*RIMS Kôkyûroku Bessatsu*
**B80** (2020), 27–46

# Stochastic approach to bounds and regularity of fundamental solutions to non-divergence form parabolic equations with irregular coefficients

By

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**Abstract**

This is a digest of a series of the author's works about the stochastic approach to Gaussian two-sided bounds and regularity of fundamental solutions to non-divergence form parabolic equations with irregular coefficients. The argument are decomposed into two parts. One is about the modulus of continuity of the solution to the parabolic equation, and the other is Gaussian two-sided bounds of the fundamental solution. Combining them, we obtain the Gaussian two-sided bounds of the fundamental solution, and the $(1-\varepsilon)$-Hölder continuity and other continuities in a spatial component.

## § 1. Introduction

In [11] the Gaussian two-sided bounds and the $(1-\varepsilon)$-Hölder continuity of the fundamental solution to non-divergence form parabolic equations with irregular coefficients are obtained. Later, it is discovered that the argument can be decomposed into two parts, and simplified. Moreover, the results are improved. One part of the argument is about the modulus of continuity of the solution to the parabolic equation, and is summarized in [13]. The other part is about the Gaussian two-sided bounds of the fundamental solution, and is summarized in [12]. In the present article we digest the works [11, 13, 12] and see the idea of proofs.

Received December 26, 2016. Revised December 28, 2019.
2010 Mathematics Subject Classification(s): 35B65, 35K10, 60H30.
*Key Words*: *parabolic equation*, *non-divergence form*, *fundamental solution*, *two-sided bounds*, *Hölder continuity*, *stochastic analysis*, *stochastic differential equation.*
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Let $a(t,x) = (a_{ij}(t,x))$ be a symmetric $d \times d$-matrix-valued bounded measurable function on $[0,\infty)\times\mathbb{R}^d$, $b(t,x) = (b_i(t,x))$ be an $\mathbb{R}^d$-valued bounded measurable function on $[0,\infty)\times\mathbb{R}^d$, and $c(t,x)$ be a bounded measurable function on $[0,\infty)\times\mathbb{R}^d$. Consider the non-divergence form linear second-order parabolic equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}u(t,x) + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}u(t,x) + c(t,x)u(t,x) \\ u(0,x) = f(x). \end{cases} \tag{1.1}$$

We assume that $a(t,x)$ is uniformly positive definite, i.e.

$$\Lambda^{-1}I \le a(t,x) \le \Lambda I \tag{1.2}$$

where $\Lambda$ is a positive constant and $I$ is the unit matrix, and assume the continuity of $a(t,\cdot)$ uniformly in $t$, i.e. for $R > 0$ there exists a continuous and nondecreasing function $\rho_R$ on $[0,\infty)$ such that $\rho_R(0) = 0$ and

$$\sup_{t\in[0,\infty)} \max_{i,j=1,2,\dots,d} |a_{ij}(t,x) - a_{ij}(t,y)| \le \rho_R(|x-y|), \quad x,y \in B(0;R). \tag{1.3}$$

We remark that in the present article we consider the case of non-regular $a(t,x)$, $b(t,x)$ and $c(t,x)$. By Stroock and Varadhan's result, under (1.2) and (1.3) we have the existence and uniqueness of the mild solution to (1.1) (See Section 2). Now we give the definition of the fundamental solution. Denote

$$L_t f(x) := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}f(x) + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}f(x) + c(t,x)f(x), \quad f \in C_b^2(\mathbb{R}^d).$$

A measurable function $p(s,x;t,y)$ defined for $(s,x),(t,y) \in [0,\infty)\times\mathbb{R}^d$ such that $s < t$ is called a fundamental solution to (1.1), if $p(s,x;t,y)$ satisfies

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} f(y)p(s,\cdot;t,y)dy = L_t\left(\int_{\mathbb{R}^d} f(y)p(s,\cdot;t,y)dy\right), \quad \lim_{r\downarrow s}\int_{\mathbb{R}^d} f(y)p(s,\cdot;r,y)dy = f$$

for $f \in C_b^2(\mathbb{R}^d)$ with a compact support.

We consider the modulus of continuity of $u(t,\cdot)$, and the existence and the Gaussian two-sided bounds of $p(s,x;t,y)$. Because the problem is very classical, there are many known results by analysis. In the case that $a$, $b$ and $c$ are Hölder continuous, parametrix method is the standard way to see the regularity of the solutions and the fundamental solutions (see e.g. [7]), and we have $u(t,\cdot) \in C^2(\mathbb{R}^d)$. Moreover, parametrix enables us to construct the fundamental solution directly, and yields the Gaussian two-sided bounds of the fundamental solution (see also Theorem 19 in [19]). Parametrix still works when $a$, $b$ and $c$ have the Dini-type continuity (see [16]), and also yields $u(t,\cdot) \in C^2(\mathbb{R}^d)$ and the Gaussian two-sided bounds of $p(s,x;t,y)$.

When $a$, $b$ and $c$ are not sufficiently regular for parametrix method, we do not have general theories for them. In particular, the fundamental solution $p(s,x;t,y)$ does not always exist for all $t \in (s,\infty)$ under the assumptions (1.2) and (1.3) (see [5].) However, for fixed $s \in [0,\infty)$ the fundamental solution $p(s,x;t,y)$ exists for almost all $t \in (s,\infty)$ (see Theorem 9.1.9 in [22]).

When all of the coefficients are independent of the time component and $a$ is uniformly continuous, the method of analytic semigroups are available (see [20] and [15]). As the result, we obtain that $u(t,\cdot) \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, dx)$ for $p > d$ (see Theorem 6 in [20]). Hence, by the Sobolev embedding theorem, we have $u(t,\cdot) \in \cap_{\varepsilon \in (0,1]} C^{2-\varepsilon}(\mathbb{R}^d)$. In this case the fundamental solution $p(s,x;t,y)$ exists for all $t \in (s,\infty)$ (see Theorem 9.2.6 in [22].) However, it is not known whether the Gaussian two-sided bounds holds or not.

The equation here is of the non-divergence form, but we also comment on the case of the divergence form equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}(\nabla \cdot a(t,x)\nabla) u(t,x) + b(t,x) \cdot \nabla u(t,x) + c(t,x)u(t,x).$$

The Gaussian two-sided bounds are obtained in the case of divergence form without the continuity of the coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ (see [1], [10] and [21]). If $a(t,x)$ is Lipschitz continuous in $x$, we are able to transform the equations to divergence form equations and obtain the Gaussian two-sided bounds. Moreover, the Hölder continuity of the fundamental solution is obtained (see [2], [4], [17], [18] and [21]). However, if $a(t,x)$ is not Lipschitz continuous in $x$, we are not able to apply the results of divergence form equations to non-divergence form equations. We remark that the Gaussian two-sided bounds of the fundamental solution to non-divergence parabolic equation with bounded measurable coefficients are studied in [3] and an abstract result is obtained.

