RIMS Kôkyûroku Bessatsu
B80 (2020), 57–80

Volume-preserving mean curvature flow for tubes in rank one symmetric spaces of non-compact type

By

Naoyuki KOIKE*

Abstract

First we investigate the evolutions of the radius function and its gradient along the volume-preserving mean curvature flow starting from a tube (of nonconstant radius) over a compact closed domain of a reflective submanifold in a symmetric space under certain condition for the radius function. Next, we prove that the tubeness is preserved along the flow in the case where the ambient space is a rank one symmetric space of non-compact type, the reflective submanifold is an invariant submanifold and the radius function of the initial tube is radial. Furthermore, in this case, we prove that the flow reaches to the invariant submanifold or it exists in infinite time and converges to another tube of constant mean curvature in the C^∞-topology in infinite time.

§ 1. Introduction

Let f_t's ($t \in [0,T)$) be a one-parameter C^∞-family of immersions of an n-dimensional compact manifold M into an $(n+1)$-dimensional Riemannian manifold $\overline{M}$, where T is a positive constant or $T = \infty$. Define a map $\widetilde{f} : M \times [0,T) \to \overline{M}$ by $\widetilde{f}(x,t) = f_t(x)$ ($(x,t) \in M \times [0,T)$). Denote by π_M the natural projection of $M \times [0,T)$ onto M. For a vector bundle E over M, denote by $\pi_M^* E$ the induced bundle of E by π_M. Also, denote by H_t, g_t and N_t the mean curvature, the induced metric and the outward unit normal vector of f_t, respectively. Define the function H over $M \times [0,T)$ by $H_{(x,t)} := (H_t)_x$ ($(x,t) \in M \times [0,T)$), the section g of $\pi_M^*(T^{(0,2)}M)$ by $g_{(x,t)} := (g_t)_x$ ($(x,t) \in M \times [0,T)$) and the section N of $\widetilde{f}^*(T\overline{M})$ by $N_{(x,t)} := (N_t)_x$ ($(x,t) \in M \times [0,T)$), where $T^{(0,2)}M$

Received January 16, 2017. Revised December 27, 2019.
2010 Mathematics Subject Classification(s): 53C44;53C35
*Department of mathematics, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku Tokyo 162-8601, Japan.
e-mail: `koike@rs.kagu.tus.ac.jp`

© 2020 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

is the tensor bundle of degree $(0,2)$ of M and $T\overline{M}$ is the tangent bundle of $\overline{M}$. The average mean curvature $\overline{H}(: [0,T) \to \mathbb{R})$ is defined by

$$\overline{H}_t := \frac{\int_M H_t dv_{g_t}}{\int_M dv_{g_t}}, \tag{1.1}$$

where dv_{g_t} is the volume element of g_t. The flow f_t's $(t \in [0,T))$ is called a *volume-preserving mean curvature flow* if it satisfies

$$\widetilde{f}_* \left(\frac{\partial}{\partial t} \right) = (\overline{H} - H)N. \tag{1.2}$$

In particular, if f_t's are embeddings, then we call $M_t := f_t(M)$'s $(0 \in [0,T))$ rather than f_t's $(0 \in [0,T))$ a volume-preserving mean curvature flow. Note that, if M has no boundary and if f is an embedding, then, along this flow, the volume of (M, g_t) decreases but the volume of the domain D_t sorrounded by $f_t(M)$ is preserved invariantly.

First we shall recall the result by M. Athanassenas ([A1,2]). Let P_i $(i = 1,2)$ be affine hyperplanes in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ meeting a affine line l orthogonally and E a closed domain of $\mathbb{R}^{n+1}$ with $\partial E = P_1 \cup P_2$. Also, let M be a hypersurface of revolution in $\mathbb{R}^{n+1}$ such that $M \subset E$, $\partial M \subset P_1 \cup P_2$ and that M meets P_1 and P_2 orthogonally. Let D be the closed domain surrouded by P_1, P_2 and M, and d the distance between P_1 and P_2. She ([A1,2]) proved the following fact.

Fact 1. *Let M_t $(t \in [0,T))$ be the volume-preserving mean curvature flow starting from M such that M_t meets P_1 and P_2 orthogonally for all $t \in [0,T)$. Then the following statements* (i) *and* (ii) *hold:*

(i) *M_t $(t \in [0,T))$ remain to be hypersurfaces of revolution.*

(ii) *If* $\mathrm{Vol}(M) \leq \frac{\mathrm{Vol}(D)}{d}$ *holds, then $T = \infty$ and as $t \to \infty$, the flow M_t converges to the cylinder C such that the volume of the closed domain surrounded by P_1, P_2 and C is equal to* $\mathrm{Vol}(D)$.

E. Cabezas-Rivas and V. Miquel ([CM1,2,3]) proved the similar result in certain kinds of rotationally symmetric spaces. Let $\overline{M}$ be an $(n+1)$-dimensional rotationally symmetric space (i.e., $SO(n)$ acts on $\overline{M}$ isometrically and its fixed point set is a one-dimensional submanifold). Note that real space forms are rotationally symmetric spaces. Denote by l the fixed point set of the action, which is an one-dimensional totally geodesic submanifold in $\overline{M}$. Let P_i $(i = 1,2)$ be totally geodesic hypersurfaces (or equidistant hypersurfaces) in $\overline{M}$ meeting l orthogonally and E a closed domain of $\overline{M}$ with $\partial E = P_1 \cup P_2$, where we note that they treat the case where P_i $(i = 1,2)$ are totally geodesic hypersurfaces (resp. equidistant hypersurfaces) in [CM1,2] (resp. [CM3]). An embedded hypersurface M in $\overline{M}$ is called a *hypersurface of revolution* if it is invariant with respect to the $SO(n)$-action. Let M be a hypersurface of revolution in $\overline{M}$ such that $M \subset E$,

$\partial M \subset P_1 \cup P_2$ and that M meets P_1 and P_2 orthogonally. Let D be the closed domain surrouded by P_1, P_2 and M, and d the distance between P_1 and P_2. They ([CM1,2,3]) proved the following fact.

Fact 2. *Assume that* $\mathrm{Sec}(v,w) < 0$ *for any* $v \in Tl$ *and* $w \in T^\perp l$ *and that* $\mathrm{Sec}(w_1,w_2) \leq 0$ *for any* $w_1,\, w_2 \in T^\perp l$, *where* $\mathrm{Sec}(\cdot, \bullet)$ *denotes the sectional curvature of the* 2*-plane spanned by* $\cdot$ *and* $\bullet$. *Let* M_t $(t \in [0,T))$ *be the volume-preserving mean curvature flow starting from* M *such that* M_t *meets* P_1 *and* P_2 *orthogonally for all* $t \in [0,T)$. *Then the following statements* (i) *and* (ii) *hold:*

(i) M_t $(t \in [0,T))$ *remain to be hypersurfaces of revolution.*

(ii) *If* $\mathrm{Vol}(M) \leq C$ *holds, where* C *is a constant depending on* $\mathrm{Vol}(D)$ *and* d, *then* $T = \infty$ *and, as* $t \to \infty$, *the flow* M_t $(t \in [0,T))$ *converges to a hypersurface of revolution* C *of constant mean curvature such that the volume of the closed domain surrounded by* P_1, P_2 *and* C *is equal to* $\mathrm{Vol}(D)$.

A *symmetric space of compact type* (resp. *non-compact type*) is a naturally reductive Riemannian homogeneous space $\overline{M}$ such that, for each point p of $\overline{M}$, there exists an isometry of $\overline{M}$ having p as an isolated fixed point and that the isometry group of $\overline{M}$ is a semi-simple Lie group each of whose irreducible factors is compact (resp. not compact) (see [He]). Note that symmetric spaces of compact type other than a sphere and symmetric spaces of non-compact type other than a (real) hyperbolic space are not rotationally symmetric. An *equifocal submanifold* in a (general) symmetric space is a compact submanifold (without boundary) satisfying the following conditions:

(E-i) the normal holonomy group of M is trivial,

(E-ii) M has a flat section, that is, for each $x \in M$, $\Sigma_x := \exp^\perp(T^\perp_x M)$ is totally geodesic and the induced metric on Σ_x is flat, where $T^\perp_x M$ is the normal space of M at x and $\exp^\perp$ is the normal exponential map of M.

(E-iii) for each parallel normal vector field v of M, the focal radii of M along the normal geodesic γ_{v_x} (with $\gamma'_{v_x}(0) = v_x$) are independent of the choice of $x \in M$, where $\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0.

