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Refinement of Strichartz estimates for Airy equation and application

Dedicated to Professor Yoshio Tsutsumi on his 60th birthday

By

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Abstract

In this paper we review our recent results [19, 20] on the refinement of the Strichartz estimate for the Airy equation. As an application of this estimate, we construct a minimal non-scattering solution to the mass sub-critical generalized Korteweg-de Vries equation.

§ 1. Introduction

We consider the space-time estimates for the solution $e^{-t\partial_x^3}f$ to the Airy equation

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & t, x \in \mathbb{R}, \\ u(0,x) = f(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given data. As with the Schrödinger equation, the Strichartz estimate for (1.1) is well-known (see [10] for instance). Grünrock [7] generalized the Strichartz estimate for (1.1) to the hat-Lebesgue space. More precisely, he obtained the following space-time estimates.

Theorem 1.1 (Stein-Tomas type estimate [7]). *Let $4/3 < p \leqslant \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L^{3p}_{t,x}} \leqslant C \|f\|_{\hat{L}^p} \tag{1.2}$$

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holds for any $f \in \hat{L}^p$*, where the space* $\hat{L}^p$ *is defined for* $1 \leqslant p \leqslant \infty$ *by*

$$\hat{L}^p = \hat{L}^p(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) | \, \|f\|_{\hat{L}^p} = \|\hat{f}\|_{L^{p'}} < \infty\},$$

where $\hat{f}$ *stands for Fourier transform of* f *with respect to space variable and* p' *denotes the Hölder conjugate of* p*.*

The space $\hat{L}^p$ has appeared in study of Fourier restriction theorem [6, 25]. Recently, well-posedness of the nonlinear dispersive equations (nonlinear Schrödinger and generalized KdV equations,...) are studied by Grünrock [7] and Hyakuna-Tsutsumi [8] in the framework of $\hat{L}^p$ space. Notice that the Hausdorff-Young inequality yields

$$\begin{aligned} L^p &\hookrightarrow \hat{L}^p && \text{if } 1 \leqslant p \leqslant 2, \\ \hat{L}^p &\hookrightarrow L^p && \text{if } 2 \leqslant p \leqslant \infty. \end{aligned}$$

The key point to prove the Stein-Tomas type estimate (1.2) is to reduce the *linear* form into a *bi-linear* form. For the reader's convenience, we give the proof of the estimate (1.2) in Appendix A.

We consider an improvement of the Stein-Tomas type estimate (1.2). To state our main result more precisely, we now introduce generalized Morrey space and generalized hat-Morrey space.

Definition 1.2. For $j, k \in \mathbb{Z}$, let $\tau_k^j = [k2^{-j}, (k+1)2^{-j})$.
(i) For $1 \leqslant q \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$, we define a generalized Morrey norm $\|\cdot\|_{M^p_{q,r}}$ by

$$\|f\|_{M^p_{q,r}} = \left\| |\tau_k^j|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\tau_k^j)} \right\|_{\ell^r_{j,k}},$$

where, the case $p = q$ and $r < \infty$ is excluded.
(ii) For $1 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant r \leqslant \infty$, we introduce a generalized hat-Morrey norm by

$$\|f\|_{\hat{M}^p_{q,r}} := \|\hat{f}\|_{M^{p'}_{q',r}} = \left\| |\tau_k^j|^{\frac{1}{q}-\frac{1}{p}} \left\|\hat{f}\right\|_{L^{q'}(\tau_k^j)} \right\|_{\ell^r_{j,k}}.$$

Banach spaces $M^p_{q,r}$ and $\hat{M}^p_{q,r}$ are defined as sets of tempered distributions of which above norms are finite, respectively.

The space $\hat{M}^p_{q,r}$ naturally appears in the context of the estimate for the maximal function [3, 22, 23] and the refinement of Stirchartz's estimate [4, 21, 12, 26, 1, 24]. Notice that the inclusion relation $\hat{L}^p \hookrightarrow \hat{M}^p_{q,r}$ holds for $1 \leqslant q' < p' < r \leqslant \infty$, see [20, Proposition A.1].

The first main theorem is as follows.

Theorem 1.3 (Refinement of Stein-Tomas type estimate [20]). *Let $4/3 \leqslant p < \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L^{3p}_{t,x}} \leqslant C \|f\|_{\hat{M}^p_{\frac{3p}{2},2(\frac{3p}{2})'}} \tag{1.3}$$

holds for any $f \in \hat{M}^p_{\frac{3p}{2},2(\frac{3p}{2})'}$.

Since the embedding $\hat{L}^p \hookrightarrow \hat{M}^p_{\frac{3p}{2},2(\frac{3p}{2})'}$ holds for $p > 4/3$, the inequality (1.3) is an improvement of Stein-Tomas estimate (1.2). This kind of refinement is known for the Schrödinger equation [4, 21, 26, 1] and the Airy equation with $p = 2$ [12, 24]. Although the inequality (1.3) holds for $p = 4/3$, the function space $\hat{M}^{4/3}_{2,4}$ is inferior in quality.