Now we turn to our results in the present article. Under the assumptions above, by means of the stochastic analysis we obtain the $(1-\varepsilon)$-Hölder continuity of the solution $u(t,x)$ in $x$ (see Section 3). For the existence and the Gaussian two-sided bounds of the fundamental solution, we need more assumptions as mentioned above. So, we prepare the following conditions. Consider a sequence $\{a^{(n)}(t,x)\}$ of $d \times d$-matrix valued smooth functions on $[0,T] \times \mathbb{R}^d$ such that $a^{(n)}(t,x)$ converges to $a(t,x)$ as $n$ goes to infinity for each $(t,x) \in [0,T] \times \mathbb{R}^d$. We assume that the fundamental solution $\check{p}^{(n)}(s,x;t,y)$ to the parabolic equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} a^{(n)}_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} u(t,x) \tag{1.4}$$

exists and satisfies

$$\begin{aligned}(1.5)\qquad \frac{C_-}{(t-s)^{d/2}}\exp\left(-\frac{\gamma_-|x-y|^2}{t-s}\right) &\le \check{p}^{(n)}(s,x;t,y)\\ &\le \frac{C_+}{(t-s)^{d/2}}\exp\left(-\frac{\gamma_+|x-y|^2}{t-s}\right)\end{aligned}$$

for $(s,x),(t,y)\in[0,T]\times\mathbb{R}^d$ such that $s<t$, and $n\in\mathbb{N}$, and

$$(1.6)\qquad \sum_{i,j=1}^{d}\sup_{t\in[0,T]}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}a_{ij}^{(n)}(t,x)\right|^{\theta}e^{-m|x|}dx\le M,\quad n\in\mathbb{N}$$

where $\theta$ is a constant in $[d,\infty)\cap(2,\infty)$, $m$ and $M$ are nonnegative constants.

Under the assumption above, we obtain the improved version of the main theorem in [11]

**Theorem 1.1.** *Assume (1.2), (1.3), (1.5), and (1.6). Then, there exist constants $\tilde{C}_-$, $\tilde{\gamma}_-$, $\tilde{C}_+$ and $\tilde{\gamma}_+$ depending on $T$, $d$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $m$, $M$, $\theta$, $\Lambda$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$\frac{\tilde{C}_-}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\tilde{\gamma}_-|x-y|^2}{t-s}\right)\le p(s,x;t,y)\le\frac{\tilde{C}_+}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\tilde{\gamma}_+|x-y|^2}{t-s}\right)$$

*for $s,t\in[0,T]$ such that $s<t$, and $x,y\in\mathbb{R}^d$. Moreover, the followings hold.*

(i) *For any $R>0$ and sufficiently small $\varepsilon>0$, there exists a constant $C$ depending on $T$, $d$, $\varepsilon$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $m$, $M$, $\theta$, $R$, $\rho_R$, $\Lambda$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;t,y)-p(0,z;t,y)|\le Ct^{-d/2-1}|x-z|^{1-\varepsilon}$$

*for $t\in(0,T]$, $x,z\in B(0;R/2)$ and $y\in\mathbb{R}^d$.*

(ii) *Additionally assume that $\int_0^1 r^{-1}\rho_R(r)dr<\infty$ for $R>0$. Then, for any $R>0$, there exists a constant $C$ depending on $T$, $d$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $m$, $M$, $\theta$, $R$, $\rho_R$, $\Lambda$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;t,y)-p(0,z;t,y)|\le Ct^{-d/2-1}|x-z|\max\{1,-\log|x-z|\}$$

*for $t\in(0,T]$, $x,z\in B(0;R/2)$ and $y\in\mathbb{R}^d$.*

(iii) *Additionally assume that $\rho_R$ appearing in (1.3) can be chosen commonly in $R$ and the common function $\rho$ satisfies $\int_0^1 r^{-1}\rho(r)dr<\infty$. Then, there exists a constant $C$ depending on $T$, $d$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $m$, $M$, $\theta$, $\rho$, $\Lambda$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;t,y)-p(0,z;t,y)|\le Ct^{-d/2-1}|x-z|$$

*for $t\in(0,T]$, $x,z\in B(0;R/2)$ and $y\in\mathbb{R}^d$.*

*Remark.* The additional assumptions (2) and (3) are same as the cases that $a(t,\cdot)$ is locally Dini continuous and Dini continuous uniformly in $t$ in Section 3, respectively.

Theorem 1.1 is obtained by combining Theorems 3.2 and 4.7 below. Theorem 3.2 is the result about the modulus of continuity of the solution, and Theorem 4.7 is the result about the Gaussian two-sided bounds of the fundamental solution. As mentioned in the beginning of this section, we see the ideas of the proofs. Both of the proofs are done by the stochastic analysis. However, the arguments are independent of each other. In Section 2 we see the relation between $u(t,x)$ and the solutions to stochastic differential equations. In Section 3 we consider the modulus of continuity of $u(t,\cdot)$. This part is based on the argument in [13]. In Section 4 we consider the Gaussian two-sided bounds of $p(s,x;t,y)$.

We regard that all the random variables appearing in the article are on a fixed probability space $(\Omega,\mathcal{F},P)$ and we denote $\min\{s,t\}$ by $s\wedge t$ for $s,t\in\mathbb{R}$.

## § 2. Probabilistic representation of the solutions

In this section, we prepare a probabilistic representation of the solution to (1.1). Define $d\times d$-matrix-valued function $\sigma(t,x)$ by the square root of $a(t,x)$. Let $T>0$. Consider the stochastic differential equations:

$$\begin{cases} dX_t^{(s,x)} = \sigma(T-t, X_t^{(s,x)})dB_t, & t\in[s,T] \\ X_s^{(s,x)} = x. \end{cases} \tag{2.1}$$

$$\begin{cases} dY_t^{(s,x)} = \sigma(T-t, Y_t^{(s,x)})dB_t + b(T-t, Y_t^{(s,x)})dt, & t\in[s,T] \\ Y_s^{(s,x)} = x. \end{cases} \tag{2.2}$$

We simply denote $X_t^{(0,x)}$ and $Y_t^{(0,x)}$ by $X_t^x$ and $Y_t^x$, respectively. The existence and the uniqueness of the solution to (2.1) and (2.2) are obtained under the assumptions (1.2) and (1.3) by Stroock and Varadhan (see [22]). Denote the transition probability measure of $X$ and $Y$ by $p^X(s,x;t,dy)$ and $p^Y(s,x;t,dy)$, respectively.

We remark that $X$, $Y$, $p^X$ and $p^Y$ are depending on $T$. Consider the following backward parabolic equation on $[0,T-s]\times\mathbb{R}^d$

$$\begin{cases} -\dfrac{\partial}{\partial t}v^s(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^d a_{ij}(T-t,x)\dfrac{\partial^2}{\partial x_i\partial x_j}v^s(t,x) \\ \qquad\qquad\qquad + \displaystyle\sum_{i=1}^d b_i(T-t,x)\dfrac{\partial}{\partial x_i}v^s(t,x) + c(T-t,x)v^s(t,x) \\ v^s(T-s,x) = f(x). \end{cases} \tag{2.3}$$

Then, we have the one-to-one correspondence between the solution $u$ to (1.1) and the solution $v$ to (2.3) by $u(t,x)=v^0(T-t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^d$. By the Feynman-Kac formula (see e.g. Theorem 7.6 of Chapter 5 in [9]), we have the following representation of $v^s(t,x)$ by $(Y_t^x)$.

$$v^s(t,x) = E\left[f(Y_{T-s}^{(t,x)})\exp\left(\int_t^{T-s} c(T-w, Y_w^{(t,x)})dw\right)\right], \quad t \in [0, T-s]. \tag{2.4}$$

In particular,

$$u(T,x) = v^0(0,x) = E\left[f(Y_T^x)\exp\left(\int_0^T c(T-s, Y_s^x)ds\right)\right]. \tag{2.5}$$

By (2.5) the solution to (1.1) is represented by the solution to (2.2). The Feynman-Kac formula which we applied above is a mathematically rigorous version of the path-integral formulation introduced by Feynman in [6], and actually the integral in time is in the right-hand side of (2.5). It is also possible to represent $u(T,x)$ by $X$ as follows. Applying the Girsanov transformation (see e.g. Theorem 4.2 of Chapter IV in [8]), we have

$$\begin{aligned} u(T,x) = E\Bigg[f(X_T^x)\exp\Bigg(&\int_0^T \langle b_\sigma(T-s, X_s^x), dB_s\rangle \\ &-\frac{1}{2}\int_0^T |b_\sigma(T-s, X_s^x)|^2 ds + \int_0^T c(T-s, X_s^x)ds\Bigg)\Bigg] \end{aligned} \tag{2.6}$$

where $b_\sigma(t,x) := \sigma(t,x)^{-1}b(t,x)$. The integral in time is also in the right-hand side of (2.6). In Sections 3 and 4, we obtain the results of the regularity of the solution and the fundamental solution, and the Gaussian two-sided bounds by stochastic approach via the solutions to (2.1) and (2.2).