In [Ko2], we showed that the mean curvature flow starting from an equifocal submanifold in a symmetric space of compact type collapses to one of its focal submanifolds in finite time. In [Ko3], we showed that the mean curvature flow starting from a certain kind of (not necessarily compact) submanifold satisfying the above conditions (E-i), (E-ii) and (E-iii) in a symmetric space of non-compact type collapses to one of its focal submanifolds in finite time. The following question arise naturally:

Question. In what case, does the volume-preserving mean curvature flow starting from a submanifold in a symmetric space of compact type (or non-compact type) converges *to a submanifold satisfying the above conditions (E-i), (E-ii) and (E-iii)?*

Let M be an equifocal hypersurface in a rank $l(\geq 2)$ symmetric space $\overline{M}$ of compact type or non-compact type. Then it admits a reflective focal submanifold F and it is a tube (of constant radius) over F, where the "*reflectivity*" means that the submanifold is a connected component of the fixed point set of an involutive isometry of $\overline{M}$ and a "*tube of constant radius $r(>0)$ over F*" means the image of $t_r(F) := \{\xi \in T^\perp F \,|\, ||\xi|| = r\}$ by the normal exponential map $\exp^\perp$ of F under the assumption that the restriction $\exp^\perp|_{t_r(F)}$ of $\exp^\perp$ to $t_r(F)$ is an embedding, where $T^\perp F$ is the normal bundle of F and $||\cdot||$ is the norm of $(\cdot)$. Any reflective submanifold in a symmetric space $\overline{M}$ of compact type or non-compact type is a singular orbit of a Hermann action (i.e., the action of the symmetric subgroup of the isometry group of $\overline{M}$) (see [KT]). Note that even if T. Kimura and M. Tanaka ([KT]) proved this fact in compact type case, the proof is valid in non-compact type case. From this fact, it is shown that M is curvature-adapted, where "the curvature-adpatedness" means that, for any point $x \in M$ and any normal vector ξ of M at x, $R(\cdot,\xi)\xi$ preserves the tangent space T_xM of M at x invariantly, and that the restriction $R(\cdot,\xi)\xi|_{T_xM}$ of $R(\cdot,\xi)\xi$ to T_xM and the shape operator A_ξ commute to each other (R : the curvature tensor of $\overline{M}$). The notion of the curvature-adaptedness was introduced in [BV]. For a non-constant positive-valued function r over F, the image of $t_r(F) := \{\xi \in T^\perp F \,|\, ||\xi|| = r(\pi(\xi))\}$ by $\exp^\perp$ is called the *tube of non-constant radius r over F* in the case where the restriction $\exp^\perp|_{t_r(F)}$ of $\exp^\perp$ to $t_r(F)$ is an embedding, where π is the bundle projection of $T^\perp F$. Note that $\exp^\perp|_{t_r(F)}$ is an embedding for a non-constant positive-valued function r over F such that $\max r$ is sufficiently small because F is homogeneous. Since F is reflective, so is also the normal umbrella $F_x^\perp := \exp^\perp(T_x^\perp F)$ of F at x and hence $F_x^\perp$ is a symmetric space.

Motivation. If $F_x^\perp$ is a rank one symmetric space, then tubes over F of constant radius satisfies the above conditions (E-i), (E-ii) and (E-iii). Hence, when $F_x^\perp$ is of rank one, it is very interesting to invesitigate in what case the volume-preserving mean curvature flow starting from a tube of non-constant radius over F converges to a tube of constant radius over F.

Under this motivation, we try to derive a result similar to those of M. Athanassenas ([A1,2]) and E. Cabezas-Rivas and V. Miquel ([CM1,2]) in this paper.

Let $\gamma : [0,\infty) \to \overline{M}$ be any normal geodesic of F. Denote by $r_{co}(\gamma)$ the first conjugate radius along the geodesic γ in $F_x^\perp$, $r_{fo}(\gamma)$ the first focal radius of F along γ. $r_\gamma := \min\{r_{co}(\gamma), r_{fo}(\gamma)\}$. It is shown that, if $F_x^\perp$ also is of rank one, then r_γ is independent of the choice of γ. Hence we denote r_γ by r_F in this case. The setting in this paper is as follows.

Setting (S). Let F be a reflective submanifold in a symmetric space $\overline{M}$ of compact type or non-compact type and B be a compact closed domain in F with smooth boundary which is star-shaped with respect to some $x_* \in B$ and does not intersect with the cut locus of x_* in F. Assume that the normal umbrellas of F are rank one symmetric spaces. Set $P := \underset{x\in\partial B}{\cup} F_x^{\perp}$ and denote by E the closed domain in $\overline{M}$ surrounded by P. Let $M := t_{r_0}(B)$ and $f := \exp^{\perp}|_{t_{r_0}(B)}$, where r_0 is a non-constant positive C^∞-function over B with $r_0 < r_F$ such that $\operatorname{grad} r_0 = 0$ holds along ∂B. Denote by D the closed domain surrouded by P and $f(M)$. See Figure 1 about this setting.

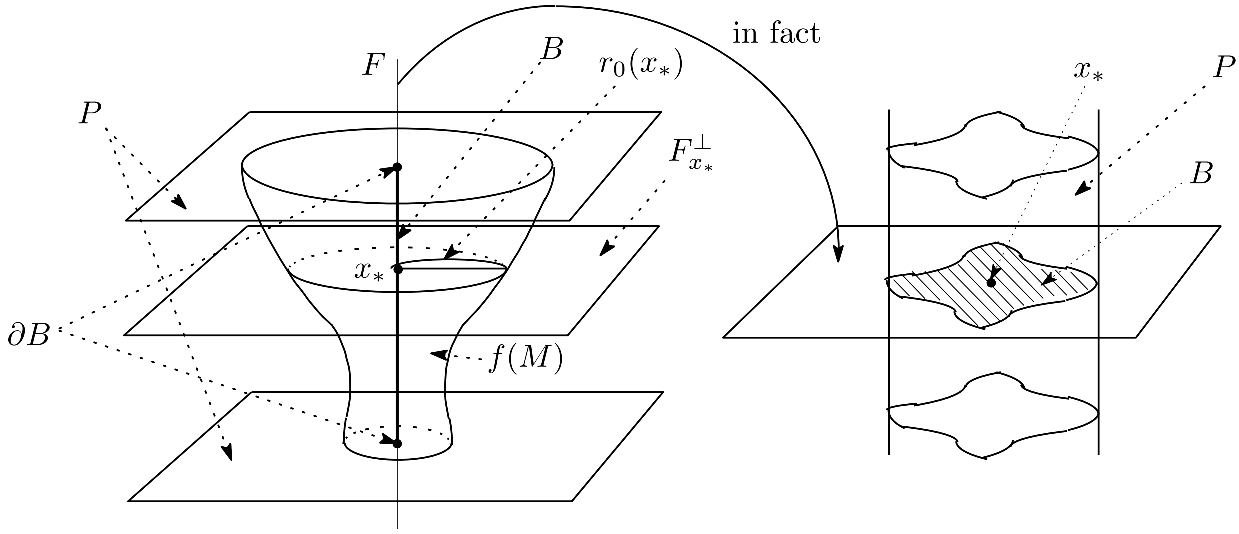

Figure 1.

Remark 1.1. (i) At least one of singular orbits of any Hermann action of cohomogeneity one on any symmetric space $\overline{M}$ without Euclidean part other than a sphere and a (real) hyperbolic space is a reflective submanifold whose normal umbrellas are rank one symmetric spaces and tubes of constant radius over the reflective singular orbit satisfy the above conditions (E-i), (E-ii) and (E-iii) (i.e., of constant mean curvature). Note that, when $\overline{M}$ is of compact type, the Hermann action has exactly two singular orbits and, when $\overline{M}$ is of non-compact type, the Hermann action has the only one singular orbit. Hermann actions of cohomogeneity one on irreducible symmetric spaces of compact type or non-compact type are classified in [BT].

(ii) At least one of singular orbits of any Hermann action of cohomogeneity greater than one on any symmetric space $\overline{M}$ of compact type or non-compact type is a reflective submanifold, but the normal umbrellas are higher rank symmetric spaces and hence tubes of constant radius over the reflective singular orbit do not satisfy the above conditions (E-i), (E-ii) and (E-iii) (i.e., not of constant mean curvature).

