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Scaling limit of a biased random walk on a critical branching random walk

By

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Abstract

Relying on powerful resistance techniques developed in [8], the recent paper 'Invariance principles for random walks in random environment on trees' [4] investigates random walks in random environment on tree-like spaces and their scaling limits in a certain regime, that is when the potential of the random walk in random environment converges. We introduce and summarise a result from [4]. We choose to review the example of a novel scaling continuum limit of a biased random walk on large critical branching random walk. In this case the diffusion that is not on natural scale is identified as a Brownian motion on a continuum random fractal tree with its canonical metric replaced by a distorted resistance metric. This example allows the least technical presentation (compared to the others covered in the main article). Moreover, it is nevertheless of current interest, given its relation to critical percolation.

§ 1. Introduction

We consider branching walk on a rooted ordered finite tree in which every edge is marked by a real-valued vector (it is equivalent to have vectors assigned to vertices instead). We associate with each vertex a trajectory of a killed walk defined by summing all the values of the edges that belong to the unique path from the root to that particular vertex. Obviously, the walk is killed after as many steps as the number of the generation of the vertex evaluated at (see (2.1)). The multiset of trajectories of the killed walk is

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called the branching walk. A branching random walk is constructed by choosing the shape of the tree and the marks at random.

Next, we consider a weakly biased random walk on branching random walk ϕ_n conditioned to have n particles, where the shape of the underlying tree T_n is generated by a critical Galton-Watson process with exponential tails for the offspring distribution, and the marks are independent, each distributed as a random variable Y in a way that ensures that the multiset of trajectories of ϕ_n is a tree when regarded as an embedded subgraph of $\mathbb{R}^d$. Here, the name weakly we gave to the biased random walk is inherited by the discrete scheme we provide that 'flattens' the cartesian bias to a single direction appropriately at every step. We show that this weakly biased random walk on the aforementioned model converges to a Brownian motion on a random Gaussian potential on a continuum random fractal tree, which in not on natural scale. See Theorem 2.1 for a definitive statement. We should stress that diffusion processes on classes of real trees that are not on natural scale have been alluded to before in the literature. In [10], the processes considered evolve by retreating back along an ancestral line or by moving among lineages of branch points according to weights chosen by a possibly infinite measure on the family of ancestral lines. The last example in [5, Example 3.8] formalizes the notion of the potential of diffusions, which are not necessarily on natural scale.

One model for which an appealing conjecture can be made is the incipient infinite cluster (IIC) of high-dimensional Bernoulli-bond percolation (each edge is open with probability p, and closed with probability $1-p$, independently of the states of other edges) on $\mathbb{Z}^d$, that is when $d > 6$. Below, the measure is denoted by $\mathbb{P}_p$. At criticality, i.e. $p = p_c(d) \in (0,1)$, it is partially confirmed that there is no infinite open cluster. Instead, one could study random walks on the IIC:

$$\mathbb{P}_{\mathrm{IIC}}(\cdot) := \lim_{n\to\infty} \mathbb{P}_{p_c}(\cdot | 0 \leftrightarrow \partial\Lambda_n),$$

where $0 \leftrightarrow \partial\Lambda_n$ if there exists a vertex of the boundary of $\Lambda_n := [-n,n]^d \cap \mathbb{Z}^d$ connected to the origin (see [12] for a construction).

It is expected that this model satisfies the same scaling properties as branching random walk (see [11] for an up-to-date survey). In particular, in high dimensions, the scaling limit of the IIC is related to the integrated super-Brownian excursion (ISE) (see [3] for a definition). In comparison, the scaling limit of a large size-conditioned critical branching random walk, after rescaling space by $n^{-1/4}$, is the ISE (see (3.4) and [9] for the construction of the Brownian motion on the latter object for $d \geq 8$). The following question was posed by Ben Arous and Fribergh in the notes [7]:

'What is the right scaling for a biased random walk on the IIC of $\mathbb{Z}^d$?'

One might anticipate that the scaling limit of a biased random walk with a weak cartesian bias to a single direction on the IIC of $\mathbb{Z}^d$ is a Brownian motion in a random Gaussian potential on an unbounded version of the continuum random fractal tree, where the latter objects appear in the definition of the limiting diffusion in Theorem 2.1. See also the recent conjecture of [8] regarding the associated simple random walks on the IIC (when individual edges have unit resistance).

§ 2. Biased random walk on the range of a branching random walk

To state our main result, we first need to formally define a biased random walk on the range of a critical branching random walk. For this, we will consider a critical Galton-Watson tree, which is a branching process with i.i.d. offspring that are copies of a random variable ξ defined under a law $\mathbf{P}$ satisfying $\mathbf{E}(\xi) = 1$. Given a random realization of a Galton-Watson tree T, we can consider a simple random walk indexed by T, which means that we assign a spatial location $\phi_T(u) \in \mathbb{R}^d$ for every $u \in T$. First, the spatial location $\phi_T(\rho)$ of the root ρ is the origin of $\mathbb{R}^d$. Then, each edge $e \in E(T)$ gets assigned, in an i.i.d. manner, a random variable $y(e)$ which is distributed according to a mean 0 continuous random variable Y. The spatial location $\phi_T(u)$ of a vertex u is the sum of the quantities $y(e)$ over all edges contained to the simple path from the root to u in the tree. In other words, given a value function $y : E(T) \to \mathbb{R}^d$, we defined a map $\phi_T : T \to \mathbb{R}^d$ by setting $\phi_T(\rho) := 0$ and $\phi_T(\vec{\rho}) := 0$ [1],