Now, letting $c = 0$ we see the relation between the fundamental solution to (1.1) and the transition probability density function of $Y$. When $c = 0$, then (2.5) and the definitions of the fundamental solution and the transition probability density function of $Y$, we have

$$\int_{\mathbb{R}^d} f(y)p(0,x;T,y)dy = \int_{\mathbb{R}^d} f(y)p^Y(0,x;T,dy).$$

The argument above is available even if the initial condition is given at time $s$, i.e. $u(0,x) = f(x)$ in (1.1) is replaced by $u(s,x) = f(x)$. Hence, if the fundamental solution to (1.1) exists and has the uniqueness, we have

$$p(s,x;T,y) = p^Y(s,x;T,y), \quad s \in [0,T),\ x,y \in \mathbb{R}^d. \tag{2.7}$$

Furthermore, in the case that $c = 0$ and $p^Y(s,x;t,y)$ is sufficiently regular, by (2.3) and

(2.4) we have

$$
\begin{aligned}
&-\frac{\partial}{\partial \tau} p^Y(\tau, x; T-s, y)\Big|_{\tau=T-t} \\
&= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(T-t,x)\frac{\partial^2}{\partial x_i \partial x_j} p^Y(T-t,x;T-s,y) \\
&\quad + \sum_{i=1}^{d} b_i(T-t,x)\frac{\partial}{\partial x_i} p^Y(T-t,x;T-s,y)
\end{aligned}
\tag{2.8}
$$

for $s, t \in [0, T]$ such that $s < t$, and $x, y \in \mathbb{R}^d$.

## § 3. Modulus of continuity

In this section, we see the results in [13] and the sketch of the proof. Before that, we introduce some classes of functions.

**Definition 3.1.** Let $f$ be a function on $\mathbb{R}^d$. If for $R > 0$ there exists a continuous and nondecreasing function $\rho_R$ on $[0, \infty)$ such that $\rho_R(0) = 0$, $\int_0^1 r^{-1}\rho_R(r)dr < \infty$ and

$$|f(x) - f(y)| \le \rho_R(|x - y|), \quad x, y \in B(0; R),$$

then $f$ is called locally Dini continuous. If $\rho_R$ is able to be chosen commonly in $R$, then $f$ is called Dini continuous.

It is easy to see that a Hölder continuous function and a locally Hölder continuous function are Dini continuous and locally Dini continuous, respectively. It is also easy to see that; for $\alpha \in (1, \infty)$ a function $f$ on $\mathbb{R}^d$ which satisfies

$$|f(x) - f(y)| \le C \min\{1, (-\log|x - y|)^{\alpha}\}, \quad x, y \in \mathbb{R}^d$$

with a positive constant $C$, is Dini continuous. Hence, the class of the Dini continuous functions is larger than the class of the Hölder continuous functions. Let $f$ be a function on $[0, \infty) \times \mathbb{R}^d$. If for $R > 0$ there exists a continuous and nondecreasing function $\rho_R$ on $[0, \infty)$ such that $\rho_R(0) = 0$ and

$$|f(t, x) - f(t, y)| \le \rho_R(|x - y|), \quad x, y \in B(0; R),$$

we call $f(t, \cdot)$ is continuous uniformly in $t$. If a function $f$ on $[0, \infty) \times \mathbb{R}^d$ such that $f(t, \cdot)$ is locally Dini continuous and the function $\rho_R$ appeared in Definition 3.1 can be chosen independently of $t$, then we call $f(t, \cdot)$ is locally Dini continuous uniformly in $t$. Similarly, we define a function Dini continuous uniformly in $t$. For a matrix-valued function $f(t, x) = (f_{ij}(t, x))$ on $[0, \infty) \times \mathbb{R}^d$, we say that $f$ is locally Dini continuous uniformly and Dini continuous function uniformly in $t$ if all components of $f$ are locally Dini continuous uniformly and Dini continuous function uniformly in $t$, respectively. In the present article, once we assume that $a(t,\cdot)$ is locally Dini continuous uniformly in $t$, then we regard that $\rho_R$ in (1.3) satisfies $\int_0^1 r^{-1}\rho_R(r)dr<\infty$. Furthermore, once we assume that $a(t,\cdot)$ is Dini continuous uniformly in $t$, then we regard that $\rho$ is the function satisfies (1.3) for all $R>0$ and $\int_0^1 r^{-1}\rho(r)dr<\infty$.

Now we describe the results in this section.

**Theorem 3.2.** *The followings hold.*

(i) *Assume (1.2) and continuity of $a(t,\cdot)$ uniformly in $t$. Then, for any $p\in[1,\infty)$, $R>0$ and sufficiently small $\varepsilon>0$, there exists a constant $C$ depending on $d$, $\Lambda$, $\varepsilon$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$\begin{aligned}|u(T,x)-u(T,z)|\le{}& Ct^{-(1+1/p)}Te^{CT}|x-z|^{(1-\varepsilon)/p}\\ &\times\max_{\eta=x,z}\left\|\int_{\mathbb{R}^d}|f(y)|p^Y(t,\cdot;T,dy)\right\|_{L^{p^*}(\mathbb{R}^d,p^Y(0,\eta;t,\cdot))}\end{aligned}$$

*for $f\in C_b(\mathbb{R}^d)$, $t\in(0,T)$, and $x,z\in B(0;R/2)$.*

(ii) *Assume (1.2) and local Dini continuity of $a(t,\cdot)$ uniformly in $t$. Then, for any $p\in[1,\infty)$ and $R>0$, there exists a constant $C$ depending on $d$, $\Lambda$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$\begin{aligned}|u(T,x)-u(T,z)|\le{}& Ct^{-(1+1/p)}Te^{CT}(|x-z|\max\{1,-\log|x-z|\})^{1/p}\\ &\times\max_{\eta=x,z}\left\|\int_{\mathbb{R}^d}|f(y)|p^Y(t,\cdot;T,dy)\right\|_{L^{p^*}(\mathbb{R}^d,p^Y(0,\eta;t,\cdot))}\end{aligned}$$

*for $f\in C_b(\mathbb{R}^d)$, $t\in(0,T)$, and $x,z\in B(0;R/2)$.*

(iii) *Assume (1.2) and Dini continuity of $a(t,\cdot)$ uniformly in $t$. Then, for any $p\in[1,\infty)$, there exists a constant $C$ depending on $d$, $\Lambda$, $\rho$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$\begin{aligned}|u(T,x)-u(T,z)|\le{}& Ct^{-(1+1/p)}Te^{CT}|x-z|^{1/p}\\ &\times\max_{\eta=x,z}\left\|\int_{\mathbb{R}^d}|f(y)|p^Y(t,\cdot;T,dy)\right\|_{L^{p^*}(\mathbb{R}^d,p^Y(0,\eta;t,\cdot))}\end{aligned}$$

*for $f\in C_b(\mathbb{R}^d)$, $t\in(0,T)$, and $x,z\in\mathbb{R}^d$.*

As mentioned in Section 1, under the assumptions in Theorem 3.2, Stroock and Varadhan obtained the existence of the transition probability density function $p^Y(0,x;t,y)$ for almost every $t$ (see [22]). On the other hand, even if $a$ is uniformly continuous, $b=0$,

$c = 0$, there is an example that the fundamental solution does not exist for a certain $t$ (see [5]). This example implies that if $a$ depends on $t$, the transition probability measure $p^Y(0,x;t,dy)$ is not always absolutely continuous with respect to the Lebesgue measure. This is the reason why the upper bounds appearing in Theorem 3.2 are complicated. However, when we know the existence of the transition probability density function $p^Y(0,x;t,y)$ and the upper bound of the $p^Y(0,x;t,y)$, we obtain simpler modulus of continuity as follows.