Under the above setting (S), we consider the volume-preserving mean curvature flow f_t $(t \in [0,T))$ starting from f and satisfying the following boundary condition:

(C1) $\mathrm{grad}\, r_t = 0$ holds along ∂B for all $t \in [0, T)$, where r_t is the radius function of $M_t := f_t(M)$ (i.e., $M_t = \exp^\perp(t_{r_t}(B))$), where r_t is possible to be multi-valued.

Furthermore, assume that the flow satisfies the following condition:

(C2) $(\mathrm{grad}\, r_t)_x$ belongs to a common eigenspace of the family $\{R(\cdot, \xi)\xi\}_{\xi \in T_x^\perp B}$ for all $(x, t) \in B \times [0, T)$.

If the initial radius function r_0 is constant over a collar neighborhood U of ∂B, then $t_{r_0}(U)$ is of constant mean curvature because the normal umbrellas of F are of rank one by the assumption (furthermore, these umbrellas are automatically isometric to one another). Hence $\overline{H}_t - H_t$ ($t \in [0, T)$) remain to be constant over $t_{r_0}(U)$, that is, r_t ($t \in [0, T)$) remain to be constant over U (see Figure 2). If $\overline{M}$ is a rank one symmetric space (other than a sphere and a (real) hyperbolic space) and if F is an invariant submanifold, then $R(\cdot, \xi)\xi|_{T_x F}$ ($x \in F$, $\xi \in T_x^\perp F$) are the constant-multiple of the identity transformation $\mathrm{id}_{T_x F}$ of $T_x F$ and hence the condition (C2) automatically holds. It is easy to show that, if the initial radius function r_0 satisfies $(\mathrm{grad}\, r_0)_x \in \mathcal{D}_x$ ($x \in B$), then r_t satisfies $(\mathrm{grad}\, r_t)_x \in \mathcal{D}_x$ ($x \in B$) for all $t \in [0, T)$, that is, the condition (C2) holds.

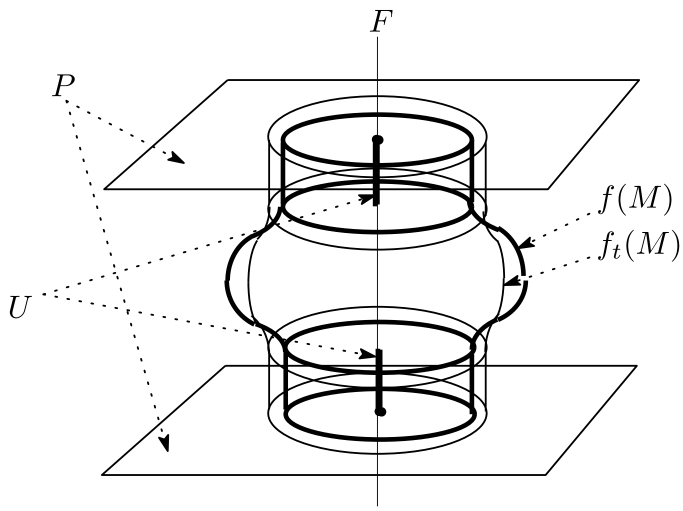

Figure 2.

If the condition (C2) holds, then it is shown that there uniquely exists the volume-preserving mean curvature flow $f_t : M \hookrightarrow \overline{M}$ starting from f as in the above setting (S) and satisfying the condition (C1) in short time (see Proposition 4.2). Under these assumptions, we first derive the evolution equations for the radius functions of the flow and some quantities related to the gradients of the functions (see Sections 4 and 5). Next, in the following special case, we derive the following preservability theorem for the tubeness along the flow by using the evolution equations.

Theorem A. *Let f be as in the above setting (S) and f_t ($t \in [0,T)$) the volume-preserving mean curvature flow starting from f and satisfying the above condition (C1). Assume that $\overline{M}$ is a rank one symmetric space of non-compact type, F is an invariant submanifold and that B is a closed geodesic ball of radius r_B centered at x_* in F, where the invariantness of F means the totally geodesicness in the case where $\overline{M}$ is a (real) hyperbolic space. If r_0 is radial with respect to x_* (i.e., r_0 is constant along each geodesic sphere centered at x_* in F), then M_t ($t \in [0,T)$) remain to be tubes over B such that the volume of the closed domain surrounded by M_t and P is equal to* $\mathrm{Vol}(D)$.

Furthermore, we derive the following results.

Theorem B. *Under the hypothesis of Theorem A, one of the following statements* (a) *and* (b) *holds:*

(a) $M_t := f_t(M)$ *reaches B as $t \to T$,*

(b) $T = \infty$ *and M_t converges to a tube of constant mean curvature over B (in C^∞-topology) as $t \to \infty$.*

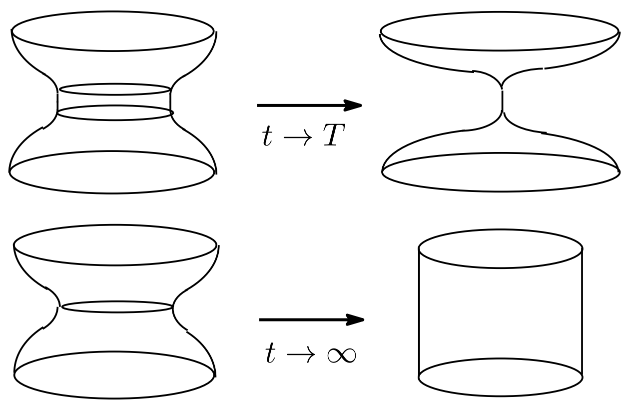

Figure 3.

Theorem C. *Under the hypothesis of Theorem A, assume that*

$$\mathrm{Vol}(M_0) \leq v_{m^H-1} v_{m^V} (\delta_2 \circ \delta_1^{-1}) \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right),$$

where $m^H := \dim F$, $m^V := \operatorname{codim} F - 1$, v_{m_H-1} (resp. v_{m^V}) is the volume of the $m^H - 1$ (resp. m^V)-dimensional Euclidean unit sphere and δ_i ($i = 1,2$) are increasing functions over $\mathbb{R}$ explicitly described (see Section 6). Then $T = \infty$ and M_t converges to a tube of constant mean curvature over B (in C^∞-topology) as $t \to \infty$.

Remark 1.2. Let $\overline{g}$ be the metric of $\overline{M}$ and c a positive constant. As $c \to \infty$, $c\overline{g}$ approches to a flat metric and δ_i ($i = 1,2$) approches to the identity transformation of $[0,\infty)$

and hence $v_{m^V}(\delta_2\circ\delta_1^{-1})\left(\dfrac{\mathrm{Vol}(D)}{v_{m^V}\mathrm{Vol}(B)}\right)$ approaches to $\dfrac{\mathrm{Vol}(D)}{\mathrm{Vol}(B)}$. Thus, as $c\to\infty$, the condition $\mathrm{Vol}(M_0)\le v_{m^H-1}v_{m^V}(\delta_2\circ\delta_1^{-1})\left(\dfrac{\mathrm{Vol}(D)}{v_{m^V}\mathrm{Vol}(B)}\right)$ approaches to the condition $\mathrm{Vol}(M)\le\dfrac{\mathrm{Vol}(D)}{d}$ in the statement (ii) of Fact 1 in the case of $\dim F=1$.

In the future, we plan to tackle the following problem.

Problem. Under the hypothesis of Theorem C, does M_t converge to a tube of constant radius over B (in C^∞-topology) as $t\to\infty$?

As the first step to solve this problem, I need to classify tubes of constant mean curvature over F other than tubes of constant radius over F.