$$\phi_T(u) := \sum_{e \in E_{\rho,u}} y(e), \qquad u \in T \setminus \{\rho\}, \tag{2.1}$$

where the sum is taken over the set of all edges belonging to the simple path between ρ and u. Also, we interpolate linearly along the edges. The couple (T, ϕ_T) is a random spatial tree under a measure that we will still denote by $\mathbf{P}$. This object will be called the critical branching random walk. This spatial tree can be viewed as a random embedding of T into $\mathbb{R}^d$, by considering the graph $\mathcal{G}$ with vertices given by

$$V(\mathcal{G}) := \{x \in \mathbb{R}^d : x = \phi_T(u) \text{ with } u \in T\}$$

and edge set

$$E(\mathcal{G}) := \{\{x_1, x_2\} \in E(\mathbb{R}^d) : x_i = \phi_T(u_i),\ i = 1, 2 \text{ with } \{u_1, u_2\} \in E(T)\}.$$

Remark. Obviously, since the increments $(y(e))_{e \in E(T)}$ of the spatial element ϕ_T are independent and identically distributed as a continuous random variable Y, the

[1] Throughout this note we use $\vec{u}$ to denote the father in the genealogy of a vertex u. In particular, when $u = \rho$, we add a new vertex b which we call the base and stick it only to the root by an edge with unit weight, so that $b = \vec{\rho}$. This yields a planted tree $\bar{T}$. For notational simplicity, even if the statements hold with respect to $\bar{T}$, we still phrase them in terms of T.

embedded subgraph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a tree. If each edge $e \in E(T)$ gets assigned instead, in an i.i.d. fashion, a random variable $x(e)$ which is distributed according to the step distribution of a simple random walk in $\mathbb{Z}^d$, still induces a spatial tree that can be similarly viewed as a subgraph of $\mathbb{Z}^d$, which is not necessarily a tree. To contrast the former and the latter cases, one usually refers to the corresponding spatial trees as critical non-lattice and lattice branching random walk respectively.

In this paper, we are interested in large critical branching random walk, which we obtain from the previous construction by conditioning the vertex cardinality of T to be n, i.e. by letting $\{(T_n, \phi_n)\}_{n\geq 1}$ be the family of random spatial trees, where T_n is generated by a Galton-Watson process with critical offspring distribution ξ defined under the measure $\mathbf{P}_n := \mathbf{P}(\cdot|\ |T| = n)$, which is asymptotically well-defined under the the inherent assumption that the distribution of ξ is aperiodic (meaning that is not supported on a sub-lattice of $\mathbb{Z}$). In this case, we denote the embedded subgraph associated with the spatial tree (T_n, ϕ_n) by $\mathcal{G}_n$.

Now, fix a bias $\beta \geq 1$, and to each edge $\{x_1, x_2\} \in E(\mathcal{G}_n)$, assign a conductance

$$c(\{x_1, x_2\}) := \beta^{\max\{\phi_n^{(1)}(u_1), \phi_n^{(1)}(u_2)\}} \tag{2.2}$$

where $\phi_n^{(1)}(u_i)$ denotes the first coordinate of $\phi_n(u_i)$. Check that $c(\{\phi_n(\vec{\rho_n}), \phi_n(\rho_n)\}) = \beta^{\max\{\phi_n^{(1)}(\vec{\rho_n}), \phi_n^{(1)}(\rho_n)\}} = 1$, which agrees with putting a unit conductance between ρ_n and its base $b_n = \vec{\rho_n}$. The biased random walk on $\mathcal{G}_n$ is then the time-homogeneous Markov chain $X = ((X_n)_{n\geq 0}, \mathbf{P}_x^{\mathcal{G}_n}, x \in V(\mathcal{G}_n))$ on $V(\mathcal{G}_n)$ with transition probabilities given by

$$P_{\mathcal{G}_n}(x_1, x_2) := \frac{c(\{x_1, x_2\})}{c(\{x_1\})},$$

where c is a measure on $V(\mathcal{G}_n)$ defined by $c(\{x\}) := \sum_{e\in E(\mathcal{G}_n):x\in e} c(e)$. It is simple to check that c is the invariant measure for X as the detailed balance equations are satisfied. Note that if $\beta > 1$, then the biased random walk X prefers to move towards the direction of the first coordinate. Besides, if $\beta = 1$, there is no bias and the preceding definition leads to the simple random walk on $\mathcal{G}_n$. Finally, using the usual terminology for random walks in random environments, for $x \in V(\mathcal{G}_n)$, we say that $\mathbf{P}_x^{\mathcal{G}^n}$ is the quenched law of X started from x. Since ρ_n is an element of $V(\mathcal{G}_n)$, we can also define an annealed law $\mathbb{P}_{\rho_n}$ for the biased random walk on $\mathcal{G}_n$ started from ρ_n by setting:

$$\mathbb{P}_{\rho_n} := \int \mathbf{P}_{\rho_n}^{\mathcal{G}_n}(\cdot)\mathrm{d}\mathbf{P}. \tag{2.3}$$

Under this law we can prove Theorem 2.1, which shows that, a weakly biased random walk on the critical non-lattice branching random walk converges to a Brownian motion in random Gaussian potential on Aldous' continuum random tree (CRT).