**Corollary 3.3.** *Assume (1.2) and that $a(t,\cdot)$ is continuous uniformly in $t$. Moreover, we assume that for each $x \in \mathbb{R}^d$, $p^X(T/2,x;T,\cdot)$ is absolutely continuous with respect to the Lebesgue measure and there exists a constant $\nu(T)$ such that*

$$\sup_{x\in\mathbb{R}^d} \frac{p^X(T/2,x;T,dy)}{dy} \le \nu(T).$$

*Then, there exists a measurable function $p(0,x;T,y)$ $x,y \in \mathbb{R}^d$ which satisfies the definition of the fundamental solution to (1.1) under the restriction that $s = 0$ and $t = T$, and the followings hold.*

(i) *For $R > 0$ and sufficiently small $\varepsilon > 0$, there exists a constant $C$ depending on $d$, $\Lambda$, $\varepsilon$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;T,y) - p(0,z;T,y)| \le CT^{-1}\nu(T)e^{CT}|x-z|^{1-\varepsilon}$$

*for almost every $y \in \mathbb{R}^d$ with respect to the Lebesgue measure, and $x,z \in B(0;R/2)$.*

(ii) *Additionally assume that $a(t,\cdot)$ is locally Dini continuous uniformly in $t$. Then, for $R > 0$, there exists a constant $C$ depending on $d$, $\Lambda$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;T,y) - p(0,z;T,y)| \le CT^{-1}\nu(T)e^{CT}|x-z|\max\{1, -\log|x-z|\}$$

*for almost every $y \in \mathbb{R}^d$ with respect to the Lebesgue measure, and $x,z \in B(0;R/2)$.*

(iii) *Additionally assume that $a(t,\cdot)$ is Dini continuous uniformly in $t$. Then, there exists a constant $C$ depending on $d$, $\Lambda$, $\rho$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;T,y) - p(0,z;T,y)| \le CT^{-1}\nu(T)e^{CT}|x-z|$$

*for almost every $y \in \mathbb{R}^d$ with respect to the Lebesgue measure, and $x,z \in \mathbb{R}^d$.*

*Remark.* If $p^X$ has the upper Gaussian bound, then we are able to choose $\nu(t)$ by $Ct^{-d/2}$. This is the typical choice of $\nu(t)$.

*Remark.* The Lipschitz continuity obtained in Corollary 3.3 (iii) is the optimal. Consider the case that $d = 1$, $a(x) = 1$, and $b(x) := -x/|x|$ for $x \neq 0$ and $b(0) := 0$. Then, the fundamental solution $p(0, x; t, y)$ of this equation is obtained explicitly, and we can see that $u(t, \cdot) \notin C^1(\mathbb{R})$ (see Remark 5.2 of Chapter 6 in [9]).

Stroock and Varadhan deeply studied the properties of the transition density functions of the solutions to stochastic differential equations with low regular coefficients in [22]. As one of their results, it is known that when the coefficients do not depend on time, then the transition probability density function exists for all time and the transition probability density function is in $L^p$ for all $p \in [1, \infty)$ (see Corollary 9.2.7 in [22]). Applying this result to Theorem 3.2, we can remove $p^X$ from the upper estimates and obtain the clearer modulus of continuity of the solutions to (1.1), as follows.

**Corollary 3.4.** *Assume that $a$ and $b$ do not depend on $t$. Assume (1.2) and that $a$ is continuous, and let $u$ be the solution of (1.1).*

(i) *For any $p \in (1, \infty]$, $R > 0$, $\tilde{T} > 0$, and sufficiently small $\varepsilon > 0$, there exist constants $\alpha$ depending on $d$ and $p$, and $C$ depending on $\tilde{T}$, $d$, $p$, $\Lambda$, $\varepsilon$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|u(T, x) - u(T, z)| \leq C T^{-\alpha} |x - z|^{1-\varepsilon} \|f\|_{L^p(\mathbb{R}^d)}$$

*for $f \in L^p(\mathbb{R}^d)$ such that $\mathrm{supp} f \subset B(0; R/2)$, $T \in (0, \tilde{T}]$ and $x, z \in B(0; R/2)$.*

(ii) *Additionally assume that $a$ is locally Dini continuous. Then, for any $p \in (1, \infty]$, $R > 0$ and $\tilde{T} > 0$, there exist constants $\alpha$ depending on $d$ and $p$, and $C$ depending on $\tilde{T}$, $d$, $p$, $\Lambda$, $R$, $\rho_R$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|u(T, x) - u(T, z)| \leq C T^{-\alpha} |x - z| \max\{1, -\log |x - z|\} \|f\|_{L^p(\mathbb{R}^d)}$$

*for $f \in L^p(\mathbb{R}^d)$ such that $\mathrm{supp} f \subset B(0; R/2)$, $T \in (0, \tilde{T}]$ and $x, z \in B(0; R/2)$.*

(iii) *Additionally assume that $a$ is Dini continuous. Then, for any $p \in (1, \infty]$ and $\tilde{T} > 0$, there exist constants $\alpha$ depending on $d$ and $p$, and $C$ depending on $\tilde{T}$, $d$, $p$, $\Lambda$, $\rho$, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|u(T, x) - u(T, z)| \leq C T^{-\alpha} |x - z| \|f\|_{L^p(\mathbb{R}^d)}$$

*for $f \in L^p(\mathbb{R}^d)$, $T \in (0, \tilde{T}]$ and $x, z \in \mathbb{R}^d$.*

*Remark.* In Corollary 3.4 $c$ can depend on the time component. If $c$ is also independent of the time component, then the theory of analytic semigroup is applicable and a better result is obtained (see Section 3.1.1 in [15]).

Now we see a sketch of the proof of Theorem 3.2. The proof is purely stochastic. In particular, we use the representation (2.5) and the coupling method introduced by Lindvall and Rogers (see [14]).

We assume that $a_{ij}(t,x) \in C_b^\infty([0,T]\times\mathbb{R}^d)$, $b_i(t,x) \in C_b^\infty([0,T]\times\mathbb{R}^d)$ and will obtain bounds with constants depending on suitable factors. Consider (2.1) with $\sigma(t,x)$ which is the square-root of $a(t,x)$ in the sense of symmetric positively definite matrices. Then, $\sigma_{ij}(t,x) \in C_b^\infty([0,T]\times\mathbb{R}^d)$, $a(t,x) = \sigma(t,x)\sigma(t,x)^T$,

$$\sup_{t\in[0,\infty)} \sup_{i,j} |\sigma_{ij}(t,x) - \sigma_{ij}(t,y)| \le C\rho_R(|x-y|), \quad x,y \in B(0;R)$$

$$\Lambda^{-1/2} I \le \sigma(t,x) \le \Lambda^{1/2} I, \quad (t,x)\in[0,T]\times\mathbb{R}^d$$

where $C$ is a constant depending on $\Lambda$. In the argument here, $\sigma$ and $b$ are Lipschitz continuous. Hence, the solution $Y^x$ to (2.2) has the pathwise uniqueness. Moreover, since $\sigma$ and $b$ are smooth, the transition probability density function $p^Y(s,x;t,y)$ exists and is smooth. In view of (2.5), to see the modulus of continuity of $u(T,\cdot)$, it is sufficient to observe the modulus of continuity of the function: $x \mapsto E\left[f(Y_T^x)\exp\left(\int_0^T c(T-s,Y_s^x)ds\right)\right]$.