We give an outline of the proof of Theorem 1.3. As with the proof of the Stein-Tomas type estimate (1.2), the key point to prove the inequality (1.3) is to reduce the *linear* form into a *bi-linear* form;

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|^2_{L^{3p}_{t,x}} = \left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\xi d\eta \right\|_{L^{\frac{3p}{2}}_{t,x}}.$$

Change of variables and Hausdorff-Young's inequality yield

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|^2_{L^{3p}_{t,x}} \leqslant C \left\{ \int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \right\}^{\frac{1}{(\frac{3p}{2})'}}, \tag{1.4}$$

where

$$m(\xi,\eta) = \frac{|\xi\eta|}{|\xi+\eta|^2|\xi-\eta|^2}.$$

Since the weight function m is singlar on the diagonal line $\eta = \xi$, we introduce a *Whitney type decomposition*. Let $\mathcal{D}_+ = \{\tau^j_k = [k2^{-j}, (k+1)2^{-j}) \mid j \in \mathbb{Z}, 0 \leqslant k \in \mathbb{Z}\}$. For τ^j_k, $\tau^j_\ell \in \mathcal{D}_+$, we define a binary relation

$$\tau^j_k \sim \tau^j_\ell \;\Leftrightarrow\; \begin{cases} \ell - k = -2, 2, 3 & \text{if } k \text{ is even}, \\ \ell - k = -3, -2, 2 & \text{if } k \text{ is odd}. \end{cases}$$

Then, we have the following Whitney-type decomposition of $\mathbb{R}_+ \times \mathbb{R}_+$;

$$\sum_{\tau^j_k \in \mathcal{D}_+} \sum_{\tau^j_\ell : \tau^j_\ell \sim \tau^j_k} \mathbf{1}_{\tau^j_k}(\xi)\mathbf{1}_{\tau^j_\ell}(\eta) = 1, \qquad (\xi,\eta) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(\xi,\xi) \mid \xi \in \mathbb{R}_+\}.$$

Since $|\xi\eta| \leqslant (\xi+\eta)^2/2$ for any $(\xi,\eta) \in \mathbb{R}_+ \times \mathbb{R}_+$, one sees that

$$m(\xi,\eta) \leqslant \frac{1}{2|\xi-\eta|^2} \leqslant \frac{1}{2|\tau^j_k|^2}$$

for any $(\xi,\eta) \in \tau_k^j \times \tau_\ell^j$ with $\tau_k^j \sim \tau_\ell^j$. We hence obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \, d\xi d\eta \\
&= \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} \int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \mathbf{1}_{\tau_k^j}(\xi) \mathbf{1}_{\tau_\ell^j}(\eta) \, d\xi d\eta \\
&\leqslant \sum_{\tau_k^j \in \mathcal{D}_+} \sum_{\tau_\ell^j : \tau_\ell^j \sim \tau_k^j} |\tau_k^j|^{-\frac{2}{3p-2}} \int_{\tau_k^j} |\hat{f}(\xi)|^{(\frac{3p}{2})'} \, d\xi \int_{\tau_\ell^j} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \, d\eta.
\end{aligned}$$

We choose a slightly larger interval containing τ_k^j and τ_ℓ^j but still of length comparable to τ_k^j. More specifically, it is enough to take $\tau_m^{j-3} \in \mathcal{D}_+$ so that $\tau_k^j, \tau_\ell^j \subset \tau_m^{j-3}$. Then we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} \, d\xi d\eta \\
&\qquad \leqslant C \sum_{\tau_k^j \in \mathcal{D}_+} |\tau_k^j|^{-\frac{2}{3p-2}} \left(\int_{\tau_k^j} |\hat{f}(\xi)|^{(\frac{3p}{2})'} \, d\xi \right)^2 \\
&\qquad \leqslant C \sum_{\tau_k^j \in \mathcal{D}_+} |\tau_k^j|^{-\frac{2}{3p-2}} \|\hat{f}\|_{L^{(\frac{3p}{2})'}(\tau_k^j)}^{\frac{6p}{3p-2}} \\
&\qquad = C \|f\|_{\hat{M}^p_{\frac{3p}{2}, 2(\frac{3p}{2})'}}^{2(\frac{3p}{2})'}.
\end{aligned} \tag{1.5}$$

By (1.4) and (1.5), we have (1.3), which completes the proof.

§ 2. Application

As an application of the refined Stein-Tomas estimate (1.3), we prove the existence of a minimal non-scattering solution to the generalized Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u = -\partial_x(|u|^{2\alpha} u) & t, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{gKdV}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given data, and $\alpha > 0$ is a constant. More precisely, in the *mass-subcritical* range $\alpha < 2$, we show existence of a threshold solution to (gKdV) which lies on the boundary of the set of scattering solutions and the set of non-scattering solutions in the framework of the scale critical space $\hat{L}^\alpha$.

Since the nonlinear term is focusing, (gKdV) admits a soliton solution

$$Q_c(t,x) = c^{\frac{1}{\alpha}} Q(c(x - c^2 t)),$$

where $Q(x)$ is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$ and $c > 0$ is a parameter describing amplitude and propagating speed of soliton.

