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# Quenched lower large deviation for the first passage time of the frog model in random initial configurations

By

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**Abstract**

We consider the so-called frog model with random initial configurations. The dynamics of this model are described as follows: Some particles are randomly assigned to any site of the multidimensional cubic lattice. Initially, only particles at the origin are active and these independently perform simple random walks. The other particles are sleeping and do not move at first. When sleeping particles are hit by an active particle, they become active and start moving in a similar fashion. In this paper, we study the behavior of the first passage time at which an active particle reaches a target site, and provide an upper bound on a quenched lower large deviation for the first passage time.

## § 1. Introduction

### § 1.1. The model

For $d \geq 2$, we write $\mathbb{Z}^d$ for the $d$-dimensional cubic lattice. Let $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$ be independent random variable with a common law on $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Furthermore, independently of $\omega$, let $\mathbb{S} = ((S_k(x,\ell))_{k=0}^{\infty})_{x \in \mathbb{Z}^d, \ell \in \mathbb{N}}$ be independent simple random walks on $\mathbb{Z}^d$ with $S_0(x,\ell) = x$. Although infinitely many simple random walks $S_\cdot(x,\ell)$, $\ell \in \mathbb{N}$ are positioned at every site $x \in \mathbb{Z}^d$ (each $S_\cdot(x,\ell)$ is the $\ell$-th simple random walk placed originally at $x$), the number of them we can use are determined at random for each site. See the explanation below (1.1) for details. The underlying probability measures of $\omega$

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and $\mathbb{S}$ is denoted by $\mathbb{P}$ and $P$, respectively. For any $x, y \in \mathbb{Z}^d$, we now introduce the *first passage time* $T(x,y) = T(x,y,\omega,\mathbb{S})$ from $x$ to $y$ as follows:

$$T(x,y) := \inf\left\{ \sum_{i=0}^{m-1} \tau(x_i, x_{i+1}) : \begin{array}{l} m \geq 1, x_0 = x, x_m = y, \\ x_1, \dots, x_{m-1} \in \mathbb{Z}^d \end{array} \right\},$$

where

$$\begin{aligned} \tau(x_i, x_{i+1}) &= \tau(x_i, x_{i+1}, \omega, \mathbb{S}) \\ &:= \inf\{k \geq 0 : S_k(x_i, \ell) = x_{i+1} \text{ for some } 1 \leq \ell \leq \omega(x_i)\} \end{aligned}$$

with the convention that $\tau(x_i, x_{i+1}) := \infty$ if $\omega(x_i) = 0$. By definition, the first passage time satisfies the triangle inequality:

$$T(x,z) \leq T(x,y) + T(y,z), \qquad x, y, z \in \mathbb{Z}^d. \tag{1.1}$$

On the event $\{\omega(0) \geq 1\}$, the intuitive meaning of the first passage time $T(0,x)$ is as follows: We now regard simple random walks as "frogs" and $\omega$ stands for an initial configuration of frogs, i.e., $\omega(y)$ frogs sit on each site $y$ (there is no frog at $y$ if $\omega(y) = 0$). Suppose that the origin 0 is occupied by at least one frog. They are active and independently perform simple random walks, but the other frogs are sleeping and do not move at first. When sleeping frogs are attacked by an active one, they become active and start doing independent simple random walks. Then, $T(0,x)$ describes the first passage time at which an active frog reaches a site $x$.

Alves et al. [1, Steps 1–6 in Section 2] obtained the following asymptotic behavior of the first passage time under the annealed law $\mathbb{P} \otimes P$: There exists a (nonrandom) norm $\mu(\cdot)$ (which is called the *time constant*) on $\mathbb{R}^d$ such that $\mathbb{P} \otimes P$-a.s. on the event $\{\omega(0) \geq 1\}$,

$$\lim_{\substack{\|x\|_1 \to \infty \\ x \in \mathbb{Z}^d}} \frac{T(0,x) - \mu(x)}{\|x\|_1} = 0, \tag{1.2}$$

where $\|\cdot\|_1$ is the $\ell^1$-norm on $\mathbb{R}^d$. Furthermore, $\mu(\cdot)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, and satisfies

$$\|x\|_1 \leq \mu(x) \leq \mu(\xi_1)\|x\|_1, \qquad x \in \mathbb{R}^d, \tag{1.3}$$

where $\xi_1$ is the first coordinate vector of $\mathbb{R}^d$.