Now we use the coupling method introduced by Lindvall and Rogers [14] and consider estimates of the oscillation of $u(T,\cdot)$. Let $Y^x$ and the Brownian motion $B$ appearing in (2.2) be given, and define the stochastic process $(Z_t^z)$ by the solution to

$$\begin{cases} Z_t^z = z + \displaystyle\int_0^{t\wedge\tau} \sigma(T-s,Z_s^z)d\tilde{B}_s + \int_{t\wedge\tau}^t \sigma(T-s,Z_s^z)dB_s + \int_0^t b(T-s,Z_s^z)ds \\ \tilde{B}_t = \displaystyle\int_0^{t\wedge\tau} H_s dB_s \end{cases} \tag{3.1}$$

where

$$H_t := I - \frac{2\left[\sigma(T-t,Z_t^z)^{-1}(Y_t^x - Z_t^z)\right]\otimes\left[\sigma(T-t,Z_t^z)^{-1}(Y_t^x - Z_t^z)\right]}{|\sigma(T-t,Z_t^z)^{-1}(Y_t^x - Z_t^z)|^2}, \quad t\in[0,\tau\wedge T)$$

and $\tau$ is a stopping time defined by $\tau := \inf\{t\ge 0; Y_t^x = Z_t^z\}$. We remark that the Lipschitz continuity of $\sigma$ and $b$ implies the pathwise uniqueness of $(Z^z(t),\tilde{B}(t); t\in[0,\tau\wedge T))$. The diffusion coefficient matrix $H_t$ of $\tilde{B}$ is orthogonal for $t\in[0,\tau\wedge T)$, and hence, $\tilde{B}_t$ is also a $d$-dimensional Brownian motion for $t\in[0,\tau\wedge T)$. From this fact we are able to extend the solution to time $\tau$ and obtain $(Z^z(t); t\in[0,T])$. Since $(\tilde{B}_t,\ t\in[0,\tau\wedge T))$ is a $d$-dimensional Brownian motion, $Y^z$ and $Z^z$ have the same law, and $Y_t^x = Z_t^z$ for $t\in[\tau,T]$ almost surely. The argument here is available if $H_t$ is orthogonal-matrix valued. The key point is the choice of the way to construct $H_t$. Above, $H_t$ was chosen so that $dB_t$ and $d\tilde{B}_t$ are symmetric with respect to the hyperplane consisting of the points which have same distance from $Y_t^x$ and $Z_t^z$. By the choice, $Y^x$ and $Z^z$ tend to coalesce, and hence, $\tau$ will be as small as possible. Therefore, under a suitable condition, when $|x-z|$ goes to 0, $\tau$ converges to 0.

Since $Y^x$ and $Z^z$ go on the same trajectory after time $\tau$, from (2.5) we have for $s \in [0,T]$

$$\begin{aligned}
&u(T,x) - u(T,z) \\
&= E\left[ (f(Y_T^x) - f(Z_T^z)) \exp\left( \int_0^T c(T-s, Y_s^x) ds \right); \tau > s \right] \\
&\quad + E\left[ f(Z_T^z) \exp\left( \int_{\tau\wedge T}^T c(T-s, Y_s^x) ds \right) \right. \\
&\qquad\qquad \left. \times \left( \exp\left( \int_0^\tau c(T-s, Y_s^x) ds \right) - \exp\left( \int_0^\tau c(T-s, Z_s^z) ds \right) \right) \right].
\end{aligned}$$

Thus, we obtain the following proposition.

**Proposition 3.5.** *Let $p \in (1,\infty)$. Then, for $f \in C_b(\mathbb{R}^d)$, $s \in (0,T)$, and $x, z \in \mathbb{R}^d$, it holds that*

$$\begin{aligned}
&|u(T,x) - u(T,z)| \\
&\le 2(s+T)(\|c\|_\infty + s^{-1}) s^{-1/p} e^{2\|c\|_\infty T} E\left[s \wedge \tau\right]^{1/p} \\
&\qquad \times \max_{\eta = x,z} E\left[ \left( \int_{\mathbb{R}^d} |f(y)| p^Y\left(s, Y_s^\eta; T, y\right) dy \right)^{p^*} \right]^{1/p^*}.
\end{aligned}$$

Proposition 3.5 implies that the modulus of continuity of $u(T,\cdot)$ is dominated by $E\left[s\wedge\tau\right]$. Hence, if $\tau$ converges to 0 as $|x-z|$ goes to 0, then the modulus of continuity of $u(T,\cdot)$ is obtained from the oscillation of the associated diffusion process. This is the idea of the proof. Next we consider the rate of the convergence of $E\left[s\wedge\tau\right]$ as $|x-z|$ goes to 0.

**Lemma 3.6.** *For $R > 0$ and sufficiently small $\varepsilon > 0$, there exists a positive constant $C$ depending on $d$, $\Lambda$, $\varepsilon$, $R$, $\rho_R$ and $\|b\|_\infty$ such that*

$$E[t\wedge\tau] \le C(1+T^2)|x-z|^{1-\varepsilon}, \quad t \in [0,T],\ x,z \in B(0;R/2). \tag{3.2}$$

The sketch of the proof is as follows. Let

$$\xi_t := Y_t^x - Z_t^z,\ \alpha_t := \sigma(T-t, Y_t^x) - \sigma(T-t, Z_t^z) H_t,\ \beta_t := b(T-t, Y_t^x) - b(T-t, Z_t^z).$$

Then, by Itô's formula we have for $t \in [0,\tau)$

$$d(|\xi_t|) = \left\langle \frac{\xi_t}{|\xi_t|}, \alpha_t dB_t \right\rangle + \frac{1}{2|\xi_t|}\left( \mathrm{tr}(\alpha_t \alpha_t^T) - \frac{|\alpha_t^T \xi_t|^2}{|\xi_t|^2} + \langle \xi_t, \beta_t \rangle \right) dt. \tag{3.3}$$

By a similar calculation to that in [14], we have a positive constant $\gamma$ depending on $d$, $\Lambda$ and $\|b\|_\infty$ such that

$$\begin{aligned} \left|\operatorname{tr}(\alpha_t\alpha_t^T) - \frac{|\alpha_t^T\xi_t|^2}{|\xi_t|^2} + \langle \xi_t, \beta_t\rangle\right| &\le \gamma\rho_R(|\xi_t|), t\in[0,\tau) \text{ s.t. } X_t^x, Z_t^z \in B(0;R), \\ \frac{|\alpha_t^T\xi_t|}{|\xi_t|} \ge \gamma^{-1}, &\qquad\qquad t\in[0,\tau) \text{ s.t. } |\sigma(t,X_t^x)-\sigma(t,Z_t^z)| \le 2\Lambda^{-1}. \end{aligned}$$

By the fact that $\rho(|\xi_t|)$ converges to 0 as $|\xi_t|$ goes to 0 and these inequalities, the comparison of (3.3) with the stochastic differential equations of the Bessel processes lets us to expect that $|\xi_t|$ is able to reach 0 until $Y_t^x$ and $Z_t^z$ locate sufficiently close positions. Hence, we hope that $E[s\wedge\tau]$ goes to 0 with the order $|x-z|$. This is an intuitive expectation in view of the theory of one-dimensional diffusion processes. What we treat here is a radius component of multi-dimensional diffusion processes, and hence the calculation is not completely same as in the case of one-dimensional diffusion processes. So, as the result, we obtain a less order $|x-z|^{1-\varepsilon}$ of the convergence as in Lemma 3.6.