For the mass critical case $\alpha = 2$, it is known that Q is orbitally unstable [15]. Hence on the analogy of the result by Kenig-Merle [9], we expect that Q will lie on the boundary between the scattering and the non-scattering sets. Concerning this, Martel-Merle-Nakanishi-Raphaël [16] classified the dynamics of solution into three cases (blow-up, soliton, away from soliton) in the small neighborhood of Q. On the other hand, for the mass sub-critical case $\alpha < 2$, it is known that Q is stable [2]. Hence, we see that Q does not belong to the boundary between the scattering and the non-scattering sets. Indeed, the conservation of energy and Stein-Tomas estimate (1.2) yield $d_+ < c_\alpha \ell(Q)$ for some $c_\alpha < 1$, where d_+ is a threshold number defined by (2.1) below [1]. As for the mass-subcritical nonlinear Schrödinger equation, the first author [17, 18] treated a minimization problem similar to (gKdV) in a framework of weighted space and showed existence of a threshold solution which is smaller than ground state solutions.

To state the result, we introduce several notations. We say that solution to (gKdV) scatters in $\hat{L}^\alpha$ forward in time if $\lim_{t\to+\infty} e^{t\partial_x^3}u(t)$ exists in $\hat{L}^\alpha$. We define a *forward scattering set* $\mathcal{S}_+$ as follows

$$\mathcal{S}_+ := \left\{ u_0 \in \hat{L}^\alpha \,\middle|\, \begin{array}{l} \text{a solution } u(t) \text{ to (gKdV) with } u_{|t=0} = u_0 \\ \text{scatters in } \hat{L}^\alpha \text{ forward in time} \end{array} \right\}.$$

A *backward scattering set* $\mathcal{S}_-$ is defined in a similar way. Thus, we consider the minimization problem for

$$\begin{aligned} d_+ = d_+(\sigma, M) &:= \inf\{\ell(u_0) \mid u_0 \in B_M \setminus \mathcal{S}_+\}, \\ \ell(u) = \ell_\sigma(u) &:= \inf_{\xi\in\mathbb{R}} \left\| e^{-ix\xi} u \right\|_{\hat{M}^\alpha_{2,\sigma}}, \end{aligned} \tag{2.1}$$

where $M > 0$ is a parameter and $B_M := \{f \in \hat{L}^\alpha \,|\, \|f\|_{\hat{L}^\alpha} \leqslant M\}$ is a ball.

Furthermore, for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t v - \partial_x^2 v = |v|^{2\alpha} v & t, x \in \mathbb{R}, \\ v(0, x) = v_0(x) & x \in \mathbb{R}, \end{cases} \tag{NLS}$$

we define

$$d_{\mathrm{NLS}} = d_{\mathrm{NLS}}(\sigma, M) := \inf\{\ell(v_0) \mid v_0 \in B_M \setminus \mathcal{S}_{\mathrm{NLS}}\} \tag{2.2}$$

with

$$\mathcal{S}_{\mathrm{NLS}} := \left\{ v_0 \in \hat{L}^\alpha \,\middle|\, \begin{array}{l} \text{a solution } v(t) \text{ to (NLS) with } v_{|t=0} = v_0 \text{ scatters} \\ \text{in } \hat{L}^\alpha \text{ forward and backward in time} \end{array} \right\}.$$

Our second main result is as follows.

[1]We prove this assertion in Appendix B.

Theorem 2.1. *Let $3/2+\sqrt{7/60}<\alpha<2$ and $\sigma\in(\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M>0$ so that $B_M\cap\mathcal{S}_+^c\neq\emptyset$. If the assumption*

$$d_+ < 2^{1-\frac{1}{\sigma}}\left(\frac{3\sqrt{\pi}\Gamma(\alpha+2)}{2\Gamma(\alpha+\frac{3}{2})}\right)^{\frac{1}{2\alpha}} d_{\mathrm{NLS}}, \tag{2.3}$$

is true, then there exists a special solution $u_c(t)$ to (gKdV) *with maximal interval $I_{\max}(u_c)\ni 0$ such that*

(i) $u_c(0)\notin\mathcal{S}_+$.

(ii) u_c attains d_+ in such a sense that one of the following two properties holds:

(a) $u_c(0)\in B_M$ and $\ell(u_c(0))=d_+$.

(b) $u_c(0)\in\mathcal{S}_-$ and scatters backward in time to $u_{c,-}$ satisfying $u_{c,-}\in B_M$ and $\ell(u_{c,-})=d_+$.

If we impose the additional regularity on the initial data, we are able to prove the existence of minimal non-scattering solution without the assumption (2.3). For fixed $8/5<\widetilde{\alpha}<\alpha$ and $0<\widetilde{s}<2\alpha+1$, define $\widetilde{B}_M=\{f\in\hat{L}^\alpha \mid \|f\|_{\hat{L}^{\widetilde{\alpha}}}+\|f\|_{\dot{H}^{\widetilde{s}}}\leqslant M\}$. It turns out that, as for a minimizing problem for

$$d'_+ = d'_+(\sigma,M) := \inf\{\ell(u_0)\mid u_0\in\widetilde{B}_M\cap\mathcal{S}_+^c\},$$

a minimizer exists *without* the assumption (2.3).