§ 2. The mean curvature of a tube over a reflective submanifold

In this section, we shall calculate the mean curvature of a tube over a reflective submanifold in a symmetric space of compact type or non-compact type. Let $\overline{M}=G/K$ be a symmetric space of compact type or non-compact type, where G is the identity component of the isometry group of $\overline{M}$ and K is the isotropy group of G at some point p_0 of $\overline{M}$. Let F be a reflective submanifold in $\overline{M}$ such that the nomal umblleras Σ_x's $(x\in F)$ are symmetric spaces of rank one. Denote by $\bar g$ (resp. g_F) the Rimannian metric of $\overline{M}$ (resp. F) and $\overline{\nabla}$ (resp. ∇^F) the Riemannian connection of $\overline{M}$ (resp. F). Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Also, let θ be the Cartan involution of $\mathfrak{g}$ with $(\mathrm{Fix}\,\theta)_0\subset K\subset\mathrm{Fix}\,\theta$ and set $\mathfrak{p}:=\mathrm{Ker}(\theta+\mathrm{id})$, which is identified with the tangent space of $T_{p_0}\overline{M}$ of $\overline{M}$ at p_0. Without loss of generality, we may assume that p_0 belongs to F. Set $\mathfrak{p}':=T_{p_0}F$ and $\mathfrak{p}'^{\perp}:=T^{\perp}_{p_0}F$. Take a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{p}'^{\perp}$ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ including $\mathfrak{b}$. Note that the dimension of $\mathfrak{b}$ is equal to 1 because the normal umbrellas of F is symmetric spaces of rank one by the assumption. For each $\alpha\in\mathfrak{a}^*$ and $\beta\in\mathfrak{b}^*$, we define a subspace $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ of $\mathfrak{p}$ by

$$\mathfrak{p}_\alpha:=\{Y\in\mathfrak{p}\,|\,\mathrm{ad}(X)^2(Y)=-\varepsilon\alpha(X)^2Y\text{ for all }X\in\mathfrak{a}\}$$

and

$$\mathfrak{p}_\beta:=\{Y\in\mathfrak{p}\,|\,\mathrm{ad}(X)^2(Y)=-\varepsilon\beta(X)^2Y\text{ for all }X\in\mathfrak{b}\},$$

respectively, where ad is the adjoint representation of $\mathfrak{g}$, $\mathfrak{a}^*$ (resp. $\mathfrak{b}^*$) is the dual space of $\mathfrak{a}$ (resp. $\mathfrak{b}$) and ε is given by

$$\varepsilon:=\begin{cases}1 & (\text{when }\overline{M}\text{ is of compact type})\\ -1 & (\text{when }\overline{M}\text{ is of non}-\text{compact type}).\end{cases}$$

Define a subset $\triangle$ of $\mathfrak{a}^*$ by

$$\triangle := \{\alpha \in \mathfrak{a}^* \,|\, \mathfrak{p}_\alpha \neq \{0\}\},$$

and subsets $\triangle'$ and $\triangle'_V$ of $\mathfrak{b}^*$ by

$$\triangle' := \{\beta \in \mathfrak{b}^* \,|\, \mathfrak{p}_\beta \neq \{0\}\}$$

and

$$\triangle'_V := \{\beta \in \mathfrak{b}^* \,|\, \mathfrak{p}_\beta \cap \mathfrak{p}'^{\perp} \neq \{0\}\}.$$

The systems $\triangle$ and $\triangle'_V$ are root systems and $\triangle' = \{\alpha|_{\mathfrak{b}} \,|\, \alpha \in \triangle\}$ holds. Let $\triangle_+$ (resp. $(\triangle'_V)_+$) be the positive root system of $\triangle$ (resp. $\triangle'_V$) with respect to some lexicographic ordering of $\mathfrak{a}^*$ (resp. $\mathfrak{b}^*$) and $\triangle'_+$ be the positive subsystem of $\triangle'$ with respect to the lexicographic ordering of $\mathfrak{b}^*$, where we take one compatible with the lexicographic ordering of $\mathfrak{b}^*$ as the lexicographic ordering of $\mathfrak{a}^*$. Also we have the following root space decomposition:

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \triangle_+} \mathfrak{p}_\alpha = \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b}) + \sum_{\beta \in \triangle'_+} \mathfrak{p}_\beta,$$

where $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$ is the centralizer of $\mathfrak{b}$ in $\mathfrak{p}$. For convenience, we set $\mathfrak{p}_0 := \mathfrak{z}_{\mathfrak{p}}(\mathfrak{b})$. Since the normal umbrellas of F are symmetric spaces of rank one, $\dim \mathfrak{b} = 1$ and this root system $\triangle'_V$ is of $(\mathfrak{a}_1)$-type or $(\mathfrak{bd}_1)$-type. Hence $(\triangle'_V)_+$ is described as

$$(\triangle'_V)_+ = \begin{cases} \{\beta\} & (\triangle'_V \,:\, (\mathfrak{a}_1)\text{−type}) \\ \{\beta, 2\beta\} & (\triangle'_V \,:\, (\mathfrak{bd}_1)\text{−type}) \end{cases}$$

for some $\beta (\neq 0) \in \mathfrak{b}^*$. However, in general, we may describe as $(\triangle'_V)_+ = \{\beta, 2\beta\}$ by interpretting as $\mathfrak{p}_{2\beta} = \{0\}$ when $\triangle'_V$ is of $(\mathfrak{a}_1)$. The system $\triangle'_+$ is described as

$$\triangle'_+ = \{k\beta \,|\, k \in \mathcal{K}\}$$

for some finite subset $\mathcal{K}$ of $\mathbb{R}_+$. Set $b := |\beta(X_0)|$ for a unit vector X_0 of $\mathfrak{b}$. Since F is curvature-adapted, $\mathfrak{p}'$ and $\mathfrak{p}'^{\perp}$ are $\mathrm{ad}(X)^2$-invariant for each $X \in \mathfrak{b}$. Hence we have the following direct sum decompositions:

$$\mathfrak{p}' = \mathfrak{p}_0 \cap \mathfrak{p}' + \sum_{k \in \mathcal{K}} (\mathfrak{p}_{k\beta} \cap \mathfrak{p}')$$

and

$$(\mathfrak{p}')^{\perp} = \mathfrak{b} + \sum_{k=1}^{2} (\mathfrak{p}_{k\beta} \cap (\mathfrak{p}')^{\perp}).$$

Assumption. Assume that there exists $k_0 \in \{0,1,2\}$ such that $\tau_x^{-1}((\mathrm{grad}\, r)_x) \in \mathfrak{p}_{k_0\beta} \cap \mathfrak{p}'$ holds for any $\xi \in M \cap T_x^{\perp} B$, where we note that $\mathfrak{p}_{k_0\beta}$ depends on the choice of ξ.

Under this assumption, we can derive the following description of the mean curvature vector H of M.

Proposition 2.1. *Under the assumption of $\tau_x^{-1}((\mathrm{grad}\, r)_x) \in \mathfrak{p}_{k_0\beta} \cap \mathfrak{p}'$, the mean curvature H_ξ of M at ξ is described as*

$$H_\xi = \frac{\cos(\sqrt{\varepsilon}k_0 br(x))}{\sqrt{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\mathrm{grad}\, r)_x||^2}} \tag{2.1}$$
$$\times \left\{ \sum_{k=1}^{2} \frac{m_k^V \sqrt{\varepsilon}kb}{\tan(\sqrt{\varepsilon}kbr(x))} - \sum_{k\in\mathcal{K}} m_k^H \sqrt{\varepsilon}kb \tan(\sqrt{\varepsilon}kbr(x)) - \frac{(\triangle_F r)(x)}{\cos^2(\sqrt{\varepsilon}k_0 br(x))} \right.$$
$$- \frac{||(\mathrm{grad}\, r)_x||^2 \sqrt{\varepsilon}k_0 b \tan(\sqrt{\varepsilon}k_0 br(x))}{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\mathrm{grad}\, r)_x||^2}$$
$$\left. + \frac{(\nabla^F dr)_x((\mathrm{grad}\, r)_x, (\mathrm{grad}\, r)_x)}{\cos^2(\sqrt{\varepsilon}k_0 br(x))(\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\mathrm{grad}\, r)_x||^2)} \right\}.$$

§ 3. The volume element of a tube over a reflective submanifold

We shall use the notations in Introduction and the previous section. In this section, we shall calculate the volume element of M. Let r_F be as in Introduction. We can derive

$$r_{co}(\gamma) = \begin{cases} \dfrac{\pi}{2\sqrt{\varepsilon}b} & (\text{when } m_2^V \neq 0) \\ \dfrac{\pi}{\sqrt{\varepsilon}b} & (\text{when } m_2^V = 0) \end{cases}$$

and

$$r_{fo}(\gamma) = \min\left\{ \frac{\pi}{2\sqrt{\varepsilon}kb} \,\middle|\, k \in \mathcal{K} \right\}.$$

Hence we obtain

$$r_F = \begin{cases} \min\left\{ \dfrac{\pi}{2\sqrt{\varepsilon}kb} \,\middle|\, k \in \mathcal{K} \cup \{1\} \right\} & (\text{when } m_2^V \neq 0) \\ \min\left\{ \dfrac{\pi}{2\sqrt{\varepsilon}kb} \,\middle|\, k \in \mathcal{K} \cup \left\{\dfrac{1}{2}\right\} \right\} & (\text{when } m_2^V = 0). \end{cases} \tag{3.1}$$

