows from (2.13) and Lemma 2.5. □

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