Theorem 2.2. *Let $3/2+\sqrt{7/60}<\alpha<2$ and $\sigma\in(\alpha', \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M>0$ so that $\widetilde{B}_M\cap\mathcal{S}_+^c\neq\emptyset$. Then, there exists a special solution $\widetilde{u}_c(t)$ to* (gKdV) *which attains d'_+ in a similar way to Theorem 2.1.*

To prove Theorem 2.1, we employ the linear profile decomposition, which is roughly speaking a decomposition of a bounded sequence of functions into a sum of characteristic profiles and a remainder by finding weak limit(s) of the sequence modulo deformations. Intuitively, this decomposition is done by a recursive use of a suitable concentration compactness result. Then, to ensure smallness of remainder as the number of detected profiles increases, a decoupling equality, so-called Pythagorean decomposition, plays a crucial role.

Let us now be more precise on the Pythagorean decomposition. Let $\{f_n\}$ be a bounded sequence of $\hat{L}^\alpha$. Since $\hat{L}^\alpha$ is reflexive as long as $1<\alpha<\infty$, by extracting subsequence, f_n converges to some function $f\in\hat{L}^\alpha$ in weak $\hat{L}^\alpha$ sense. Now we suppose that $f\neq 0$. Then, the Pythagorean decomposition is a decoupling equality of the form

$$\|\hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} = \|\hat{f}\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + \|\hat{f}-\hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + o(1) \tag{2.4}$$

as $n \to \infty$, It is well known that the above decoupling holds for $\alpha = 2$ and may fail for $\alpha \neq 2$. Due to this reason, we prove the linear profile decomposition in a weaker space $\hat{M}^{\alpha}_{2,\sigma}$.

The space $\hat{L}^{\alpha}$-norm is invariant under the following group actions:

- Translation in Physical side: $(T(y)f)(x) := f(x - y)$.
- Translation in Fourier side: $(P(\xi)f)(x) := e^{-ix\xi} f(x)$.
- Airy flow: $(A(t)f)(x) = (e^{-t\partial_x^3} f)(x)$.
- Dilation (scaling): $(D(h)f)(x) = h^{\alpha} f(hx)$.

Hence, we define a set of deformations as follows

$$G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}. \tag{2.5}$$

We often identify $\mathcal{G} \in G$ with a corresponding parameter $\Gamma \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if there is no fear of confusion. Let us now introduce a notion of orthogonality between two families of deformations.

Definition 2.3. We say two families of deformations $\{\mathcal{G}_n\} \subset G$ and $\{\widetilde{\mathcal{G}}_n\} \subset G$ are *orthogonal* if corresponding parameters $\Gamma_n, \widetilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfy

$$\begin{aligned} \lim_{n\to\infty} \Bigg(\left|\log \frac{h_n}{\widetilde{h}_n}\right| + \left|\xi_n - \frac{\widetilde{h}_n}{h_n}\widetilde{\xi}_n\right| + \left|s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right| (1 + |\xi_n|) \\ + \left|y_n - \frac{h_n}{\widetilde{h}_n}\widetilde{y}_n - 3\left(s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right)(\xi_n)^2\right| \Bigg) = +\infty. \end{aligned} \tag{2.6}$$

Theorem 2.4 (Linear profile decomposition for real valued functions)**.** *Let* $4/3 < \alpha < 2$ *and* $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$*. Let* $u = \{u_n\}_n$ *be a sequence of real-valued functions in* B_M*. Then, there exist* $\psi^j \in B_M$*,* $r^j_n \in B_{(2j+1)M}$ *and pairwise orthogonal families of deformations* $\{\mathcal{G}^j_n\}_n \subset G$ *(*$j = 1, 2, \dots$*) parametrized by* $\{\Gamma^j_n = (h^j_n, \xi^j_n, s^j_n, y^j_n)\}_n$ *such that, extracting a subsequence in* n*,*

$$u_n = \sum_{j=1}^{l} \mathrm{Re}(\mathcal{G}^j_n \psi^j) + r^l_n \tag{2.7}$$

for all $l \geqslant 1$ *and*

$$\limsup_{n\to\infty} \left\| e^{-t\partial_x^3} r^l_n \right\|_{L^{3\alpha}_{t,x}} \to 0 \tag{2.8}$$

as $l \to \infty$*. For all* $j \geqslant 1$*,*

$$\textit{either} \quad \xi^j_n = 0, \ \forall n \geqslant 0 \quad \textit{or} \quad \xi^j_n \to \infty \ \textit{as} \ n \to \infty.$$

Moreover, a decoupling inequality

$$\limsup_{n\to\infty} \ell(u_n) \geqslant \left(\sum_{j=1}^{J} c_j^{1-\sigma} \ell(\psi^j)^\sigma\right)^{1/\sigma} + \limsup_{n\to\infty} \ell(r_n^J) \tag{2.9}$$

holds for all $J \geqslant 1$, where

$$c_j = \begin{cases} 1 & \text{if } \xi_n^j \equiv 0, \\ 2 & \text{if } \xi_n^j \to \infty \text{ as } n \to \infty. \end{cases}$$

Furthermore, it holds that

$$c_j \left\|\psi^j\right\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha} \tag{2.10}$$

for any j.