More formally, denote by $e := (e(t))_{0\le t\le 1}$ a normalized Brownian excursion. Informally, e is a Brownian motion started at the origin and conditioned to remain positive over the time interval $(0,1)$, and to return back to 0 at time 1. By convention, let $e(t) := 0$ if $t > 1$. For every $s,t \ge 0$, we set

$$d_{\mathcal{T}}(s,t) := e(s) + e(t) - 2 \inf_{r\in[s\wedge t, s\vee t]} e(r).$$

The random real tree coded by e (see [15] for more details)

$$\mathcal{T} := [0,\infty)/\sim^e, \tag{2.4}$$

where $s \sim^e t$ if and only if $d_{\mathcal{T}}(s,t) = 0$ (or equivalently if and only if $e(s) = e(t) = \inf_{r\in[s\wedge t, s\vee t} e(r))$, is called Aldous' continuum random tree and will be denoted by $(\mathcal{T}, d_{\mathcal{T}})$ when endowed with its canonical metric $d_{\mathcal{T}}$. Furthermore, the equivalence class of t with respect to the relation defined above will be denoted by $p_e(t)$. We may consider an $\mathbb{R}^d$-valued tree-indexed Gaussian process $(\phi(\sigma))_{\sigma\in\mathcal{T}}$ with

$$\begin{aligned} &E\phi(\sigma) = 0,\\ &\mathrm{Cov}(\phi(\sigma),\phi(\sigma')) = d_{\mathcal{T}}(\rho, \sigma\wedge\sigma')I, \end{aligned} \tag{2.5}$$

where I is the d-dimensional identity matrix and $\sigma\wedge\sigma'$ is the most recent common ancestor of σ and σ'. For almost-every realization of $\mathcal{T}$ (w.r.t the normalized Itô excursion measure $\mathbb{N}_1$), this process has a continuous modification. Our main result is the following.

Theorem 2.1. *Assume that $\mathbf{E}(\xi) = 1$, $\sigma_\xi^2 := \mathrm{Var}(\xi) \in (0,\infty)$ and $\mathbf{E}(e^{\lambda\xi}) < \infty$, for $\lambda > 0$ and that the distribution of ξ is aperiodic. The increments of the spatial element ϕ_n, conditional on T_n, are assumed to be i.i.d. copies of a mean 0 continuous random variable Y with finite variance $\Sigma_Y^2 < \infty$ that furthermore satisfies the tail condition*

$$\mathbf{P}(d_E(0,Y) \ge y) = o(y^{-4}).$$

Consider the weakly biased random walk $X^n = (X^n_m)_{m\ge 1}$ on T_n with bias parameter $\beta^{n^{-1/4}}$, for $\beta > 1$. There exists constant $\sigma_T > 0$ and a positive definite $d\times d$-matrix Σ_ϕ such that the following convergence in law holds

$$\left(n^{-1/4}\phi_n(X^n_{n^{3/2}t})\right)_{t\ge 0} \to \left(\Sigma_\phi\phi(X_{t\sigma_T^{-1}})\right)_{t\ge 0},$$

where X^n is chosen under the annealed law $\mathbb{P}_{\rho_n}$ and the limiting process $(X_t)_{t\ge 0}$ is a Brownian motion in a random Gaussian potential $\phi^{(1)}$ on the CRT, where $\phi^{(1)}$ is the first coordinate of the tree-indexed Gaussian process $(\phi(\sigma))_{\sigma\in\mathcal{T}}$. The convergence is annealed and occurs in $D(\mathbb{R}_+,\mathbb{R}^d)$.

Remark. As suggested by [13, Theorem 2], the fourth order polynomial decay of the step variable Y is necessary to obtain the convergence of a two-dimensional process $(\hat{C}_n(i), \hat{R}_n(i))$. Here, $\hat{C}_n(i)$ is the distance to the root of the position of a particle that traverses the outline of T_n from left to right at unit speed (notice that we visit every vertex apart from the root a number of times given by its degree), and $\hat{R}_n(i) := \phi_n(\hat{u}_i^n)$, where $\hat{u}_i^n$ denotes the i-th visited vertex in exploration of T_n by the particle. It will be convenient to define a somewhat shifted version of $(\hat{C}_n(i) : 0 \le i \le 2(n-1))$ to correspond to the exploration of the outline of the planted tree $\bar{T}_n$: let $C_n(0) = C_n(2n) = 0$ and, for $0 \le i \le 2n-1$, $C_n(i) := 1 + \hat{C}_n(i-1)$.

§ 3. Biased random walk as a random walk in a random environment

The aim of this section is to describe how a biased random walk on a non-lattice branching random walk, which is essentially a biased random walk on a self-avoiding path can be expressed as a random walk in a random envrironment on a tree. As we will demonstrate, this will enable us to transfer results known for the latter model to the particular model we study.