By Proposition 3.5 and Lemma 3.6 we have the estimate in Theorem 3.2 (i) with constants which depend on the suitable factors. The case of non-smooth $a$ and $b$ is proved by approximation of smooth coefficient and applying Theorem 11.3.4 in [22].

When $a(t,\cdot)$ is locally Dini continuous uniformly in $t$, the scale function with a base point 0 of the one-dimensional diffusion process which appears in the comparison to $\xi_t$

$$f(\eta) = \int_0^\eta \exp\left(-\int_0^{\theta_1} \frac{2\gamma^3\rho_R(\theta_2)}{\theta_2} d\theta_2\right) d\theta_1, \quad \eta\in[0,\infty)$$

exists. By using this function we obtain that $E[s\wedge\tau]$ converges to 0 with the order $|x-z|\max\{1,-\log|x-z|\}$. So, Lemma 3.6 is improved and obtain the desired estimate. The Dini continuous case is similar.

The argument above is a sketch of the proof of Theorem 3.2.

## § 4. Two-sided bounds

In [12] Gaussian two-sided bounds and the regularity of the transition probability density of solutions to stochastic differential equations are concerned. Recall that the issue here is the case that coefficients have low regularities and that stochastic differential equations are associated to non-divergence form parabolic partial differential equations as we have seen in Section 2. While the concerned stochastic differential equation in [12] is path-dependent one, to focus on the application to partial differential equations we deduce the argument in [12] to stochastic differential equations of Markov type. Furthermore, some settings are simplified compared with [12].

Consider the stochastic differential equations

$$\begin{cases} dX_t^x = \sigma(t, X_t^x) dB_t, \quad t \in [0,T] \\ X_0^x = x, \end{cases} \tag{4.1}$$

$$\begin{cases} dY_t^x = \sigma(t, Y_t^x) dB_t + b(t, Y_t^x) dt, \quad t \in [0,T] \\ Y_0^x = x, \end{cases} \tag{4.2}$$

where $\sigma(t,x) = (\sigma_{ij}(t,x))$ be a $d \times d$-matrix-valued bounded measurable function on $[0,\infty) \times \mathbb{R}^d$, $b(t,x) = (b_i(t,x))$ be an $\mathbb{R}^d$-valued bounded measurable function on $[0,T] \times \mathbb{R}^d$. Denote the transition probability measure of $X$ and $Y$ by $p^X(s,x;t,dy)$ and $p^Y(s,x;t,dy)$, respectively. Let $a(t,x) := \sigma(t,x)\sigma(t,x)^T$. We remark that (4.1) and (4.2) are different from (2.1) and (2.2) by the time reversal.

First, we concern a sufficiently good case. We assume (1.2) and that there exists a continuous increasing function $M_a$ on $[0,\infty)$ such that $M_a(0) = 0$,

$$\sup_{t \in [0,\infty)} |a(t,x) - a(t,y)| \le M_a(|x-y|), \qquad \int_0^1 \frac{1}{\tilde{r}} \left( \int_0^{\tilde{r}} \frac{1}{r} M_a(r) dr \right) d\tilde{r} < \infty. \tag{4.3}$$

Then, for given $\tau \in (0,T]$ we are able to apply the parametrix method to the partial differential equation

$$\frac{\partial}{\partial t} \check{u}(t,x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(\tau - t, x) \frac{\partial^2}{\partial x_i \partial x_j} \check{u}(t,x), \quad t \in [0,\tau] \tag{4.4}$$

(see [16]) and obtain the existence and uniqueness of the fundamental solution $q^\tau(s,x;t,y)$ to (4.4) which satisfies $q^\tau(s,x;\cdot,y) \in C^1((s,\tau))$, $q^\tau(s,\cdot;t,y) \in C_b^2(\mathbb{R}^d)$. Moreover, by [16] and [19], we have some positive constants $C_-$, $\gamma_-$, $C_+$ and $\gamma_+$ depending on $T$, $d$, $\Lambda$ and $M_a$ such that

$$\frac{C_-}{(t-s)^{d/2}} \exp\left( -\frac{\gamma_-|x-y|^2}{t-s} \right) \le q^\tau(s,x;t,y) \le \frac{C_+}{(t-s)^{d/2}} \exp\left( -\frac{\gamma_+|x-y|^2}{t-s} \right),$$

$$|\nabla_x q^\tau(s,x;t,y)| \le \frac{C_+}{(t-s)^{(d+1)/2}} \exp\left( -\frac{\gamma_+|x-y|^2}{t-s} \right)$$

for $(s,x), (t,y) \in [0,\tau] \times \mathbb{R}^d$ such that $s < t$. From (2.7) and the fact that inequalities above hold for any $\tau \in (0,T]$ with constants independent of $\tau$, we obtain

$$\begin{aligned} \frac{C_-}{(t-s)^{d/2}} \exp\left( -\frac{\gamma_-|x-y|^2}{t-s} \right) &\le p^X(s,x;t,y) \\ &\le \frac{C_+}{(t-s)^{d/2}} \exp\left( -\frac{\gamma_+|x-y|^2}{t-s} \right), \end{aligned} \tag{4.5}$$

$$|\nabla_x p^X(s,x;t,y)| \le \frac{C_+}{(t-s)^{(d+1)/2}} \exp\left( -\frac{\gamma_+|x-y|^2}{t-s} \right) \tag{4.6}$$

for $(s,x),(t,y)\in[0,T]\times\mathbb{R}^d$ such that $s<t$. Similarly to (2.8), noting that we did not reverse time now, we have

$$\frac{\partial}{\partial s}p^X(s,x;t,y)+\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(s,x)\frac{\partial^2}{\partial x_i\partial x_j}p^X(s,x;t,y)=0 \tag{4.7}$$

for $(s,x),(t,y)\in[0,T]\times\mathbb{R}^d$ such that $s<t$.

By the Girsanov transformation (see e.g. Theorem 4.2 of Chapter IV in [8]) we have

$$\begin{aligned}&p^Y(0,x;t,y)\\ &=p^X(0,x;t,y)E^{X_t^x=y}\left[\exp\left(\int_0^t\langle b_\sigma(s,X^x),dB_s\rangle-\frac{1}{2}\int_0^t|b_\sigma(s,X^x)|^2ds\right)\right],\end{aligned} \tag{4.8}$$

for almost every $y$ with respect to the Lebesgue measure, $t\in(0,T)$ and $x\in\mathbb{R}^d$, where $b_\sigma(t,x):=\sigma^{-1}(t,x)b(t,x)$ and $E^{X_t^x=y}[\cdot]$ is the expectation with respect to the regular conditional probability measure given by $X_t^x=y$ (see Theorem 3.3 of Chapter I in [8]). In view of (4.8), we regard $Y^x$ as a perturbation of $X^x$ and show two-sided bounds also hold for $p^Y(0,x;t,y)$. Let

$$\mathcal{E}(t,X^x):=\exp\left(\int_0^t\langle\tilde{b}_\sigma(s,X^x),dB_s\rangle-\frac{1}{2}\int_0^t|\tilde{b}_\sigma(s,X^x)|^2ds\right)$$

for $t\in[0,T]$ and $x\in\mathbb{R}^d$. To show the two-sided bounds of $p^Y(0,x;t,y)$, it is sufficient to show bounds of $E^{X_t^x=y}[\mathcal{E}(t,X^x)]$. However, the conditional probability measure $P^{X_t^x=y}$ is singular to the original probability measure $P$. So, we need some approximation. In particular, we show the following lemma.