Fix $\xi \in M \cap T_x^\perp B$ and $X \in T_x B$. Without loss of generality, we may assume that $\tau_x^{-1}\xi \in \mathfrak{b}$. By the assumption (C2), we may assume that $\tau_x^{-1}((\mathrm{grad}\, r)_x) \in \mathfrak{p}_{k_0\beta} \cap \mathfrak{p}'$ for some $k_0 \in \mathcal{K} \cup \{0\}$. Let $\widetilde{S}(x, r(x))$ be the hypersphere of radius $r(x)$ in $T_x^\perp B$ centered the origin and $S(x, r(x))$ the geodesic hypersphere of radius $r(x)$ in $F_x^\perp := \exp^\perp(T_x^\perp B)$ centered x. Denote by $dv_{(\cdot)}$ the volume element of the induced metric on $(\cdot)$. Take $v \in T_\xi \widetilde{S}(x, r(x))$. Define a function ψ_r over B by

$$\psi_r(x) := \left(\prod_{k=1}^{2} \left(\frac{\sin(\sqrt{\varepsilon}kbr(x))}{\sqrt{\varepsilon}kb} \right)^{m_k^V} \right) \left(\prod_{k\in\mathcal{K}\setminus\{k_0\}} \cos^{m_k^H}(\sqrt{\varepsilon}kbr(x)) \right) \tag{3.2}$$
$$\times \cos^{m_{k_0}^H - 1}(\sqrt{\varepsilon}k_0 br(x)) \sqrt{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\mathrm{grad}\, r)_x||^2}.$$

We have $r < r_F$ because $\exp^\perp|_{t_r(B)}$ is not an immersion when $r(x) = r_F$ for some $x \in B$. Hence ψ_r is positive by (3.1). Define a function ρ_r over B by

$$
\begin{aligned}
\rho_r(x) := & \frac{\cos(\sqrt{\varepsilon}k_0 br(x))}{\sqrt{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2}} \\
& \times \left(\sum_{k=1}^{2} \frac{m_k^V \sqrt{\varepsilon}kb}{\tan(\sqrt{\varepsilon}kbr(x))} - \sum_{k \in \mathcal{K}} m_k^H \sqrt{\varepsilon}kb \tan(\sqrt{\varepsilon}kbr(x)) \right. \\
& \left. + \frac{||(\operatorname{grad} r)_x||^2 \sqrt{\varepsilon}k_0 b \tan(\sqrt{\varepsilon}k_0 br(x))}{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2} \right).
\end{aligned} \tag{3.3}
$$

According to (2.1), we have

$$
\begin{aligned}
H_\xi = & \rho_r(x) - \frac{(\triangle_F r)(x)}{\cos(\sqrt{\varepsilon}k_0 br(x))\sqrt{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2}} \\
& + \frac{(\nabla^F dr)_x((\operatorname{grad} r)_x, (\operatorname{grad} r)_x)}{\cos(\sqrt{\varepsilon}k_0 br(x))(\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2)^{3/2}}.
\end{aligned} \tag{3.4}
$$

We can derive the following relation for the volume element of M.

Proposition 3.1. *The volume element dv_M is given by*

$$
(dv_M)_\xi = \psi_r(x) \left(((\exp^\perp|_{\widetilde{S}(x,r(x))})^{-1})^* dv_{\widetilde{S}(x,r(x))} \wedge (\pi|_M)^* dv_F \right), \tag{3.5}
$$

where $\xi \in M \cap T_x^\perp B$.

From (3.5), we can derive the following relation for the volume of M.

Proposition 3.2. *The volume* $\mathrm{Vol}(M)$ *of M and the average mean curvature $\overline{H}$ of M are given by*

$$
\mathrm{Vol}(M) = v_{m^V} \int_B r^{m^V} \psi_r dv_F \tag{3.6}
$$

and

$$
\begin{aligned}
\overline{H} = & \frac{1}{\int_B r^{m^V} \psi_r dv_F} \\
& \times \int_{x \in B} r^{m^V} \left(\rho_r(x) - \frac{(\triangle_F r)(x)}{\cos(\sqrt{\varepsilon}k_0 br(x))\sqrt{\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2}} \right. \\
& \left. + \frac{(\nabla^F dr)_x((\operatorname{grad} r)_x, (\operatorname{grad} r)_x)}{\cos(\sqrt{\varepsilon}k_0 br(x))(\cos^2(\sqrt{\varepsilon}k_0 br(x)) + ||(\operatorname{grad} r)_x||^2)^{3/2}} \right) \psi_r \, dv_F,
\end{aligned} \tag{3.7}
$$

respectively, where $\xi \in M \cap T_x^\perp B$ and v_{m^V} is the volume of the m^V-dimensional Euclidean unit sphere.

§ 4. The evolution of the radius function

Let F, B, $M = t_{r_0}(B)$ and f be as in Setting (S) of Introduction. Assume that the volume-preserving mean curvature flow f_t $(t \in [0,T))$ starting from f satisfies the conditions (C1) and (C2). We use the notations in Introduction and Sections 1-3. Denote by $S^\perp B$ the unit normal bundle of of B and $S^\perp_x B$ the fibre of this bundle over $x \in B$. Define a positive-valued function $\widehat{r}_t : M \to \mathbb{R}$ $(t \in [0,T))$ and a map $w^1_t : M \to S^\perp B$ $(t \in [0,T))$ by $f_t(\xi) = \exp^\perp(\widehat{r}_t(\xi) w^1_t(\xi))$ $(\xi \in M)$. Also, define a map $c_t : M \to B$ by $c_t(\xi) := \pi(w^1_t(\xi))$ $(\xi \in M)$ and a map $w_t : M \to T^\perp B$ $(t \in [0,T))$ by $w_t(\xi) := \widehat{r}_t(\xi) w^1_t(\xi)$ $(\xi \in M)$. Here we note that c_t is surjective by the boundary condition in Theorem A, $\widehat{r}_0(\xi) = r_0(\pi(\xi))$ and that $c_0(\xi) = \pi(\xi)$ $(\xi \in M)$. Define a function $\bar{r}_t$ over B by $\bar{r}_t(x) := \widehat{r}_t(\xi)$ $(x \in B)$ and a map $\bar{c}_t : B \to B$ by $\bar{c}_t(x) := c_t(\xi)$ $(x \in B)$, where ξ is an arbitrary element of $M \cap S^\perp_x B$. It is clear that they are well-defined. This map $\bar{c}_t$ is not necessarily a diffeomorphism. In particular, if $\bar{c}_t$ is a diffeomorphism, then $M_t := f_t(M)$ is equal to the tube $\exp^\perp(t_{r_t}(B))$, where $r_t := \bar{r}_t \circ \bar{c}_t^{-1}$. It is easy to show that, if $c_t(\xi_1) = c_t(\xi_2)$, then $\widehat{r}_t(\xi_1) = \widehat{r}_t(\xi_2)$ and $\pi(\xi_1) = \pi(\xi_2)$ hold. Also, let $a : M \times [0,T) \to G$ be a smooth map with $a(\xi,t)p_0 = c_t(\xi)$ $(\xi,t) \in M \times [0,T)$. In this section, we shall calculate the evolution equations for the functions r_t and $\widehat{r}_t$. Define $\widetilde{f} : M \times [0,T) \to \overline{M}$, $r : B \times [0,T) \to \mathbb{R}$, $w^1 : M \times [0,T) \to M$ and $c : M \times [0,T) \to B$ by $\widetilde{f}(\xi,t) := f_t(\xi)$, $r(x,t) := r_t(x)$, $w^1(\xi,t) := w^1_t(\xi)$, $w(\xi,t) := w_t(\xi)$ and $c(\xi,t) := c_t(\xi)$, respectively, where $\xi \in M, x \in B$ and $t \in [0,T)$ (see Figure 7). Fix $(\xi_0,t_0) \in M \times [0,T)$ and set $x_0 := c(\xi_0,t_0)$. According to the condition (C2), we may assume that $\tau_{x_0}^{-1}((\mathrm{grad} r_{t_0})_{x_0})$ belong to $\mathfrak{p}_{k_0\beta}$ for some $k_0 \in \mathcal{K} \cup \{0\}$, where $\mathcal{K}$ and $\mathfrak{p}_{k_0\beta}$ are the quantities defined as in Section 2 for the maximal ableian subspace $\mathfrak{b} := \mathrm{Span}\{\tau_{x_0}^{-1}(w(\xi_0,t_0))\}$ of $\mathfrak{p}'^\perp$.