For $u \in T_n$, we denote its children by $u_1, ..., u_{\xi(u)}$ and its parent by u_0. Let us define the random walk in a random environment of interest to us in this section. The random environment ω will be represented by a random sequence $\{(\omega_{uu_i})_{i=0}^{\xi(u)} : u \in T_n\}$ in $(0,1)^{\xi(u)+1}$ such that $\sum_{i=0}^{\xi(u)} \omega_{uu_i} = 1$, and will be built on the probability space with probability measure $\mathbf{P}$ as well. The random walk in a random environment will be the time-homogeneous Markov chain $X' = ((X'_n)_{n\ge 0}, \mathbf{P}_u^\omega, u \in T_n)$ on T_n with transition probabilities

$$\{(P_\omega(X'_{n+1} = u_i | X'_n = u))_{i=0}^{\xi(u)} : u \in T_n\} = \{(\omega_{uu_i})_{i=0}^{\xi(u)} : u \in T_n\}.$$

For $u \in T_n$, the law $\mathbf{P}_u^\omega$ is the quenched law of X' started from u. Furthermore, we can define an annealed law for X' similarly to (2.3). To connect this model with the biased random walk on the critical non-lattice branching random walk conditioned to have n particles, suppose that the transition probabilities are defined by setting

$$\{(\omega_{uu_i})_{i=0}^{\xi(u)} : u \in T_n\} := \{(P_{\mathcal{G}_n}(\phi_n(u), \phi_n(u_i)))_{i=0}^{\xi(u)} : u \in T_n\}.$$

For this choice of random environment, it is immediate that the quenched law of $\phi_n(X')$ is the same as that of X. A corresponding identity holds for the relevant annealed laws.

It is also important to connect the first coordinate of the random embedding ϕ_n with the potential of the random walk in the random environment. To be more precice, let $(\Delta_n(u))_{u\in T_n}$ be its increments process, i.e.

$$\Delta_n(u) := \phi_n^{(1)}(u) - \phi_n^{(1)}(\vec{u}).$$

Then, if $\rho_{\vec{u}u} := \omega_{\vec{\vec{u}}\vec{u}}/\omega_{\vec{u}u}$ [2], where ω is defined as in the previous paragraph, the potential $(V_n(u))_{u\in T_n}$ of the random walk in a random environment, is obtained by setting

$$V_n(u) := \begin{cases} \sum_{e\in E_{\rho_n,u}} \log \rho_e, & \text{if } u \in T_n \setminus \{\rho_n\}, \\ 0, & \text{if } u = \rho_n. \end{cases} \tag{3.1}$$

The elementary calculation involving the telescopic sum

$$\sum_{e\in E_{\rho_n,u}} \log \rho_e = -\log\beta \cdot \max\{\phi_n^{(1)}(\vec{u}), \phi_n^{(1)}(u)\}, \tag{3.2}$$

yields that the potential satisfies

$$V_n(u) = -\log\beta(\phi_n^{(1)}(\vec{u}) + \max\{0, \Delta_n(u)\}). \tag{3.3}$$

Therefore, if the individual increments are negligible, $\phi_n^{(1)}$ (multiplied by a negative constant that depends on β) nearly gives the leading order of the potential of the random walk in the random environment.

The potential is relevant when understanding the behaviour of the random walk in a random environment in Sinai's regime. When ϕ_n rescales in such a way that it incorporates a functional invariance principle on the first coordinate, we demonstrate through the connection between V_n and $\phi_n^{(1)}$ derived above, that the potential V_n converges to an embedding of the CRT into the Euclidean space, so that an arc of length t in the CRT is mapped to the range of a Brownian motion run for time t. In other words, V_n converges to ϕ as defined in (2.5).

For trees that satisfy the assumptions in Theorem 2.1, [9, Corollary 10.3] ensures the following convergence in law. If d_{T_n} gives the length of the shortest path between elements of T_n and μ_{T_n} is the empirical measure on T_n which puts mass n^{-1} on each vertex, we have that

$$\Big((T_n, n^{-1/2}d_{T_n}, \mu_{T_n}, \rho_n), n^{-1/4}\phi_n\Big) \to ((\mathcal{T}, \sigma_T d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho), \Sigma_\phi \phi), \tag{3.4}$$

where $\sigma_T := 2/\sigma_\xi$ and $\Sigma_\phi := \Sigma_Y \sqrt{\sigma_T}$. Combining (3.3) with (3.4) yields

$$((T_n, n^{-1/2}d_{T_n}, \mu_{T_n}, \rho_n), n^{-1/4}\phi_n, n^{-1/4}V_n) \to ((\mathcal{T}, \sigma_T d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho), \Sigma_\phi\phi, \sigma_{\beta,\phi}\phi^{(1)}), \tag{3.5}$$

in the spatial Gromov-Hausdorff-vague topology, where $\phi^{(1)}$ denotes the first coordinate of ϕ and $\sigma_{\beta,\phi} = -\log\beta \cdot (\Sigma_\phi)_{11}$.

In Theorem 2.1, on account of the 'flattening' that the bias has to undergo, we referred to X^n as the weakly biased random walk on $\mathcal{G}_n$ with bias parameter $\beta_n := \beta^{n^{-1/4}}$, for some $\beta > 1$. Observing that $n^{-1/4}V_n$ is the potential of the random walk

[2] The fraction is well-defined for every node of T_n except the root ρ_n.

in a random environment X' when β is replaced by β_n, and in combination with (3.5), indicates why we chose to change the bias at every step in that way. From now on, we will solely work with this random walk in a random environment and keep referring to it as such. One of the crucial facts for the random walk in a random environment is that, for fixed ω, the random walk is a reversible Markov chain and can therefore be described as an electrical network. The conductances are given by $c(\{\vec{u},u\}) = e^{-n^{-1/4}V_n(u)}$ (see (3.1), (3.2) and cf. (2.2)) and the stationary reversible measure which is unique up to multiplication by constant is given pointwise in u by

$$\nu_n(\{u\}) = e^{-n^{-1/4}V_n(u)} + \sum_{i=1}^{\xi(u)} e^{-n^{-1/4}V_n(u_i)}, \tag{3.6}$$

where the sum is taken over the set of all vertices contained in the neighborhood of u excluding its parent. The reversibility means that, for all $m \in \mathbb{N}_0$ and $u_1, u_2 \in T_n$, we have that

$$\nu_n(\{u_1\})P_\omega^{u_1}(X'_m = u_2) = \nu_n(\{u_2\})P_\omega^{u_2}(X'_m = u_1).$$