**Lemma 4.1.** *For any $r\in\mathbb{R}$, there exist positive constants $\tilde{C}_r$ and $\tilde{\gamma}_r$ depending on $r$, $T$, $d$, $\Lambda$, $\|b\|_\infty$ and $M_a$ such that*

$$\sup_{s\in[0,t)}E\left[\mathcal{E}(s,X^x)^r p^X(0,X_s^x;t-s,y)\right]\le\tilde{C}_r t^{-d/2}\exp\left(-\frac{\tilde{\gamma}_r|x-y|^2}{t}\right)$$

*for $t\in(0,T]$ and $x,y\in\mathbb{R}^d$.*

Sketch of the proof of Lemma 4.1 is as follows. For $s\in[0,t)$, applying Itô's formula and (4.7) we have

$$\begin{aligned}&E\left[\mathcal{E}(s,X^x)^r p^X(s,X_s^x;t,y)\right]-p^X(0,x;t,y)\\ &=\frac{r(r-1)}{2}\int_0^s E\left[\mathcal{E}(w,X^x)^r p^X(w,X_w^x;t,y)|b_\sigma(w,X_w^x)|^2\right]dw\\ &\quad+r\int_0^s E\left[\mathcal{E}(w,X^x)^r\left\langle\left[\nabla p^X(w,\cdot;t,y)\right](X_w^x),b(w,X_w^x)\right\rangle\right]dw.\end{aligned} \tag{4.9}$$

Hence, by (4.5), (4.6) and the fact that

$$\sup_{t\in[0,T]} E[\mathcal{E}(t,X^x)^r] \le Ce^{|r|\cdot|r-1|T}$$

with a constant $C$ depending on $\Lambda$, we have the assertion.

By the definition of the regular conditional probability measure $P^{X_t^x=y}$, we have

$$\int_{\mathbb{R}^d} f(y)P^{X_t^x=y}(A)\,p^X(0,x;t,y)dy = \int_{\mathbb{R}^d} f(y)E\left[\mathbb{I}_A\,p^X(s,X_s^x;t,y)\right]dy$$

for an event $A$ measurable with respect to $\{(X_w^x, B_w); w\le s\}$ and $f\in C_b(\mathbb{R}^d)$. From this equality and Lemma 4.1 we obtain

$$\begin{aligned}
p^X(0,x;t,y)E^{X_t^x=y}\left[\mathcal{E}(t,X^x)^r\right] &= \lim_{s\uparrow t} p^X(0,x;t,y)E^{X_t^x=y}\left[\mathcal{E}(s,X^x)^r\right]\\
&= \lim_{s\uparrow t} E\left[\mathcal{E}(s,X^x)^r p^X(0,X_s^x;t-s,y)\right]\\
&\le \tilde{C}_r t^{-d/2}\exp\left(-\frac{\tilde{\gamma}_r|x-y|^2}{t}\right)
\end{aligned}$$

for $r\in\mathbb{R}$, $t\in(0,T]$ and $x,y\in\mathbb{R}^d$. By letting $r=1$ in the inequality and applying (4.8), we obtain the Gaussian upper bound of $p^Y(0,x;t,y)$. By shift of time, we also have the Gaussian bounds of $p^Y(s,x;t,y)$. The Gaussian lower bound via the estimate with $r=-1$ (see the proof of Theorem 2.5 in [12]). The obtained result is as follows.

**Theorem 4.2.** *For each $s,t\in[0,T]$ such that $s<t$ and $x\in\mathbb{R}^d$, choose a suitable version of $p(s,x;t,\cdot)$. Then, there exist positive constants $\tilde{C}_-$, $\tilde{\gamma}_-$, $\tilde{C}_+$ and $\tilde{\gamma}_+$ depending on $T$, $d$, $\Lambda$, $\|b\|_\infty$, and $M_a$ such that*

$$\tilde{C}_-(t-s)^{-d/2}\exp\left(-\frac{\tilde{\gamma}_-|x-y|^2}{t-s}\right) \le p^Y(s,x;t,y) \le \tilde{C}_+(t-s)^{-d/2}\exp\left(-\frac{\tilde{\gamma}_+|x-y|^2}{t-s}\right)$$

*for $(s,x),(t,y)\in[0,T]\times\mathbb{R}^d$ such that $s<t$.*

The argument above is associated to the perturbation in the operator theory. Indeed, if $a(t,x)$ and $b(t,x)$ are not depending on time $t$, by defining as

$$A^X := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j},\quad A^Y := \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i},$$

$T_t^X := e^{tA^X}$ and $T_t^Y := e^{tA^Y}$, and letting $r=1$ in (4.9), we obtain the variance of parameters formula

$$T_t^Y f - T_t^X f = \int_0^t T_{t-s}^Y(A^Y-A^X)T_s^X f ds,\quad f\in C_\infty^2(\mathbb{R}^d).$$

We also obtain the following typical result by the perturbation.

**Theorem 4.3.** *Let $t \in (0,T]$. Assume that there exist $\beta \in [0,1]$ and a continuous increasing function $\rho$ on $[0,\infty)$ such that $\rho(0)=0$,*

$$|p^X(s,x;t,y_1) - p^X(s,x;t,y_2)| \le (t-s)^{-(d+\beta)/2}\rho(|y_1-y_2|) \tag{4.10}$$

$$|\nabla_x p^X(s,x;t,y_1) - \nabla_x p^X(s,x;t,y_2)| \le (t-s)^{-(d+2\beta)/2}\rho(|y_1-y_2|) \tag{4.11}$$

*for $s \in [0,t)$ and $x, y_1, y_2 \in \mathbb{R}^d$. Then, for any $\varepsilon \in (0,1)$, there exist positive constants $\tilde{C}$ and $\tilde{\gamma}$ depending on $\varepsilon$, $T$, $d$, $\Lambda$, $\|b\|_\infty$ and $M_a$ such that*

$$\begin{aligned}&|p^Y(s,x;t,y_1) - p^Y(s,x;t,y_2)|\\&\le \tilde{C}(t-s)^{-[d+\beta(1-\varepsilon)]/2}\rho(|y_1-y_2|)^{1-\varepsilon}\exp\left(-\frac{\tilde{\gamma}\min\{|x-y_1|^2, |x-y_2|^2\}}{t-s}\right)\end{aligned}$$

*for $s \in [0,t)$ and $x, y_1, y_2 \in \mathbb{R}^d$.*

*Remark.* If $a(t,x)$ is sufficiently smooth, we can choose $\beta = 1$ and $\rho(r) = cr$ with a constant $c$. Then, we obtain the $(1-\varepsilon)$-Hölder continuity of $p^Y(0,x;t,\cdot)$. This choice of the pair $(\beta,\rho)$ is a typical one.

In [11] and [12], the case of less regular $a(t,x)$ is also concerned. Now we consider the case. Letting $\sigma$ be sufficiently smooth, we obtain estimates with constants depending on suitable factors. We assume (1.2), (1.6), and the Gaussian two-sided bounds (4.5) on $p^X(s,x;t,y)$.

In this case the parametrix method is not available. In particular, we do not have such an estimate as (4.6). So we have to treat

$$\left|\int_0^t E\left[\mathcal{E}(s,X^x)^r \langle [\nabla q(s,\cdot;t,y)](X_s^x), \tilde{b}(s,X_s^x)\rangle\right] ds\right| \tag{4.12}$$

by a different way. Because of the reason, in [11] the following estimate is prepared.