Proof. (Outline) To prove Theorem 2.4, we first prove a decomposition of sequence of *complex-valued* functions. The desired decomposition for real-valued functions then follows as a corollary. As in [5], we split the proof into two parts. The first part is the procedure of finding profiles and obtaining pairwise orthogonality between profiles and remainder term. By employing a successive notion of smallness of remainder term, used in [13, 1, 5], this part can be shown in an abstract way. The second part is concentration compactness. Intuitively, the meaning of the concentration compactness is as follows. Let us consider a bonded sequence $\{u_n\}_n \subset X$. Here, X is a Banach space. In addition to the boundedness with respect to X, we make some additional assumption on the sequence. If the additional assumption is so strong that it removes almost all possible deformations for $\{u_n\}_n$ with few exceptions, say G, then we can find a *non-zero* weak limit modulo G. In our case, $X = \hat{M}^\alpha_{2,\sigma}$ and we use

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_n \right\|_{L^{3\alpha}_{t,x}} \geqslant m$$

as an additional assumption, where m is some positive constant. It will turn out that this assumption removes almost all deformations. The exception is G given in (2.5). This is the reason why we use the set G of deformations in Theorems 2.4. Note that the refined Stein-Tomas estimate (1.3) plays an important rule to show the concentration compactness. See [20, Theorem 4.3] for the detail of the proof. □

The second tool to prove Theorem 2.1 is uniform boundedness of solutions with highly oscillating initial data. The assumption (2.3) is necessary for this boundedness.

Theorem 2.5. *Let $12/7 < \alpha < 2$. Assume (2.3). Let $\phi \in \hat{L}^\alpha_x(\mathbb{R})$ be a complex valued function such that*

$$\ell(\phi) < 2^{1-\frac{1}{\sigma}} d_+.$$

Let $\{\xi_n\}_{n\geqslant 1} \subset (0,\infty)$ with $\xi_n \to \infty$ and let $\{t_n\}_{n\geqslant 1} \subset \mathbb{R}$ be such that $-3t_n\xi_n$ converges to some $T_0 \subset [-\infty,\infty]$. Then for n sufficiently large, a corresponding $\hat{L}^\alpha$-solution u_n to (gKdV) with the initial condition

$$u_n(t_n,x) = A(t_n) Re[P(\xi_n)\phi(x)] \tag{2.11}$$

exists globally in time. Moreover, the solution u_n satisfies a uniform space-time bound

$$\|u_n\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R};L_t^{5\alpha}(\mathbb{R}))} + \|u_n\|_{L^{3\alpha}_{t,x}(\mathbb{R}^2)} \leqslant C, \tag{2.12}$$

where C is a positive constant depending only on ϕ.

Proof. (Outline) We prove existence of a global solution u_n to (gKdV) by constructing approximating solution via the solution to the one dimensional nonlinear Schrödinger equation (NLS). More precisely, let v_n be a solution of (NLS) with initial (or final) condition

$$\begin{cases} v_n(T_0) = P_{|\xi|\leqslant \xi_n^{1/4}} e^{-iT_0\partial_x^2}\phi & \text{if } |T_0| < \infty, \\ \lim\limits_{t\to T_0} \|v_n(t) - P_{|\xi|\leqslant \xi_n^{1/4}} e^{-it\partial_x^2}\phi\|_{\hat{L}^\alpha_x} = 0 & \text{if } T_0 = \pm\infty, \end{cases} \tag{2.13}$$

where $P_{|\xi|\leqslant a} = \mathcal{F}^{-1}\varphi(\xi)\mathcal{F}$ with even bump function φ satisfying $supp\varphi \subset [-a,a]$. As in [14], we introduce an approximate solution $\tilde{u}_n$ to (gKdV):

$$\tilde{u}_n(t,x) := \begin{cases} Re[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)], & \text{if } |t| \leqslant \dfrac{T}{3\xi_n}, \\ e^{-(t-\frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n - \frac{i}{3}T\xi_n^2} v_n(-T, x+\xi_n T)], & \text{if } t > \dfrac{T}{3\xi_n}, \\ e^{-(t+\frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n + \frac{i}{3}T\xi_n^2} v_n(T, x-\xi_n T)], & \text{if } t < -\dfrac{T}{3\xi_n}, \end{cases} \tag{2.14}$$

where T is a large parameter independent of n. By applying the long time stability [20, Proposition 3.2] for $\tilde{u}_n$ with a suitable T, we construct a global solution u_n to (gKdV). For the detail of the proof, see [20, Theorem 4.4]. □

We now give an outline of the proof of Theorem 2.1.

Step 1 Take a minimizing sequence $\{u_n\}_n$ as follows;

$$u_n \in B_M \setminus \mathcal{S}_+, \quad \ell(u_n) \leqslant d_+ + \frac{1}{n}. \tag{2.15}$$

We apply the linear profile decomposition theorem (Theorem 2.4) to the sequence $\{u_n\}_n$. Then, up to subsequence, we obtain a decomposition

$$u_n = \sum_{j=1}^{l} \mathrm{Re}(\mathcal{G}_n^j \psi^j) + r_n^l \tag{2.16}$$

for $n, l \geqslant 1$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^j\}_{n,j} = \{\log h_n^j\}_{n,j}, \{t_n^j\}_{n,j}, \{y_n^j\}_{n,j}, \{3\xi_n^j t_n^j\}$, either $x_n^j \equiv 0$, $x_n^j \to \infty$ as $n \to \infty$, or $x_n^j \to -\infty$ as $n \to \infty$ holds.

Step 2 We show that $\psi^j \equiv 0$ except for at most one j.