Let $(c(e))_{e\in E(T_n)}$ be the collection of edge weights given by the conductances described above. The electrical resistance between $u_1 \neq u_2$ is given by

$$r_n(u_1,u_2)^{-1} = \inf\{\mathcal{E}_n(f,f) : f(u_1) = 0, f(u_2) = 1\},$$

where

$$\mathcal{E}_n(f,g) = \frac{1}{2}\sum_{x\sim y} c(\{x,y\})(f(x)-f(y))(g(x)-g(y))$$

is a quadratic form on T_n. One can check that r_n is a metric on T_n (e.g. [16] and characterises the weights and the quadratic form uniquely [14]). Moreover, since T_n is a graph tree, one has that r_n is identical to the weighted shortest path metric, with edges weighted according to $(1/c(e))_{e\in E(T_n)}$, i.e.

$$r_n(u_1,u_2) = \sum_{u\in[u_1,u_2]]} e^{n^{-1/4}V_n(u)}, \qquad u_1,u_2 \in T_n, \tag{3.7}$$

where for u_1, u_2 in a rooted metric tree (T, r, ρ) by

$$[[u_1,u_2]] := \{u \in T : r(u_1,u_2) = r(u_1,u) + r(u,u_2)\},$$

$[u_1,u_2]] := [[u_1,u_2]]\setminus\{u_1\}$, $[u_1,u_2] := [[u_1,u_2]]\setminus\{u_1,u_2\}$, we denoted the path intervals.

In the rest of the paper we verify that (3.5) also holds when the shortest path metric d_{T_n} and the empirical measure which puts mass n^{-1} on each of the vertices of T_n are perturbed by exponential functionals of the potential of the random walk in a random environment as can be seen by the form of the stationary reversible measure

ν_n and the electrical resistance r_n in (3.6) and (3.7) respectively. Before stating this result, we need to introduce the notion of a length measure on separable rooted metric spaces (T, r, ρ). Denote the skeleton of T as $\mathrm{Sk}(T) := \cup_{u \in T} [\rho, u] \cup \mathrm{Is}(T)$, where $\mathrm{Is}(T) := \{u_1, u_2 \in T \setminus \{\rho\} : u_1 \neq u_2 \text{ and } [[u_1, u_2]] = \{u_1, u_2\}\}$ is the set of isolated points of T, excluding ρ. Then, the length measure of T is the unique σ-finite measure λ, such that $\lambda(T \setminus \mathrm{Sk}(T)) = 0$ and for all $u \in T$,

$$\lambda([\rho, u]]) = r(\rho, u). \tag{3.8}$$

Since all the points in (T_n, r_n, ρ_n) are isolated, note that the length measure of T_n shifts the length of an edge to the endpoint that is further away from ρ_n, and therefore it does depend on ρ_n.

§ 4. Convergence of the distorted metric measure trees

The convergence of the processes mentioned in Theorem 2.1 will essentially emanate from the convergence of the distorted metrics (3.7) and measures (3.6) that provide the scale functions and speed measures in this framework. Indeed, as an application of the main results in [6] and [8], this will lead to the convergence of the associated random walks in random environment. We prove the former below and leave the latter as a concluding remark to the article.

Theorem 4.1. *The following convergence in law holds*

$$((T_n, n^{-1/2} r_n, (2n)^{-1} \nu_n, \rho_n), n^{-1/4} \phi_n, n^{-1/4} V_n) \to ((\mathcal{T}, \sigma_T r_{\phi^{(1)}}, \nu_{\phi^{(1)}}, \rho), \Sigma_\phi \phi, \sigma_{\beta,\phi} \phi^{(1)})$$

in the spatial Gromov-Hausdorff-vague topology, where

$$r_{\phi^{(1)}}(u_1, u_2) := \int_{u_1}^{u_2} e^{\sigma_{\beta,\phi} \phi^{(1)}(v)} \mathrm{d}\lambda(v), \tag{4.1}$$

and $\nu_{\phi^{(1)}}$ is the image measure $(p_{\tilde{e}})_* \ell$ by the canonical projection $p_{\tilde{e}}$ of the Lebesgue measure ℓ on $[0, 1]$, where

$$\tilde{e} := \left(\int_{p_e(0)}^{p_e(t)} e^{-\sigma_{\beta,\phi} \phi^{(1)}(v)} \mathrm{d}\lambda(v) \right)_{0 \le t \le 1}. \tag{4.2}$$

Remark. Note that $\tilde{e}$ is a random excursion, which means that $\tilde{e} \in C([0, 1], \mathbb{R}_+)$ and $\tilde{e}(0) = \tilde{e}(1) = 0$. Thus, $p_{\tilde{e}}(t)$ is well-defined to represent the equivalence class of t with respect to the relation

$$s \sim^{\tilde{e}} t \iff \tilde{e}(s) = \tilde{e}(t) = \inf_{r \in [s \wedge t, s \vee t]} \tilde{e}(r).$$

It is easy to check that $r_{\phi^{(1)}}$ defines a metric on $\mathcal{T}$. In addition, $d_{\mathcal{T}}$ and $r_{\phi^{(1)}}$ are topologically equivalent and the metric space $(\mathcal{T}, \sigma_T r_{\phi^{(1)}})$ is also a compact real tree coded by $\tilde{e}$ in the sense of (2.4).