The frog model has two randomnesses: the initial configuration $\omega$ and the trajectories $\mathbb{S}$ of frogs. Thus, compared with the aforementioned work [1], in this paper we treat the first passage time for the quenched setting in which the trajectory $\mathbb{S}$ of frogs is fixed.

## § 1.2. Main results

To mention our main results, we introduce the following quantity $a_\lambda(x, y)$ (which is regarded as the cost to make an active frog reach a site $y$ in the case where only frogs sitting on a site $x$ are active at first and we have already known trajectories $\mathbb{S}$ of frogs): For $\lambda \geq 0$ and $x, y \in \mathbb{Z}^d$,

$$a_\lambda(x, y) = a_\lambda(x, y, \mathbb{S}) := -\log \mathbb{E}[e^{-\lambda T(x,y)}].$$

Note that since we are now working on the setting that $\omega$ is the sequence of i.i.d. random variables, the process $(a_\lambda(x, y))_{x,y\in\mathbb{Z}^d}$ is stationary under $P$. In addition, if $\omega(0)$ is a constant $\mathbb{P}$-almost surely, then our model is the frog model with a nonrandom initial configuration, and the above expectation has no meaning. Hence, throughout this paper, we assume that $\omega(0)$ is not concentrated in a nonnegative integer.

The first result is the following asymptotic behavior of the cost $a_\lambda(0, x)$.

**Theorem 1.1.** *Let $\lambda \geq 0$. There exists a norm $\alpha_\lambda(\cdot)$ (which is called the* Lyapunov exponent*) on $\mathbb{R}^d$ such that for all $x \in \mathbb{Z}^d$, $P$-almost surely and in $L^1(P)$,*

$$\alpha_\lambda(x) = \lim_{k\to\infty} \frac{1}{k} a_\lambda(0, kx) = \lim_{k\to\infty} \frac{1}{k} E[a_\lambda(0, kx)] = \inf_{k\geq 1} \frac{1}{k} E[a_\lambda(0, kx)]. \tag{1.4}$$

*In particular, $P$-almost surely,*

$$\lim_{\substack{\|x\|_1\to\infty \\ x\in\mathbb{Z}^d}} \frac{a_\lambda(0, x) - \alpha_\lambda(x)}{\|x\|_1} = 0. \tag{1.5}$$

*Furthermore, $\alpha_\lambda(\cdot)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, and has the bounds*

$$\lambda\|x\|_1 \leq \alpha_\lambda(x) \leq \alpha_\lambda(\xi_1)\|x\|_1, \qquad x \in \mathbb{R}^d. \tag{1.6}$$

For a subset $A$ of $\mathbb{R}^d$, the point-to-set first passage time $T(0, A)$ is defined as

$$T(0, A) := \inf_{x\in A} T(0, x).$$

Then, the following theorem is our second result. This gives an upper bound for the lower tail estimate of the point-to-set first passage time.

**Theorem 1.2.** *$P$-almost surely, for any closed subset $A$ of $\mathbb{R}^d$,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T(0,tA) \le t) \le -\inf_{x\in A} I(x), \tag{1.7}$$

*where $I(x) := \sup_{\lambda\ge 0}(\alpha_\lambda(x) - \lambda)$.*

Theorems 1.1 and 1.2 follow from the same argument which Zerner [4, 5] used to obtain the large deviation principles for random walks in random environments. Since that argument is not so long, for the convenience of the reader, we repeat it and give the complete proofs of Theorems 1.1 and 1.2.

Finally, let us comment on Theorem 1.2. Theorem 1.2 seems to give an upper bound of the lower large deviation principle with the rate function $I$ for the point-to-set first passage time in the present setting. Unfortunately, we cannot directly apply the argument in [4, 5] to obtain the following lower bound of the lower large deviation principle: $P$-almost surely, for any open subset $B$ of $\mathbb{R}^d$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T(0,tB) \le t) \ge -\inf_{x\in B} I(x). \tag{1.8}$$

The main difficulty is that the first passage times $T(x,y)$ and $T(y,z)$ have correlation under $\mathbb{P}$. However, we expect that (1.8) is also true and the lower large deviation principle holds with the rate function $I$ given in Theorem 1.2.