Suppose not. Then, by means of (2.9), we have $c_j^{\frac{1}{\sigma}-1} \ell(\psi^j) < d_+$ for all j. Let us define $V_n^j(t,x)$ as follows:

- When $\xi_n \equiv 0$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3 t + t_n^j)$, where $\Psi^j(t)$ is a nonlinear profile associated with $(\mathrm{Re}\,\psi^j, t_n^j)$, that is,
  - if $t_n^j \equiv 0$ then $\Psi^j(t)$ is a solution to (gKdV) with $\Psi^j(0) = \mathrm{Re}\,\psi^j$;
  - if $t_n^j \to \infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters forward in time to $e^{-t\partial_x^3} \mathrm{Re}\,\psi^j$;
  - if $t_n^j \to -\infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters backward in time to $e^{-t\partial_x^3} \mathrm{Re}\,\psi^j$;
- When $\xi_n \to \infty$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi_n^j((h_n^j)^3 t + t_n^j)$, where Ψ_n^j is a solution to (gKdV) with the initial condition

$$\Psi_n^j(t_n^j) = A(t_n^j)\,\mathrm{Re}(P(\xi_n^j)\psi^j).$$

Here, we define an approximate solution

$$\tilde{u}_n^J(t,x) = \sum_{j=1}^{J} V_n^j(t,x) + e^{-t\partial_x^3} r_n^J. \tag{2.17}$$

We apply long time stability [20, Proposition 3.2] for $\tilde{u}_n^J$ defined by (2.17) to see that $\left\|\Psi_n^j\right\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R}_+; L_t^{5\alpha}(\mathbb{R}))} < \infty$ for sufficiently large n. Then, the scattering criterion in $\hat{L}^\alpha$ implies that $u_n \in \mathcal{S}_+$, which contradicts with the definition of $\{u_n\}_n$.

Step 3 We now see that there exists j_0 such that $c_j^{\frac{1}{\sigma}-1} \ell(\psi^{j_0}) = d_+$. Then, one sees from the definition of $\{u_n\}_n$ and (2.9) that $\psi^j \equiv 0$ for $j \neq j_0$. For simplicity, we drop index j_0 and write

$$u_n = \mathcal{G}_n \psi + r_n, \qquad \tilde{u}_n(t) = V_n(t) + e^{-t\partial_x^3} r_n$$

in what follows. Further, we have $\lim_{n\to\infty} \|r_n\|_{\hat{M}^{\alpha}_{2,\sigma}} = 0$.

When $|\xi_n| \to \infty$, as in the previous step, we see from assumption (2.3) and Theorem 2.5 that $u_n \in \mathcal{S}_+$ for large n, a contradiction. Hence, $\xi_n \equiv 0$. Recall that

$$V_n = D(h_n)T(y_n)\Psi((h_n)^3 t + t_n)$$

where $\Psi(t)$ is a nonlinear profile associated with (ψ, t_n). Let us now show that $u_c := \Psi$ is the solution which has the desired property. We have $\Psi(t_n) \notin \mathcal{S}_+$, otherwise $u_n \in \mathcal{S}_+$ for large n by long time stability.

The case $t_n \to \infty$ $(n \to \infty)$ is excluded since this implies $\Psi(t_n) \in \mathcal{S}_+$. If $t_n \equiv 0$ then $\Psi(0) = \psi$ and so $\ell(\Psi(0)) = d_+$. Finally, if $t_n \to -\infty$ as $n \to \infty$ then $\lim_{t\to-\infty} e^{t\partial_x^3}\Psi(t) = \psi$ and putting $u_{c,-} := \lim_{t\to-\infty} e^{t\partial_x^3}\Psi(t)$, we have $\ell(u_{c,-}) = d_+$. This completes the proof of Theorem 2.1.

§ Appendix A. Proof of Stein-Tomas estimate

In this appendix, we show the Stein-Tomas estimate (1.2).

Proof. (of Theorem 1.1.) Following [19, Lemma 2.2], we give a direct proof which is based on the fact that the exponents for space-variable and time-variable in the left hand side coincide. The case $p = \infty$ follows from the Hausdorff-Young inequality. Let $p < \infty$. As explained in the introduction, the key point to prove the inequality (1.2) is to reduce the linear form into a bi-linear form;

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L^{3p}_{t,x}}^2 = \left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\xi d\eta \right\|_{L^{\frac{3p}{2}}_{t,x}}.$$

Changing variables by $a = \xi - \eta$ and $b = \xi^3 - \eta^3$, we have

$$\begin{aligned}\left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\xi d\eta \right\|_{L^{\frac{3p}{2}}_{t,x}} \\ = \left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)} \frac{1}{3|\xi^2-\eta^2|}\, dadb \right\|_{L^{\frac{3p}{2}}_{t,x}}.\end{aligned}$$