Proof. Using Skorohod's representation, we can assume that we are working in a probability space under which the exploration of the outline of T_n (for a definition, see the remark that lies below the statement of Theorem 2.1), which is normalized by setting:

$$(C_{(n)}(t))_{0\le t\le 1} := \left(n^{-1/2}C_n(2nt) : 0 \le t \le 1\right),$$

converges almost-surely (in the uniform norm) to a normalized Brownian excursion e. The convergence in law

$$C_{(n)} \to \sigma_T e \text{ in } C([0,1], \mathbb{R}_+) \tag{4.3}$$

was originally shown by Aldous in [2]. We construct an appropriate correspondence

$$R_n := \{(u_i^n, p_e(t)) : \exists t \in [0,1] \text{ s.t. } i = \lfloor 2nt \rfloor\},$$

where u_i^n is the i-th visited vertex in the exploration of the outline of T_n and $p_e(t)$ is the equivalence class of t with respect to the relation $s \sim^e t$. The trees are defined as projections from $[0,1]$ (with some additional equivalence structure), so taking pairs of projections from $[0,1]$ gives a natural correspondence.

The first part of the proof shows that the distortion of the natural correspondence

$$\operatorname{dis}(R_n) := \sup\{|n^{-1/2}r_n(u_i^n, u_j^n) - r_{\phi^{(1)}}(p_e(s), p_e(t))| : (u_i^n, p_e(s)), (u_j^n, p_e(t)) \in R_n\}$$

converges to 0. Now, suppose that $(u_i^n, p_e(s)) \in R_n$. Then

$$\begin{aligned} &\left|n^{-1/2}r_n(u_0^n, u_i^n) - r_{\phi^{(1)}}(p_e(0), p_e(s))\right| \\ &= \left|n^{-1/2}\sum_{v\in[u_0^n,u_i^n]]} e^{n^{-1/4}V_n(v)} - \int_{[[p_e(0),p_e(s)]]} e^{\sigma_{\beta,\phi}\phi^{(1)}(v)}\mathrm{d}\lambda(v)\right| \\ &= \left|\int_{[u_0^n,u_i^n]]} e^{n^{-1/4}V_n(v)}\mathrm{d}\lambda_n(v) - \int_{[[p_e(0),p_e(s)]]} e^{\sigma_{\beta,\phi}\phi^{(1)}(v)}\mathrm{d}\lambda(v)\right|, \end{aligned} \tag{4.4}$$

where λ_n is the length measure of the discrete rooted metric tree $(T_n, n^{-1/2}d_{T_n}, u_0^n)$. The second equality follows from the fact that λ_n shifts the length of an edge to the endpoint that is further away from u_0^n, while also observing that $\lambda_n([u_k^n, u_{k'}^n]]) = n^{-1/2}$, for all $u_k^n \sim u_{k'}^n$.

Recall here that $C_n := (C_n(i) : 0 \le i \le 2n)$ is a random excursion with length $2n$. For every $i, j \ge 0$, setting

$$d_{C_n}(i,j) := C_n(i) + C_n(j) - 2 \inf_{k \in [i \wedge j, i \vee j]} C_n(k),$$

we identify i and j if and only if $d_{C_n}(i,j) = 0$ (or equivalently if and only if $C_n(i) = C_n(j) = \inf_{k \in [i\wedge j, i \vee j]} C_n(k)$). In the terminology as well as in the notation of (2.4), we have that T_n coincides with the tree coded by the function C_n. Furthermore, the equivalence class of i with respect to the relation defined above will be denoted by $p_{C_n}(i)$. Note that the previous quantity represents u_i^n. It is not difficult to convince oneself that

$$\int_{[u_0^n, u_i^n]]} e^{n^{-1/4} V_n(v)} \mathrm{d}\lambda_n(v) = \int_{A_{(n)}^s} e^{n^{-1/4} V_n(u^n_{\lfloor 2nr \rfloor})} \mathrm{d}\lambda_{(n)}(r), \tag{4.5}$$

where $A_{(n)}^s := \{r \le s : \inf_{u \in [r,s]} C_{(n)}(u) = C_{(n)}(r)\}$ and $\lambda_{(n)} := \lambda_n \circ (p_{C_{(n)}})$. Similarly,

$$\int_{[[p_e(0), p_e(s)]]} e^{\sigma_{\beta,\phi} \phi^{(1)}(v)} \mathrm{d}\lambda(v) = \int_{A_e^s} e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))} \mathrm{d}\lambda_e(r), \tag{4.6}$$

where $A_e^s := \{r \le s : \inf_{u \in [r,s]} e(u) = e(r)\}$ and $\lambda_e := \lambda \circ (p_e)$.