## § 2. Proof of Theorem 1.1

The goal of this section is to show Theorem 1.1. First of all, to this end, we refer the following upper tail estimate for the first passage time, which was obtained by the author [3, Proposition 2.4]: There exist constants $0 < C_1, C_2, C_3 < \infty$ and $0 < \alpha \le 1$ such that for all $x \in \mathbb{Z}^d$ and $t \ge C_1\|x\|_1$,

$$\mathbb{P} \otimes P(T(0,x) \ge t | \omega(0) \ge 1) \le C_2 e^{-C_3 t^\alpha}. \tag{2.1}$$

Let us next check some properties of the cost $a_\lambda(x,y)$: the triangle inequality, the integrability and the maximal lemma.

**Lemma 2.1.** *For $\lambda \ge 0$ and $x, y, z \in \mathbb{Z}^d$,*

$$a_\lambda(x,z) \le a_\lambda(x,y) + a_\lambda(y,z).$$

**Proof.** The triangle inequality (1.1) for the first passage time proves

$$a_\lambda(x,z) \le -\log \mathbb{E}\big[e^{-\lambda T(x,y)} e^{-\lambda T(y,z)}\big].$$

Therefore, it suffices to show that

$$\mathbb{E}\big[e^{-\lambda T(x,y)}e^{-\lambda T(y,z)}\big] \geq \mathbb{E}\big[e^{-\lambda T(x,y)}\big]\mathbb{E}\big[e^{-\lambda T(y,z)}\big]. \tag{2.2}$$

To prove (2.2), fix $\mathbb{S}$ and define for each $N \in \mathbb{N}$,

$$X_N(x,y,\omega) := e^{-\lambda T(x,y,\omega,\mathbb{S})}\mathbf{1}_{\{T(x,y,\omega,\mathbb{S})\leq N\}}.$$

Note that $X_N(x,y,\omega)$ depends only on the configuration $(\omega(v))_{v\in B_1(x,N)}$ and is increasing in the sense that $X_N(x,y,\omega_1) \leq X_N(x,y,\omega_2)$ if $\omega_1(v) \leq \omega_2(v)$ for all $v \in \mathbb{Z}^d$. This enables us to use Chebyshev's association inequality (see [2, Theorem 2.14]):

$$\mathbb{E}[X_N(x,y)X_N(y,z)] \geq \mathbb{E}[X_N(x,y)]\mathbb{E}[X_N(y,z)].$$

Since $|X_N(x,y)| \leq 1$ and $X_N(x,y)$ converges to $e^{-\lambda T(x,y)}$ as $N \to \infty$, (2.2) follows from Lebesgure's dominated convergence theorem, and the proof is complete. □

**Lemma 2.2.** *For $\lambda \geq 0$ and $x, y \in \mathbb{Z}^d$, $a_\lambda(x,y)$ is integrable with respect to $P$.*

**Proof.** We have

$$\begin{aligned} E[a_\lambda(x,y)] &= E\big[-\log \mathbb{E}\big[e^{-\lambda T(x,y)}\mathbf{1}_{\{\omega(x)\geq 1\}}\big]\big] \\ &= E\big[-\log \mathbb{E}\big[e^{-\lambda T(x,y)}\big|\omega(x)\geq 1\big]\big] - \log \mathbb{P}(\omega(0)\geq 1). \end{aligned}$$

If $\mathbb{E}[\lambda T(x,y)|\omega(x) \geq 1]$ is finite, then the fact that $\mathbb{E}[e^{-\lambda T(x,y)}|\omega(x) \geq 1] \leq 1$ and Jansen's inequality imply

$$-\log \mathbb{E}[e^{-\lambda T(x,y)}|\omega(x)\geq 1] \leq \mathbb{E}[\lambda T(x,y)|\omega(x)\geq 1].$$