We now use the Hausdorff-Young inequality to deduce that

$$\begin{aligned}&\left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)} \frac{1}{3|\xi^2-\eta^2|}\, dadb \right\|_{L^{\frac{3p}{2}}_{t,x}} \\ &\leqslant C \left\| |\xi\eta|^{\frac{1}{3p}} \hat{f}(\xi)\overline{\hat{f}(\eta)} |\xi^2-\eta^2|^{-1} \right\|_{L^{(\frac{3p}{2})'}_{a,b}} \\ &\leqslant C \left\{ \int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \right\}^{\frac{1}{(\frac{3p}{2})'}},\end{aligned}$$

where

$$m(\xi,\eta) = \frac{|\xi\eta|}{|\xi+\eta|^2|\xi-\eta|^2}.$$

Notice that $3p/2 \geqslant 2$. From the elementary inequality $\xi\eta \leqslant (\xi+\eta)^2/2$, we have

$$m(\xi,\eta) \leqslant \frac{1}{2|\xi-\eta|^2}.$$

Hence we obtain

$$\int_0^\infty \int_0^\infty m(\xi,\eta)^{\frac{1}{3p-2}} |\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'} d\xi d\eta \leqslant C \int_0^\infty \int_0^\infty \frac{|\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi-\eta|^{\frac{2}{3p-2}}} d\xi d\eta.$$

By the Hölder and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{|\hat{f}(\xi)|^{(\frac{3p}{2})'} |\hat{f}(\eta)|^{(\frac{3p}{2})'}}{|\xi-\eta|^{\frac{2}{3p-2}}} d\xi d\eta &\leqslant \left\| |\hat{f}|^{\frac{3p}{3p-2}} \right\|_{L^{\frac{3p-2}{3p-3}}} \left\| (|\xi|^{-\frac{2}{3p-2}} * |\hat{f}|^{\frac{3p}{3p-2}}) \right\|_{L^{3p-2}} \\ &\leqslant C \left\| \hat{f} \right\|_{L^{p'}}^{\frac{6p}{3p-2}} = C \left\| f \right\|_{\hat{L}^p_x}^{\frac{6p}{3p-2}}\end{aligned}$$

as long as $2/(3p-2) < 1$, that is, $p > 4/3$. Combining the above inequalities, we obtain the inequality (1.2).

□

§ Appendix B. Upper Bound for d_+

In this appendix, we show the following upper bound for d_+;

$$\text{(B.1)} \qquad d_+ \leqslant c_\alpha \ell(Q), \qquad c_\alpha = \left(\frac{(\alpha+1) \left\| Q' \right\|_{L^2}^2}{\left\| Q \right\|_{L^{2\alpha+2}}^{2\alpha+2}} \right)^{\frac{1}{2\alpha}} < 1.$$

To show (B.1), we give a criteria for scattering in terms of the energy. Here we remark that an $\hat{L}^\alpha$-solution has conserved quantities, provided the solution has appropriate regularity. More precisely, when $u_0 \in \hat{L}^\alpha \cap L^2$, a solution $u(t)$ has a conserved mass

$$M[u(t)] := \left\| u(t) \right\|_{L^2}^2.$$

Similarly, if $u_0 \in \hat{L}^\alpha \cap \dot{H}^1$ then energy

$$E[u(t)] := \frac{1}{2} \left\| \partial_x u(t) \right\|_{L^2}^2 - \frac{1}{2\alpha+2} \left\| u(t) \right\|_{L^{2\alpha+2}}^{2\alpha+2}$$

is invariant.

We note that if an $\hat{L}^\alpha$-solution $u(t)$ scatters (in $\hat{L}^\alpha$ sense) as $t \to \pm\infty$ and if $u_0 \in \hat{L}^{\alpha_0}$ (resp. if $u_0 \in \dot{H}^\sigma$) then $u(t)$ scatters as $t \to \pm\infty$ also in $\hat{L}^{\alpha_0}$ sense (resp. $\dot{H}^\sigma$ sense).

Theorem Appendix B.1 ([19]). *Let $8/5 < \alpha < 10/3$. If $u_0 \in \hat{L}^\alpha \cap H^1$ satisfies $u_0 \neq 0$ and $E[u_0] \leqslant 0$ then $u(t)$ does not scatter as $t \to \pm\infty$.*

The inequality (B.1) easily follows from Theorem Appendix B.1. Indeed, since $u_0 = c_\alpha Q$ satisfies $E[u_0] = 0$, we have $u_0 \notin \mathcal{S}_+$ by Theorem Appendix B.1. Hence we have (B.1).

Proof. We suppose for contradiction that $u(t)$ scatters to $u_+ \in \hat{L}^\alpha$ as $t \to \infty$. Since $u_0 \in H^1$, [11] imply that $u(t) \in C(\mathbb{R}; H^1)$. Further, $u(t)$ scatters also in H^1 and so we see that

$$\|\partial_x u(t)\|_{L^2} = \left\|\partial_x e^{t\partial_x^3} u(t)\right\|_{L^2} \to \|u_+\|_{\dot{H}^1}$$

as $t \to \infty$.