Let us set $g_n(r) = e^{n^{-1/4} V_n(u^n_{\lfloor 2nr \rfloor})}$ and $g(r) = e^{\sigma_{\beta,\phi}\phi^{(1)}(p_e(r))}$. Combining (4.5) and (4.6) with (4.4), for $(u_i^n, p_e(s)) \in R_n$, we deduce

$$\begin{aligned}\left| \int_{A_{(n)}^s} g_n(r)\mathrm{d}\lambda_{(n)}(r) - \int_{A_e^s} g(r) \mathrm{d}\lambda_e(r) \right| &\le \left| \int_{A_{(n)}^s} g_n(r)\mathrm{d}\lambda_{(n)}(r) - \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_{(n)}(r) \right| \\ &+ \left| \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_{(n)}(r) - \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_e(r) \right| \\ &+ \left| \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_e(r) - \int_{A_e^s} g(r)\mathrm{d}\lambda_e(r) \right|.\end{aligned}$$

Hence

$$\begin{aligned}\left| n^{-1/2} r_n(u_0^n, u_i^n) - r_{\phi^{(1)}}(p_e(0), p_e(s)) \right| &\le \left| \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_{(n)}(r) - \int_{A_{(n)}^s} g(r)\mathrm{d}\lambda_e(r) \right| \\ &+ ||g||_\infty \int |\mathbf{1}_{A_{(n)}^s}(r) - \mathbf{1}_{A_e^s}(r)| \mathrm{d}\lambda_e(r) \\ &+ ||g_n - g||_\infty \lambda_{(n)}(A_{(n)}^s).\end{aligned} \tag{4.7}$$

There are several steps along the way to show that the right-hand side converges to 0 uniformly in $s \in [0,1]$. First, by definition

$$|\lambda_{(n)}(A_{(n)}^s) - \lambda_e(A_e^s)| = |C_{(n)}(s) - e(s)| \xrightarrow{n \to \infty} 0, \tag{4.8}$$

uniformly in $s \in [0,1]$. Now suppose that $(u_i^n, p_e(s))$, $(u_j^n, p_e(t)) \in R_n$ with $s \le t$. In a second place, it is not hard to see that

$$
\begin{aligned}
\left|\lambda_{(n)}([s,t]) - \lambda_e([s,t])\right| = \bigg| &\Big(\lambda_{(n)}(A^s_{(n)}) + \lambda_{(n)}(A^s_{(n)}) - 2\lambda_{(n)}(A^r_{(n)})\Big) \\
&- \big(\lambda_e(A^s_e) + \lambda_e(A^t_e) - 2\lambda_e(A^r_e)\big)\bigg|,
\end{aligned}
\tag{4.9}
$$

if r is any time which achieves the minimum between s and t of $C_{(n)}$ and e, when the latter processes are coupled in such a way that (4.3) holds almost-surely. Together with (4.8), this shows that (4.9) converges to 0 uniformly in $s, t \in [0,1]$, and consequently that $\lambda_{(n)}$ converges strongly to λ_e. This entails that the first term on the right-hand side of the inequality in (4.7) converges to 0 uniformly in $s \in [0,1]$. The same can be said of the second term since

$$
|\mathbf{1}_{A^s_{(n)}}(r) - \mathbf{1}_{A^s_e}(r)| \xrightarrow{n\to\infty} 0,
$$

uniformly in $s \in [0,1]$. Its convergence simply follows by applying the dominated convergence theorem. Finally, the convergence to 0 of the third term on the right-hand side of the inequality in (4.7) stems from a combination of (3.5) and (4.8).

It remains to bound $\mathrm{dis}(R_n)$. The roots enable an orientation sensitive integration given by

$$
\begin{aligned}
|n^{-1/2} r_n(u_i^n, u_j^n) - r_{\phi^{(1)}}(p_e(s), p_e(t))| \le{}& 2|n^{-1/2} r_n(u_0^n, u_k^n) - r_{\phi^{(1)}}(p_e(0), p_e(r))| \\
&+ |n^{-1/2} r_n(u_0^n, u_i^n) - r_{\phi^{(1)}}(p_e(0), p_e(s))| \\
&+ |n^{-1/2} r_n(u_0^n, u_j^n) - r_{\phi^{(1)}}(p_e(0), p_e(t))|,
\end{aligned}
$$

where $(u_k^n, p_e(r)) \in R_n$ and $r \in [s,t]$ as in the previous paragraph. As we have shown on our work on the bound of (4.7), each individual term converges to 0 uniformly in $r, s, t \in [0,1]$ respectively, and this completes the first part of the proof about the convergence to 0 of the distortion $\mathrm{dis}(R_n)$ of the natural correspondence.

We introduce what we call the distorted exploration of the outline of T_n. It actually collects a weight equal to $e^{-n^{-1/4} V_n(u_i^n)}$ whenever the directed edge connecting the parent of u_i^n to u_i^n is traversed in the canonical exploration of the outline of T_n. More specifically, we set

$$
\tilde{C}_n(i) := \sum_{u \in [u_0^n, u_i^n]]} e^{-n^{-1/4} V_n(u)}, \qquad 0 < i < 2n.
$$

By convention: $\tilde{C}_n(0) = \tilde{C}_n(2n) = 0$. Extend $\tilde{C}_n$ by linear interpolation to non-integers. Then, (T_n, r_n) when endowed with the distorted metric r_n coincides with the real tree coded by the function $\tilde{C}_n$. In parallel with this, $(\mathcal{T}, r_{\phi^{(1)}})$ coincides with the real tree coded by the function $\tilde{e}$ as defined in (4.2).