Otherwise, this inequality is trivial. Hence,

$$E[a_\lambda(x,y)] \leq \lambda \mathbb{E}\otimes E[T(x,y)|\omega(x)\geq 1] - \log \mathbb{P}(\omega(0)\geq 1),$$

and the integrability immediately follows from (2.1). □

**Lemma 2.3.** *Let $\lambda \geq 0$. There exist positive constants $C_4$, $C_5$ and $C_6$ such that for all $\epsilon > 0$ and for all large $\|x\|_1$,*

$$P\Big(\sup\{a_\lambda(x,y)\vee a_\lambda(y,x) : y \in \mathbb{Z}^d,\ \|x-y\|_1 \leq \epsilon\|x\|_1\} \geq C_4\epsilon\|x\|_1\Big) \leq C_5 e^{-C_6(\epsilon\|x\|_1)^\alpha}.$$

**Proof.** Fix $\lambda \geq 0$ and $\epsilon > 0$. We set $C_4 := 2\lambda C_1 C_3^{1/\alpha}$ and use Jensen's inequality to obtain that

$$\begin{aligned} &P(a_\lambda(x,y) \geq C_4\epsilon\|x\|_1) \\ &\leq P\Big(\lambda\mathbb{E}[T(0,y-x)|\omega(0)\geq 1] \geq C_4\epsilon\|x\|_1 + \log\mathbb{P}(\omega(0)\geq 1)\Big). \end{aligned} \tag{2.3}$$

To bound this, let us define the following increasing, convex, nonnegative function $g(t)$ on $[0,\infty)$:

$$g(t) := \begin{cases} e^{(1-\alpha)/\alpha}, & \text{if } 0 \leq t < \left(\frac{2(1-\alpha)}{C_3\alpha}\right)^{1/\alpha}, \\ e^{C_3 t^\alpha/2}, & \text{if } t \geq \left(\frac{2(1-\alpha)}{C_3\alpha}\right)^{1/\alpha}. \end{cases}$$

Since $g(t)$ is increasing and nonnegative, Markov's inequality shows that the right side of (2.3) is not greater than

$$g\big(\lambda^{-1}(C_4\epsilon\|x\|_1 + \log\mathbb{P}(\omega(0)\geq 1))\big)^{-1} E\big[g(\mathbb{E}[T(0,y-x)|\omega(0)\geq 1])\big].$$

Furthermore, the convexity of $g(t)$ combined with Jensen's inequality yields that this is smaller than or equal to

$$g\big(\lambda^{-1}(C_4\epsilon\|x\|_1 + \log\mathbb{P}(\omega(0)\geq 1))\big)^{-1} \mathbb{E}\otimes E\big[g(T(0,y-x))\big|\omega(0)\geq 1\big].$$

To shorten notation, set $a := \exp\{(C_3/2)(C_1\epsilon\|x\|_1)^\alpha\}$. Then, the definition of $g(t)$ and (2.1) prove that for any $x,y\in\mathbb{Z}^d$ with $\|x-y\|_1 \leq \epsilon\|x\|_1$, the last expectation is bounded from above by

$$\begin{aligned} &e^{(1-\alpha)/\alpha} + \mathbb{E}\otimes E\big[e^{(C_3/2)T(0,y-x)^\alpha}\big|\omega(0)\geq 1\big] \\ &\leq e^{(1-\alpha)/\alpha} + a + \int_a^\infty \mathbb{P}\otimes P\big(T(0,y-x) \geq \{(2/C_3)\log t\}^{1/\alpha}\big|\omega(0)\geq 1\big)\,dt \\ &\leq e^{(1-\alpha)/\alpha} + a + \int_0^\infty C_2 t^{-2}\,dt. \end{aligned}$$

Thus, for all $x,y\in\mathbb{Z}^d$ with $\|x-y\|_1 \leq \epsilon\|x\|_1$ and for all large $\|x\|_1$,

$$\begin{aligned} &P(a_\lambda(x,y) \geq C_4\epsilon\|x\|_1) \\ &\leq 3\exp\left\{-\frac{C_1}{2\lambda^\alpha}\big(C_4\epsilon\|x\|_1 + \log\mathbb{P}(\omega(0)\geq 1)\big)^\alpha + \frac{C_3}{2}(\epsilon C_1\|x\|_1)^\alpha\right\} \\ &\leq 3\exp\left\{-\frac{C_1\wedge C_3}{2}\left(\frac{C_4^\alpha}{\lambda^\alpha} - C_1^\alpha C_3\right)(\epsilon\|x\|_1)^\alpha\right\}. \end{aligned}$$

Since $C_4^\alpha/\lambda^\alpha - C_1^\alpha C_3 > 0$, the lemma follows from the union bound. $\square$

After the preparation above, let us prove Theorem 1.1.