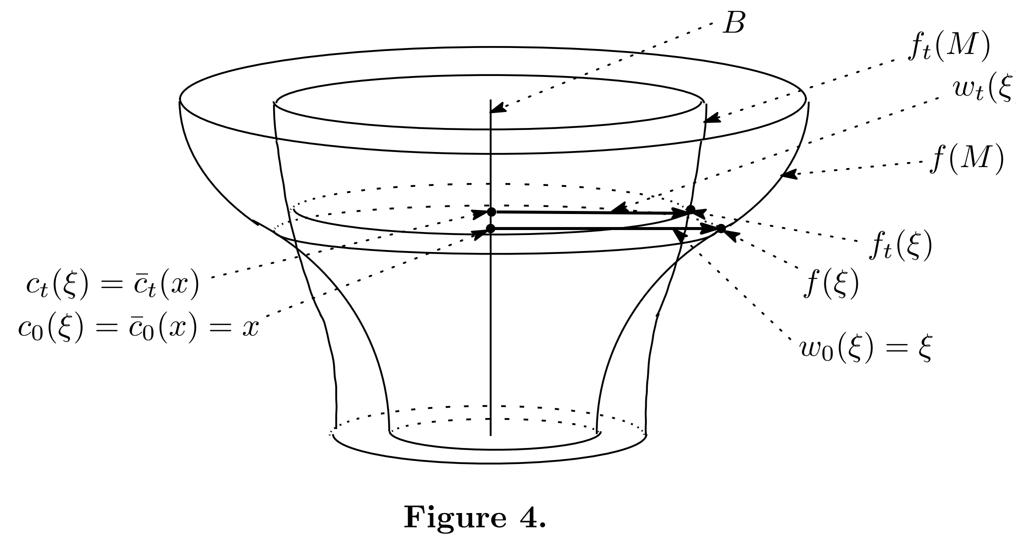

Figure 4.

Notation. Set

$$T_1 := \sup\{t' \in [0,T) \mid M_t := f_t(M)\ (0 \leq t \leq t')\ :\ \text{tubes over } B\}.$$

(Note that $\bar{c}_t$ $(0 \leq t < T_1)$ are diffeomorphisms.)

Assume that $t_0 < T_1$. Denote by $\triangle_F r_t$ the Laplacian of r_t with respect to g_F. Then we have the following evolution equations.

Lemma 4.1. (i) *The radius functions r_t's satisfies the following evolution equation:*

$$\begin{aligned}
&\frac{\partial r}{\partial t}(x,t) - \frac{(\triangle_F r_t)(x)}{\cos^2(\sqrt{\varepsilon}k_0 b r_t(x))}\\
&= \frac{\sqrt{\cos^2(\sqrt{\varepsilon}k_0 b r_t(x)) + ||(\mathrm{grad}\, r_t)_x||^2}}{\cos(\sqrt{\varepsilon}k_0 b r_t(x))} \cdot (\overline{H}_t - \rho_{r_t}(x))\\
&\quad - \frac{(\nabla^F dr_t)_x((\mathrm{grad}\, r_t)_x, (\mathrm{grad}\, r_t)_x)}{\cos^2(\sqrt{\varepsilon}k_0 b r_t(x))(\cos^2(\sqrt{\varepsilon}k_0 b r_t(x)) + ||(\mathrm{grad}\, r_t)_x||^2)}
\end{aligned} \tag{4.1}$$

$((x,t) \in M \times [0,T_1))$.

(ii) *The radius functions $\widehat{r}_t$'s satisfies the following evolution equation:*

$$\begin{aligned}
&\frac{\partial \widehat{r}}{\partial t}(\xi,t) - (\triangle_t \widehat{r}_t)(\xi)\\
&= \frac{\cos(\sqrt{\varepsilon}k_0 b \widehat{r}_t(\xi))(\overline{H}_t - \rho_{r_t}(c(\xi,t)))}{\sqrt{\cos^2(\sqrt{\varepsilon}k_0 b \widehat{r}_t(\xi)) + ||(\mathrm{grad}\, r_t)_{c(\xi,t)}||^2}}\\
&\quad - \frac{\sqrt{\varepsilon}k_0 b \sin(\sqrt{\varepsilon}k_0 b \widehat{r}_t(\xi)) \cos(\sqrt{\varepsilon}k_0 b \widehat{r}_t(\xi)) ||(\mathrm{grad}\, r_t)_{c(\xi,t)}||^2}{(\cos^2(\sqrt{\varepsilon}k_0 b \widehat{r}_t(\xi)) + ||(\mathrm{grad}\, r_t)_{c(\xi,t)}||^2)^2}
\end{aligned} \tag{4.2}$$

$((\xi,t) \in M \times [0,T_1))$.

Replacing $\overline{H}$ in (4.1) to any $C^{1,\alpha/2}$ real-valued function ϕ such that $\phi(0) = \overline{H}(0)$, we obtain a parabolic equation, which has a unique solution r_t such that $\mathrm{grad}\, r_t = 0$ along ∂B in short time for any initial data r_0 such that $\mathrm{grad}\, r_0 = 0$ holds along ∂B. By using a routin fixed point argument (see [M]), we can establish the short time existence and uniqueness also for (4.1) with the same boundary condition. From this fact, we can derive the following statement.

Proposition 4.2. *Under Setting (S), assume that $\overline{M}$ is a rank one symmetric space and that F is an invariant submanifold. Then there uniquely exists the volume-preserving mean curvature flow $f_t : M \hookrightarrow \overline{M}$ starting from f and satisfying the condition (C1) in short time.*

§ 5. The evolution of the gradient of the radius function

We use the notations in Introduction and Sections 1-4. Let T_1 be as in Section 4. Define a function $\widehat{u}_t : M \to \mathbb{R}$ $(t \in [0, T_1))$ by

$$\widehat{u}_t(\xi) := \bar{g}(N_{(\xi,t)}, \tau_{\gamma_{w(\xi,t)}|_{[0,1]}}(w^1(\xi,t))) \quad (\xi \in M)$$

and a map $\widehat{v}_t : M \to \mathbb{R}$ by $\widehat{v}_t := \frac{1}{\widehat{u}_t}$ $(0 \le t < T_1)$. Define a map $\widehat{u} : M \times [0, T_1) \to \mathbb{R}$ by $\widehat{u}(\xi, t) := \widehat{u}_t(\xi)$ $((\xi, t) \in M \times [0, T_1))$ and a map $\widehat{v} : M \times [0, T_1) \to \mathbb{R}$ by $\widehat{v}(\xi, t) := \widehat{v}_t(\xi)$ $((\xi, t) \in M \times [0, T_1))$. Define a function $\bar{u}_t$ (resp. $\bar{v}_t$) over B by $\bar{u}_t(x) := \widehat{u}_t(\xi)$ $(x \in B)$ (resp. $\bar{v}_t(x) := \widehat{v}_t(\xi)$ $(x \in B)$, where ξ is an arbitrary element of $M \cap S^\perp_x B$. It is clear that these functions are well-defined. Set $u_t := \bar{u}_t \circ \bar{c}_t^{-1}$ and $v_t := \bar{v}_t \circ \bar{c}_t^{-1}$. We have only to show $\inf_{(x,t)\in B\times[0,T_1)} u(x, t) > 0$, that is, $\sup_{(x,t)\in B\times[0,T_1)} v(x, t) < \infty$. In the sequel, assume that $t < T_1$. Bet ween $\widehat{u}_t$ and $\operatorname{grad} r_t$, the following relation holds:

$$\widehat{u}_t(\xi) = \frac{\cos(\sqrt{\varepsilon}k_0 b\widehat{r}(\xi,t))}{\sqrt{\cos^2(\sqrt{\varepsilon}k_0 b\widehat{r}(\xi,t)) + ||(\operatorname{grad} r_t)_{c(\xi,t)}||^2}}. \tag{5.1}$$

We can derive the following evolution equations for $\widehat{u}_t$ and $\widehat{v}_t$.