On the other hand, by the Gagliardo-Nirenberg inequality and mass conservation,

$$\|u(t)\|_{L_x^{2\alpha+2}} \leqslant C \|u_0\|_{L_x^2}^{\frac{1}{\alpha+1}} \left\||\partial_x|^{\frac{1}{3\alpha}} u(t)\right\|_{L_x^{3\alpha}}^{\frac{\alpha}{\alpha+1}}.$$

Since $u(t)$ scatters as $t \to \infty$, as in the proof of [19, Theorem 1.9], we can show that $|\partial_x|^{\frac{1}{3\alpha}} u \in L_{t,x}^{3\alpha}$. Therefore, we can take a sequence $\{t_n\}_n$ with $t_n \to \infty$ as $n \to \infty$ so that $\|u(t_n)\|_{L^{2\alpha+2}} \to 0$ as $n \to \infty$. Thus, by conservation of energy,

$$0 \geqslant E[u_0] = E[u(t_n)] = \frac{1}{2}\|\partial_x u(t_n)\|_{L^2}^2 - \frac{1}{2\alpha+2}\|u(t_n)\|_{L^{2\alpha+2}}^{2\alpha+2} \to \frac{1}{2}\|u_+\|_{\dot{H}^1}^2$$

as $n \to \infty$. Hence, $E[u_0] < 0$ yields a contradiction. If $E[u_0] = 0$ then we see that $u_+ = 0$, and so that $\|u_0\|_{L^2} = \|u_+\|_{L^2} = 0$. This contradicts to $u_0 \neq 0$. □

References

[1] Bégout P. and Vargas A., *Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation.* Trans. Amer. Math. Soc. **359** (2007), 5257–5282.

[2] Benjamin T.B. *The stability of solitary waves.* Proc. Roy. Soc. (London) Ser. A **328** (1972), 153–183.

[3] Bourgain J., *On the restriction and multiplier problems in* $\mathbb{R}^3$. Geometric aspects of functional analysis (1989–90), Lecture Notes in Math., **1469**, Springer, Berlin, (1991), 179–191.

[4] Bourgain J., *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity.* Internat. Math. Res. Notices **1998** (1998), 253–283.

[5] Carles R., and Keraani S., *On the role of quadratic oscillations in nonlinear Schrödinger equations II. The L^2-critical case*. Trans. Amer. Math. Soc. **359** (2007), 33–62.

[6] Fefferman C., *Inequalities for strongly singular convolution operators*. Acta Math. **124** (1970) 9–36.

[7] Grünrock A., *An improved local well-posedness result for the modified KdV equation*. Int. Math. Res. Not. **2004** (2004), 3287–3308.

[8] Hyakuna R. and Tsutsumi M., *On existence of global solutions of Schrödinger equations with subcritical nonlinearity for $\hat{L}^p$-initial data*. Proc. Amer. Math. Soc. **140** (2012), 3905–3920.

[9] Kenig C.E. and Merle F., *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*. Invent. Math. **166** (2006), 645–675.

[10] Kenig C.E., Ponce G. and Vega L., *Oscillatory integrals and regularity of dispersive equations*. Indiana Univ.math J. **40** (1991), 33–69.

[11] Kenig C.E., Ponce G. and Vega L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*. Comm. Pure Appl. Math. **46** (1993), 527–620.

[12] Kenig C.E., Ponce G. and Vega L., *On the concentration of blow up solutions for the generalized KdV equation critical in L^2*. Nonlinear wave equations (Providence, RI 1998), Contemp. Math. **263**, Amer. Math. Soc., Providence, RI (2000), 131–156.

[13] Keraani S., *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*. J. Differential Equations **175** (2001), 353–392.

[14] Killip R, Kwon S., Shao S. and Visan M., *On the mass-critical generalized KdV equation*. Discrete Contin. Dyn. Syst. **32** (2012), 191–221.

[15] Martel Y. and Merle F., *Instability of solitons for the critical generalized Korteweg-de Vries equation*. Geom. Funct. Anal. **11** (2001), 74–123.

[16] Martel Y., Merle F., Nakanishi K. and Raphaël P., *Codimension one threshold manifold for the critical gKdV equation*. Comm. Math. Phys. **342** (2016), 1075–1106.

[17] Masaki S., *On minimal non-scattering solution for focusing mass-subcritical nonlinear Schrödinger equation*. preprint available at arXiv:1301.1742 (2013).

[18] Masaki S., *A sharp scattering condition for focusing mass-subcritical nonlinear Schrödinger equation*. Commun. Pure Appl. Anal. **14** (2015), 1481–1531.

[19] Masaki S. and Segata J., *On well-posedness of generalized Korteweg-de Vries equation in scale critical $\hat{L}^r$ space*. Anal. and PDE. **9** (2016), 699–725.

[20] Masaki S. and Segata J., *Existence of a minimal non-scattering solution to the mass-subcritical generalized Korteweg-de Vries equation*, preprint available at arXiv:1602.05331.

[21] Merle F. and Vega L., *Conpactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D*. Internat. Math. Res. Notices **1998** (1998), 399-425.

[22] Moyua, A., Vargas, A., and Vega, L., *Schrödinger maximal function and restriction properties of the Fourier transform*. Internat. Math. Res. Notices **1996** (1996), 793–815.

[23] Moyua, A., Vargas, A., and Vega, L., *Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$*. Duke Math. J. **96** (1999), no. 3, 547–574.

[24] Shao S., *The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality*. Anal. PDE **2** (2009), 83–117.

[25] Tomas P. A., *A restriction theorem for the Fourier transform*. Bull. Amer. Math. Soc. **81** (1975), 477–478.

[26] Vargas A. and Vega L., *Global wellposedness for 1D non-linear Schrödinger equation for*

data with an infinite L^2 norm. J. Math. Pures Appl. (9) **80** (2001), 1029–1044.