**Proof of Theorem 1.1.** As mentioned above, the proof is exactly same as in [4, Proposition 4 and Theorem 8] and [5, Lemma 7 and Theorem A]. Note that $\omega = (\omega(x))_{x \in \mathbb{Z}^d}$ is a family of i.i.d. random variables and $\mathbb{S} = (S.(x,\ell))_{x \in \mathbb{Z}^d, \ell \in \mathbb{N}}$ are independent simple random walks. This, combined with Lemmata 2.1 and 2.2, allows us to use the subadditive ergodic theorem for the process $a_\lambda(ix, jx)$, $0 \le i < j$, $i, j \in \mathbb{N}_0$. Hence, we can find a (nonrandom) constant $\alpha_\lambda(x)$ satisfying (1.4) and (1.6) for any $x \in \mathbb{Z}^d$. Setting $\alpha_\lambda(x/q) := \alpha_\lambda(x)/q$, one extends $\alpha_\lambda(\cdot)$ to a function on $\mathbb{Q}^d$, and then by continuity to a norm $\mathbb{R}^d$ which satisfies (1.6) for any $x \in \mathbb{R}^d$.

For the proof of (1.5), it suffices to prove that for any $0 < \epsilon \in \mathbb{Q}$, the following holds $P$-almost surely: There exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{Z}^d$ with $\|x\|_1 \ge N$,

$$|a_\lambda(0,x) - \alpha_\lambda(x)| \le \epsilon \|x\|_1.$$

To this end, suppose that the above statement is false. Then, there exists $\epsilon_0 > 0$ such that with positive $P$-probability, we can take a sequence $(x_i)_{i=1}^\infty$ of $\mathbb{Z}^d$ satisfying that $\|x_i\|_1 \to \infty$ as $i \to \infty$ and

$$|a_\lambda(0,x_i) - \alpha_\lambda(x_i)| > \epsilon_0 \|x_i\|_1, \qquad i \ge 1.$$

There is no loss of generality in assuming that $x_i/\|x_i\|_1 \to v$ as $i \to \infty$ for some $v \in \mathbb{R}^d$ with $\|v\|_1 = 1$. Let $\eta$ be a positive number to be chosen later. Take $v' \in \mathbb{Q}^d$ and $M \in \mathbb{N}$ with with $\|v'\|_1 = 1$, $\|v - v'\|_1 < \eta$ and $Mv' \in \mathbb{Z}^d$. Furthermore, define for $i \ge 1$,

$$x_i' := \left\lfloor \frac{\|x_i\|_1}{M} \right\rfloor Mv'.$$

Then, for all large $i$,

$$\|x_i - x_i'\|_1 < \eta \|x_i\|_1 + M \le 2\eta \|x_i\|_1.$$

We use Lemma 2.1 to obtain

$$\begin{aligned} \epsilon_0 \|x_i\|_1 &< |a_\lambda(0,x_i) - \alpha_\lambda(x_i)| \\ &\le a_\lambda(x_i, x_i') \vee a_\lambda(x_i', x_i) + |a_\lambda(0, x_i') - \alpha_\lambda(x_i')| + |\alpha_\lambda(x_i') - \alpha_\lambda(x_i)|. \end{aligned}$$

Thanks to the fact that $\alpha_\lambda(\cdot)$ is a norm on $\mathbb{R}^d$ and (1.6), if $i$ is large enough, then the last term of the most right side is not greater than

$$\alpha_\lambda(x_i' - x_i) \le \alpha_\lambda(\xi_1)\|x_i - x_i'\|_1 \le 2\alpha_\lambda(\xi_1)\eta \|x_i\|_1,$$

and consequently

$$\epsilon_0\|x_i\|_1 < a_\lambda(x_i, x_i') \vee a_\lambda(x_i', x_i) + |a_\lambda(0, x_i') - \alpha_\lambda(x_i')| + 2\alpha_\lambda(\xi_1)\eta\|x_i\|_1.$$