Lemma 5.1. *The functions $\widehat{u}_t$'s $(t \in [0, T))$ satisfy the following evolution equation:*

$$\begin{aligned}
&\frac{\partial \widehat{u}}{\partial t}(\xi,t) - (\triangle_t \widehat{u}_t)(\xi)\\
&= \overline{H}_t \sqrt{\varepsilon}k_0 b \tan(\sqrt{\varepsilon}k_0 b\widehat{r}(\xi,t))(1 - \widehat{u}(\xi,t)^2)\\
&\quad -\widehat{u}(\xi,t)(1-\widehat{u}(\xi,t)^2)\sum_{k=1}^{2}\frac{m_k^V(\sqrt{\varepsilon}kb)^2}{\sin^2(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}\\
&\quad -\widehat{u}(\xi,t)(1-\widehat{u}(\xi,t)^2)\sqrt{\varepsilon}k_0 b\tan(\sqrt{\varepsilon}k_0 b\widehat{r}(\xi,t))\sum_{k=1}^{2}\frac{m_k^V\sqrt{\varepsilon}kb}{\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}\\
&\quad -\widehat{u}(\xi,t)(1-\widehat{u}(\xi,t)^2)\sqrt{\varepsilon}k_0 b\tan(\sqrt{\varepsilon}k_0 b\widehat{r}(\xi,t))\\
&\qquad \times\sum_{k\in\mathcal{K}} m_k^H\sqrt{\varepsilon}kb\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))\\
&\quad +\widehat{u}(\xi,t)\left(\lambda_t(\xi) + \widehat{u}(\xi,t)\sum_{k\in\mathcal{K}} m_k^H\sqrt{\varepsilon}kb\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))\right)^2\\
&\quad +\widehat{u}(\xi,t)(1-\widehat{u}(\xi,t)^2)\sum_{k\in\mathcal{K}}\frac{m_k^H(\sqrt{\varepsilon}kb)^2||(\operatorname{grad} r_t)_{c(\xi,t)}||}{\cos^2(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}
\end{aligned} \tag{5.2}$$

$((\xi, t) \in M \times [0, T_1))$. Also, the functions $\widehat{v}_t$'s $(t \in [0, T))$ satisfy the following evolution

equation:

$$
\begin{aligned}
&\frac{\partial \widehat{v}}{\partial t}(\xi,t)-(\triangle_t\widehat{v}_t)(\xi)\\
&=-\overline{H}_t\sqrt{\varepsilon}k_0b\tan(\sqrt{\varepsilon}k_0b\widehat{r}(\xi,t))(\widehat{v}(\xi,t)^2-1)\\
&\quad+\widehat{v}(\xi,t)\left(1-\frac{1}{\widehat{v}(\xi,t)^2}\right)\sum_{k=1}^{2}\frac{m_k^V(\sqrt{\varepsilon}kb)^2}{\sin^2(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}\\
&\quad+\widehat{v}(\xi,t)\left(1-\frac{1}{\widehat{v}(\xi,t)^2}\right)\sqrt{\varepsilon}k_0b\tan(\sqrt{\varepsilon}k_0b\widehat{r}(\xi,t))\sum_{k=1}^{2}\frac{m_k^V\sqrt{\varepsilon}kb}{\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}\\
&\quad-\widehat{v}(\xi,t)\left(1-\frac{1}{\widehat{v}(\xi,t)^2}\right)\sqrt{\varepsilon}k_0b\tan(\sqrt{\varepsilon}k_0b\widehat{r}(\xi,t))\\
&\qquad\times\sum_{k\in\mathcal{K}}m_k^H\sqrt{\varepsilon}kb\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))\\
&\quad-\widehat{v}(\xi,t)\left(\lambda_t(\xi)+\frac{1}{\widehat{v}(\xi,t)}\sum_{k\in\mathcal{K}}m_k^H\sqrt{\varepsilon}kb\tan(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))\right)^2\\
&\quad-\widehat{v}(\xi,t)\left(1-\frac{1}{\widehat{v}(\xi,t)^2}\right)\sum_{k\in\mathcal{K}}\frac{m_k^H(\sqrt{\varepsilon}kb)^2||(\mathrm{grad}\,r_t)_{c(\xi,t)}||}{\cos^2(\sqrt{\varepsilon}kb\widehat{r}(\xi,t))}\\
&\quad-\frac{2}{\widehat{v}(\xi,t)}||(\mathrm{grad}_t\widehat{v}_t)_\xi||^2
\end{aligned}
\tag{5.3}
$$

$((\xi,t)\in M\times[0,T_1))$.

§ 6. Estimate of the volume

We use the notations in Introduction and Sections 1-5. Assume that $\overline{M}$ is a rank one symmetric space, F is invariant, B is a closed geodesic ball of radius r_B centered at x_* in F and that r_0 is radial with respect to x_*, where r_B is a positive number smaller than the injective radius of F at x_*. Note that the invariantness of F means the totally geodesicness in the case where $\overline{M}$ is a sphere or a (real) hyperbolic space. Since F is of rank one, each geodesic sphere in F is homogeneous. Hence, since r_0 is radial by the assumption, so are also r_t. For $X\in\widetilde{S}'(x_*,1)$, denote by γ_X the geodesic in F having X as the initial velocity vector (i.e., $\gamma_X(z)=\exp^F_{x_*}(zX)$). Then, since r_t is radial, it is described as $r_t(\gamma_X(z))=r_t^\circ(z)$ $(X\in\widetilde{S}'(x_*,1),\ z\in[0,r_B))$ for some function r_t° over $[0,r_B)$, where $\widetilde{S}'(x_*,1)$ denotes the unit geodesic sphere in F centered x_* and r_B denotes the radius of B. Then we have

$$
\begin{aligned}
&(\mathrm{grad}\,r_t)_{\gamma_X(z)}=(r_t^\circ)'(z)\gamma_X'(z),\quad (\triangle_F r_t)(\gamma_X(z))=(r_t^\circ)''(z)\\
&\text{and}\quad (\nabla^F dr_t)((\mathrm{grad}\,r_t)_{\gamma_X(z)},(\mathrm{grad}\,r_t)_{\gamma_X(z)})=(r_t^\circ)'(z)^2(r_t^\circ)''(z).
\end{aligned}
\tag{6.1}
$$

Since F is invariant, we have $\mathcal{K}=\{1\}$ (hence $k_0=1$) and

$$
r_{fo}(\gamma)=r_{co}(\gamma)=\begin{cases}\dfrac{\pi}{2b}\ (\varepsilon=1)\\ \infty\ (\varepsilon=-1)\end{cases}
$$

for any normal geodesic γ of F and hence

$$r_F = \begin{cases} \frac{\pi}{2b} \,(\varepsilon = 1) \\ \infty \,(\varepsilon = -1), \end{cases}$$

where $r_{fo}(\gamma), r_{co}(\gamma)$ and r_F are as in Introduction. Denote by ∇^t the Levi-Civita connection of g_t. In the sequel, assume that $t < T_1$. In this section, we estimate the volume of M_t from below in terms of the infimum and the maximum of the radius function r_t. Furthermore, we show that r_t and the average mean curvature $\overline{H}_t$ are uniform bounded in terms of the estimate. Denote by $\pi_F^\perp$ the bundle projection of the normal bundle $T^\perp F$ of F. Set

$$\widetilde{W} := \{\xi \,|\, \xi \in T^\perp F \text{ s.t. } ||\xi|| < r_F\}$$

and $W := \exp^\perp(\widetilde{W})$. Define a submersion pr_F of W onto F by $\mathrm{pr}_F(\exp^\perp(\xi)) := \pi_F^\perp(\xi)$, where $\xi \in \widetilde{W}$. Let $\widetilde{r} : \overline{M} \to \mathbb{R}$ be the distance function from F, where we note that $\widetilde{r}(\exp^\perp(\xi)) = ||\xi||$ holds for $\xi \in \widetilde{W}$. Define a function $\overline{\psi}$ over $[0, r_F)$ by

$$\overline{\psi}(s) := \left(\prod_{k=1}^{2} \left(\frac{\sin(\sqrt{\varepsilon}kbs)}{\sqrt{\varepsilon}kbs} \right)^{m_k^V} \right) \cos^{m^H}(\sqrt{\varepsilon}bs).$$