**Lemma 4.4.** *Let $t \in (0,\infty)$ and $\phi$ be a nonnegative continuous function on $(0,t)\times\mathbb{R}^d$ such that $\phi(\cdot,x) \in W^{1,1}_{\mathrm{loc}}((0,t),ds)$ for $x \in \mathbb{R}^d$ and $\phi(s,\cdot) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d,dx)$ for*

*$s \in (0,t)$. Then, for $s_1, s_2 \in (0,t)$ such that $s_1 \leq s_2$*

$$
\begin{aligned}
&\int_{s_1}^{s_2} \int_{\mathbb{R}^d} \frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2} \phi(u,\xi) d\xi du \\
&\leq C(1+|\log(t-s_1)|) \int_{\mathbb{R}^d} \phi(s_1,\xi) d\xi + C(t-s_1)^{-1} \int_{\mathbb{R}^d} |y-\xi|^2 \phi(s_1,\xi) d\xi \\
&\quad + C(1+|\log(t-s_2)|) \int_{\mathbb{R}^d} \phi(s_2,\xi) d\xi + C(t-s_2)^{-1} \int_{\mathbb{R}^d} |y-\xi|^2 \phi(s_2,\xi) d\xi \\
&\quad + C \int_{s_1}^{s_2} \int_{\mathbb{R}^d} \phi(u,\xi) d\xi du + C \int_{s_1}^{s_2} \int_{\mathrm{supp}\phi} \frac{|\nabla_\xi \phi(u,\xi)|^2}{\phi(u,\xi)} d\xi du \\
&\quad + C \int_{s_1}^{s_2} (1+|\log(t-u)|) \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial u} \phi(u,\xi) \right| d\xi du \\
&\quad + C \int_{s_1}^{s_2} (t-u)^{-1} \int_{\mathbb{R}^d} |y-\xi|^2 \left| \frac{\partial}{\partial u} \phi(u,\xi) \right| d\xi du \\
&\quad + C \sum_{i,j=1}^{d} \int_0^t \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial \xi_j} a_{ij}(u,\xi) \right|^2 \phi(u,\xi) d\xi du
\end{aligned}
$$

*where $C$ is a constant depending on $d$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, and $\Lambda$, and $\mathrm{supp}\phi$ is the support of $\phi$.*

For fixed $t$, $x$ and $y$, let $\phi$ be a function on $(0,t) \times \mathbb{R}^d$ defined by

$$
\phi(u,\xi) := [u(t-u)]^{1/4} \cdot \frac{C_+}{[u(t-u)]^{d/2}} \exp\left[ -\gamma^+ \left( \frac{|\xi - x|^2}{u} + \frac{|y-\xi|^2}{t-u} \right) \right].
$$

Applying Lemma 4.4 we obtain the following lemma.

**Lemma 4.5.** *For sufficiently small $\varepsilon > 0$ it holds that*

$$
\int_0^t [s(t-s)]^{1/4} \int_{\mathbb{R}^d} \frac{\left|\nabla_\xi p^X(s,\xi;t,y)\right|^2}{p^X(s,\xi;t,y)} p^X(0,x;s,\xi) d\xi ds \leq C t^{-d/2+1/2}(1+|\log t|),
$$

*where $C$ is a constant depending on $d$, $\Lambda$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$ $m$, $M$ and $\theta$.*

By using Lemma 4.5 we are able to estimate the term (4.12) and we obtain the Gaussian two-sided bounds of $p^Y(0,x;t,y)$.

**Proposition 4.6.** *Assume the smoothness of $a(t,x)$, (1.2), (1.6) and (4.5). Then, there exist positive constants $\tilde{C}_-$, $\tilde{\gamma}_-$, $\tilde{C}_+$ and $\tilde{\gamma}_+$ depending on $T$, $d$, $\Lambda$, $\|\tilde{b}\|_\infty$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $\theta$, $m$ and $M$, such that*

$$
\tilde{C}_- t^{-d/2} \exp\left( -\frac{\tilde{\gamma}_- |x-y|^2}{t} \right) \leq p^Y(0,x;t,y) \leq \tilde{C}_+ t^{-d/2} \exp\left( -\frac{\tilde{\gamma}_+ |x-y|^2}{t} \right)
$$

*for $t \in (0,T]$ and $x, y \in \mathbb{R}^d$.*

Now we consider the case of non-smooth $a(t,x)$. Let $a(t,x)$ be a function satisfying (1.2) and (1.3). Consider a sequence $\{a^{(n)}(t,x)\}$ of $d\times d$-matrix valued smooth functions on $[0,T]\times\mathbb{R}^d$ such that $a^{(n)}(t,x)$ converges to $a(t,x)$ as $n$ goes to infinity for each $(t,x)\in[0,T]\times\mathbb{R}^d$. Let $\check{p}^{(n)}(s,x;t,y)$ be the fundamental solution to (1.4). We assume that (1.2) with $a^{(n)}(t,x)$ in place of $a(t,x)$, (1.5) and (1.6). Then, we have the Gaussian two-sided bounds of $p^{Y,(n)}(0,x;t,y)$ with constants independent of $n$ by Proposition 4.6. Hence, in view of Theorem 11.3.4 in [22], we see that $p^Y(0,x;t,y)$ also satisfies the Gaussian two-sided bounds with same constants. By shifting the time of the initial condition, we have the same bounds of $p^Y(s,x;t,y)$. As the result, we obtain the following theorem.

**Theorem 4.7.** *Assume (1.2), (1.3), and the existence of a sequence $\{a^{(n)}(t,x)\}$ of $d\times d$-matrix valued smooth functions on $[0,T]\times\mathbb{R}^d$ such that $a^{(n)}(t,x)$ converges to $a(t,x)$ as $n$ goes to infinity for each $(t,x)\in[0,T]\times\mathbb{R}^d$. Denote by $\check{p}^{(n)}(s,x;t,y)$ the fundamental solution to (1.4). Additionally assume (1.2) with $a^{(n)}(t,x)$ in place of $a(t,x)$, (1.5) and (1.6). Then, there exist positive constants $\tilde{C}_-$, $\tilde{\gamma}_-$, $\tilde{C}_+$, $\tilde{\gamma}_+$ depending on $T$, $r$, $d$, $\Lambda$, $\|\tilde{b}\|_\infty$, $C_-$, $\gamma_-$, $C_+$, $\gamma_+$, $\theta$, $m$ and $M$ such that*

$$\tilde{C}_-(t-s)^{-d/2}\exp\left(-\frac{\tilde{\gamma}_-|x-y|^2}{t-s}\right)\le p^Y(s,x;t,y)\le \tilde{C}_+(t-s)^{-d/2}\exp\left(-\frac{\tilde{\gamma}_+|x-y|^2}{t-s}\right)$$

*for $(s,x),(t,y)\in[0,T]\times\mathbb{R}^d$ such that $s<t$.*

*Remark.* In Theorem 4.7 we assumed uniform estimates in approximation sequence as (1.6) and (1.5), because we do not know whether it is possible to choose such a sequence $\{a^{(n)}(t,x)\}$ or not under the assumption that (1.5) holds with $\check{p}(s,x;t,y)$ in place of $\check{p}^{(n)}(s,x;t,y)$. On the other hand, we are able to choose such a sequence $\{a^{(n)}(t,x)\}$ satisfying (1.2) and (1.3) with $a^{(n)}(t,x)$ in place of $a(t,x)$, and (1.6), if (1.6) is satisfied with $a(t,x)$ in place of $a^{(n)}(t,x)$.

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