Since $\|x_i\|_1 \to \infty$, (1.4) proves that with probability one, for all large $i$,

$$|a_\lambda(0, \lfloor\|x_i\|_1/M\rfloor Mv') - \alpha_\lambda(\lfloor\|x_i\|_1/M\rfloor Mv')| < \eta\left\lfloor \frac{\|x_i\|_1}{M}\right\rfloor \le \eta\|x_i\|_1.$$

Moreover, Lemma 2.3 guarantees that if $i$ is large enough, then since $\|x_i - x_i'\|_1 \le 2\eta\|x_i\|_1$,

$$a_\lambda(x_i, x_i') \vee a_\lambda(x_i', x_i) < 2C_4\eta\|x_i\|_1$$

holds with high probability. With these observations, if $i$ is large enough, then one has with positive probability,

$$\epsilon_0\|x_i\|_1 < (2C_4 + 1 + 2\alpha_\lambda(\xi_1))\eta\|x_i\|_1,$$

which leads to a contradiction as long as we take $\eta < \epsilon_0(2C_4+1+2\alpha_\lambda(\xi_1))^{-1}$. Therefore, (1.5) is established. □

## § 3. Proof of Theorem 1.2

This section is dedicated to proving Theorem 1.2.

**Proof of Theorem 1.2.** Let $A$ be a closed subset of $\mathbb{R}^d$. Without loss of generality, we can assume $A \subset B_1(0,1)$. Indeed, since $T(0,ty) > t$ holds for all $y \notin B_1(0,1)$. It is clear that for a given $\lambda \ge 0$,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(T(0,tA)\le t) &\le \lambda + \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\big[e^{-\lambda T(0,tA)}\big]\\ &\le \lambda + \limsup_{t\to\infty}\frac{1}{t}\log\sum_{z\in tA\cap\mathbb{Z}^d}\mathbb{E}\big[e^{-\lambda T(0,z)}\big].\end{aligned}$$

Let $z_t$ be a site in $tA\cap\mathbb{Z}^d$ attaining $\max_{z\in tA\cap\mathbb{Z}^d}\mathbb{E}[e^{-\lambda T(0,z)}]$, and note that the cardinality $\#(tA\cap\mathbb{Z}^d)$ is of order $t^d$. We use Theorem 1.1 to bound the last term in the above inequality from above by

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\Big(\#(tA\cap\mathbb{Z}^d)\mathbb{E}\big[e^{-\lambda T(0,z_t)}\big]\Big) &= -\liminf_{t\to\infty}\frac{1}{t}a_\lambda(0,z_t)\\ &\le -\inf_{y\in A}\alpha_\lambda(y).\end{aligned}$$

Therefore, for all $\lambda \geq 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T(0,tA) \leq t) \leq -\sup_{\lambda\geq 0} \inf_{y\in A} (\alpha_\lambda(y) - \lambda). \tag{3.1}$$

It remains to exchange the infimum and the supremum in (3.1). To do this, note that $A = \bigcup_{\lambda\geq 0} A_\lambda$ holds for any $\epsilon > 0$, where

$$A_\lambda := \Big\{ z \in A : \alpha_\lambda(z) - \lambda > \inf_{y\in A} I(y) - \epsilon \Big\}.$$

From the compactness of $A$, we can choose $\lambda_1, \ldots, \lambda_m$ such that $A$ is covered by the finite collection $A_{\lambda_i}$, $i = 1, \ldots, m$. Hence, the union bound and (3.1) show

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T(0,tA) \leq t) &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sum_{i=1}^{m} \mathbb{P}(T(0,t\overline{A}_{\lambda_i}) \leq t) \\
&\leq \sup_{1\leq i\leq m} \Big( -\sup_{\lambda\geq 0} \inf_{y\in \overline{A}_{\lambda_i}} (\alpha_\lambda(y) - \lambda) \Big) \\
&\leq -\inf_{y\in A} I(y) + \epsilon,
\end{aligned}$$

where $\overline{A}_{\lambda_i}$ is the closure of the set $A_{\lambda_i}$. Thus, letting $\epsilon \searrow 0$ leads to the desired conclusion. □

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