The last step in the proof consists of showing that the Prokhorov distance $d^P_{(T_n,r_n)}$ between $(2n)^{-1}\mu_{\tilde{C}_n} := (p_{\tilde{C}_n})_*\ell$ and $\nu_{\phi^{(1)}}$ is negligible. This is sufficient since

$$d^P_{(T_n,r_n)}\left((2n)^{-1}\mu_{\tilde{C}_n}, (2n)^{-1}\nu_n\right) \leq (2n)^{-1}.$$

Consulting the proof of [1, Proposition 2.10], there exists a common metric space (K, d_K) such that

$$d^P_{(K,d_K)}\left((2n)^{-1}\mu_{\tilde{C}_n}, \nu_{\phi^{(1)}}\right) \leq \frac{1}{2}\mathrm{dis}(R_n) + |\mathrm{supp}(\tilde{C}_n) - \mathrm{supp}(\tilde{e})|.$$

The right-hand side converges to 0 as $n \to \infty$, and since

$$d_{\mathrm{GHP}}\left((T_n, n^{-1/2}r_n, (2n)^{-1}\nu_n), (\mathcal{T}, \sigma_T r_{\phi^{(1)}}, \nu_{\phi^{(1)}})\right) \leq |\mathrm{supp}(\tilde{C}_n) - \mathrm{supp}(\tilde{e})| + \mathrm{dis}(R_n),$$

the desired result follows. □

The process associated with $(\mathcal{T}, \sigma_T r_{\phi^{(1)}})$, or the $\nu_{\phi^{(1)}}$-Brownian motion in a random Gaussian potential $\phi^{(1)}$ on the CRT as we called it, is a new object that appears as the scaling limit of the weakly biased random walk X^n on T_n with bias parameter β_n. To make this precise, we assume that the random elements involved in the statement of Theorem 4.1 belong to a probability space with probability measure $\mathbf{P}$ under which the probability measure such that the pair of the normalized discrete tours $(\hat{C}_n, \hat{R}_n)$ is in its support, converges weakly to a probability measure defined in such a way that $(\mathcal{T}, \phi)$ has it as its marginal (see [13, Theorem 2]). Using Skorohod's representation, the annealed laws $\mathbb{P}_{\rho_n}$ and $\mathbb{P}_\rho$ of the weakly biased random walk X^n and the $\nu_{\phi^{(1)}}$-Brownian motion in a random Gaussian potential $\phi^{(1)}$ are constructed by integrating the randomness out of the state spaces with respect to $\mathbf{P}$.

Finally, we are able to prove our result. The subspace of continuous functions $f : \mathcal{T} \to \mathbb{T}$ vanishing at infinity will be denoted by $\mathcal{C}_\infty$. A continuous function $f : \mathcal{T} \to \mathbb{R}$ is called locally absolutely continuous if for every $\varepsilon > 0$ and all subsets $\mathcal{T}' \subseteq \mathcal{T}$ with $\lambda(\mathcal{T}') < \infty$, there exists a $\delta \equiv \delta(\mathcal{T}', \varepsilon)$, such that if $[[u_i, v_i]]_{i=1}^n \subseteq \mathcal{T}$ is a disjoint collection with $\sum_{i=1}^n r(u_i, v_i) < \delta$, then $\sum_{i=1}^n |f(u_i) - f(v_i)| < \varepsilon$. Denote the subspace of locally absolutely continuous functions by $\mathcal{A}$.

Proof of Theorem 2.1. As a consequence of Theorem 4.1 and since the spaces involved in the spatial Gromov-Hausdorff-vague convergence are compact, for Lebesgue a.e. $R \geq 0$,

$$\lim_{R\to\infty} \liminf_{n\to\infty} \mathbf{P}\left(r_n(\rho_n, B_n(\rho_n, R)^c) \geq \lambda\right) = 1, \qquad \forall \lambda \geq 0.$$

Therefore, as a corollary of [8, Theorem 1.2], the ν_n-speed motion on (T_n, r_n) converges to the $\nu_{\phi^{(1)}}$-Brownian motion on $(\mathcal{T}, \sigma_T r_{\phi^{(1)}})$ in the annealed sense weakly on $D(\mathbb{R}_+, \mathbb{R}^d)$, when continuously embedded into $\mathbb{R}^d$ by $n^{-1/4}\phi_n$ and $\Sigma_\phi \phi$ respectively. The ν_n-speed motion on (T_n, r_n) is the continuous-time nearest neighbor random walk on (T_n, r_n) with jump rates

$$q_n(u_1, u_2)^{-1} := 2 \cdot \nu_n(\{u_1\}) \cdot r_n(u_1, u_2), \qquad u_1 \sim u_2,$$

which is equal in law to $(X^n_{n^{3/2}t})_{t \geq 0}$. Finally, the $\nu_{\phi^{(1)}}$-Brownian motion on $(\mathcal{T}, \sigma_T r_{\phi^{(1)}})$ is the $\nu_{\phi^{(1)}}$-symmetric strong Markov process associated with the regular Dirichlet form $(\mathcal{E}, \bar{\mathcal{D}}(\mathcal{E}))$ on $(\mathcal{T}, \sigma_T r_{\phi^{(1)}}, \nu_{\phi^{(1)}})$ given by

$$\mathcal{E}(f, g) := \frac{1}{2} \int \nabla f(v) \cdot \nabla g(v) e^{\sigma_{\beta,\phi}\phi^{(1)}(v)} \mathrm{d}\lambda(v), \qquad f, g \in \mathcal{D}(\mathcal{E}),$$

where

$$\mathcal{D}(\mathcal{E}) := \{f \in L^2(\nu_{\phi^{(1)}}) \cap \mathcal{C}_\infty \cap \mathcal{A} : \nabla f \in L^2(e^{\sigma_{\beta,\phi}\phi^{(1)}} \mathrm{d}\lambda)\},$$

and this concludes the proof. □

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