Since $r_F = \frac{\pi}{2\sqrt{\varepsilon}b}$, $\overline{\psi}$ is positive. Define a function δ_1 and δ_2 over $[0, r_F)$ by

$$\delta_1(s) := \int_0^s s^{m^V} \overline{\psi}(s) ds$$

and

$$\delta_2(s) := \int_0^s \frac{s^{m^V} \overline{\psi}(s)}{\cos(\sqrt{\varepsilon}bs)} ds,$$

respectively. According to (6.3) below, we have $\frac{\mathrm{Vol}(D_t)}{v_{m^V} \mathrm{Vol}(B)} \in \delta_1([0, r_F))$. Since r_B is smaller than the injective radius of F , $\overline{\psi}(s)$ and $\frac{\overline{\psi}(s)}{\cos(\sqrt{\varepsilon}bs)}$ are positive over $[0, r_B)$. Hence δ_i $(i = 1, 2)$ are increasing. Set

$$\hat{r}_1 := \delta_1^{-1} \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right)$$

and

$$\hat{r}_2 := \delta_2^{-1} \left(\frac{\mathrm{Vol}(M)}{v_{m^V}} + \delta_2(\hat{r}_1) \right).$$

Denote by $(r_t)_{\max}$ (resp. $(r_t)_{\min}$) the maximum (resp. the minimum) of r_t. Then we have

$$\begin{aligned} &\delta_1(\widehat{r}_1) v_{m^V} \mathrm{Vol}(B) = \mathrm{Vol}(D_t) \\ &= v_{m^V} \int_{x \in B} \delta_1(r_t(x))\, dv_F \begin{cases} \geq v_{m^V} \mathrm{Vol}(B) \delta_1((r_t)_{\min}) \\ \leq v_{m^V} \mathrm{Vol}(B) \delta_1((r_t)_{\max}) \end{cases} \end{aligned}$$

and hence $\delta_1((r_t)_{\min}) \leq \delta_1(\widehat{r}_1) \leq \delta_1((r_t)_{\max})$, that is, $(r_t)_{\min} \leq \widehat{r}_1 \leq (r_t)_{\max}$. Set

$$a_{r_B} := \begin{cases} \prod_{k=1}^{2} \left(\dfrac{\sin(\sqrt{\varepsilon} k b r_B)}{\sqrt{\varepsilon} k b} \right)^{m_k^H} & (\overline{M} \ : \text{ of compact type}) \\ 1 & (\overline{M} \ : \text{ of noncompact type}). \end{cases}$$

Then we can estimate the volume of M_t from below as follows:

$$\begin{aligned} \mathrm{Vol}(M_t) &\geq a_{r_B} v_{m^V} v_{m^H-1} \int_{(r_t)_{\min}}^{(r_t)_{\max}} \frac{s^{m^V} \overline{\psi}(s)}{\cos(\sqrt{\varepsilon} bs)} \, ds \\ &= a_{r_B} v_{m^V} v_{m^H-1} (\delta_2((r_t)_{\max}) - \delta_2((r_t)_{\min})). \end{aligned} \tag{6.2}$$

On the other hand, since $\mathrm{Vol}(D_t)$ preserves invariantly along the volume-preserving mean curvature flow and $\mathrm{Vol}(M_t)$ is decreasing along the flow, we have $\mathrm{Vol}(D_t) = \mathrm{Vol}(D_0)$ and $\mathrm{Vol}(M_t) \leq \mathrm{Vol}(M_0)$. Hence we have

$$\begin{aligned} (r_t)_{\max} &\leq \delta_2^{-1} \left(\delta_2(\hat{r}_1) + \frac{\mathrm{Vol}(M_0)}{a_{r_B} v_{m^V} v_{m^H-1}} \right) \\ &\leq \delta_2^{-1} \left(\frac{\mathrm{Vol}(D_0)}{v_{m^V} \mathrm{Vol}(B)} + \frac{\mathrm{Vol}(M_0)}{a_{r_B} v_{m^V} v_{m^H-1}} \right). \end{aligned}$$

Thus we obtain the following result.

Proposition 6.1. *The family $\{r_t\}_{t \in [0,T_1)}$ is uniform bounded as follows:*

$$\sup_{(x,t) \in B \times [0,T_1)} r_t(x) \leq \delta_2^{-1} \left(\delta_2(\hat{r}_1) + \frac{\mathrm{Vol}(M_0)}{a_{r_B} v_{m^V} v_{m^H-1}} \right). \tag{6.3}$$

For uniform boundedness of the average mean curvatures $|\overline{H}|$, we have the following result.

Proposition 6.2. *If $0 < a_1 \leq r_t(x) \leq a_2 < r_F$ holds for all $(x,t) \in M \times [0,T_0]$ $(T_0 < T_1)$, then $\max_{t \in [0,T_0]} \overline{H}_t \leq C(a_1,a_2)$ holds for some constant $C(a_1,a_2)$ depending only on a_1 and a_2.*

For uniform positivity of the average mean curvatures $|\overline{H}|$, we have the following result.

Proposition 6.3. *Assume that $\overline{M}$ is of non-compact type. If $r_t(x) \geq a > 0$ holds for all $(x,t) \in M \times [0,T_0]$ $(T_0 < T_1)$, then $\min_{t \in [0,T_0]} \overline{H}_t \geq \widehat{C}(a)$ holds for some constant $\widehat{C}(a)$ depending only on a.*

§ 7. Proof of Theorem A

In this section, we shall prove Theorem A in terms of the evolution equation (5.3) of $\widehat{v}_t$ and Propositions 6.2 and 6.3.

Proof of Theorem A. Suppose that $T_1 < T$. Take any $t_0 \in (T_1, T)$. Set

$$\beta_1(t_0) := \min_{(x,t)\in B\times[0,t_0]} r_t(x)\ (>0) \ \text{ and } \ \beta_2(t_0) := \max_{(x,t)\in B\times[0,t_0]} r_t(x)\ (<\infty).$$

According to Propostions 6.2 and 6.3, we have

$$0 < \widehat{C}(\beta_1(t_0)) < \overline{H}_t < C(\beta_1(t_0), \beta_2(t_0)) \qquad (t \in [0, T_1)),$$

where we note that $r_F = \infty$ because $\overline{M}$ is of non-compact type. By using the evolution equation (5.3) for $\widehat{v}_t$, this inequality, $(1 - 1/\widehat{v}_t^2)\widehat{v}_t \le \widehat{v}_t$ and $\widehat{v}_t \ge 1$, we can derive

$$\begin{aligned}
&\frac{\partial \widehat{v}}{\partial t}(\xi, t) - (\triangle_t \widehat{v}_t)(\xi) \\
&\le \widehat{v}(\xi,t)\left(\sum_{k=1}^{2} \frac{m_k^V (kb)^2}{\sinh^2(kb\widehat{r}(\xi,t))} + b\tanh(b\widehat{r}(\xi,t)) \sum_{k=1}^{2} \frac{m_k^V kb}{\tanh(kb\widehat{r}(\xi,t))}\right) \\
&\quad -\widehat{v}(\xi,t)^2 \widehat{C}(\beta_1(t_0)) b\tanh(b\widehat{r}(\xi,t)) + C(\beta_1(t_0), \beta_2(t_0))
\end{aligned} \tag{7.1}$$

$(t \in [0, T_1))$. For simplicity, set

$$K_1(\beta_1(t_0), \beta_2(t_0)) := \sum_{k=1}^{2} \frac{m_k^V (kb)^2}{\sinh^2(kb\beta_1(t_0))} + b\tanh(b\beta_2(t_0)) \sum_{k=1}^{2} \frac{m_k^V kb}{\tanh(kb\beta_1(t_0))}$$

and

$$K_2(\beta_1(t_0)) := \widehat{C}(\beta_1(t_0)) b\tanh(b\beta_1(t_0)).$$

Take any $t_1 \in [0, T_1)$. Let $(\xi_2, t_2) \in M \times [0, t_1]$ be a point attaining the maximum of $\widehat{v}$ over $M \times [0, t_1]$. Since $\widehat{v}_{t_2} = 1$ along ∂M, (ξ_2, t_2) belongs to the interior of $M \times [0, t_1]$. Then we have $\frac{\partial \widehat{v}}{\partial t}(\xi_2, t_2) = 0$ and $(\triangle_{t_2} \widehat{v}_{t_2})(\xi_2) \le 0$, that is, $\frac{\partial \widehat{v}}{\partial t}(\xi_2, t_2) - (\triangle_{t_2} \widehat{v}_{t_2})(\xi_2) \ge 0$. Hence, from (7.1), we can derive

$$\begin{aligned}
&\max_{(\xi,t)\in M\times[0,t_1]} \widehat{v}(\xi,t) = \widehat{v}(\xi_2, t_2) \\
&\le \frac{K_1(\beta_1(t_0), \beta_2(t_0)) + \sqrt{K_1(\beta_1(t_0), \beta_2(t_0))^2 + 4K_2(\beta_1(t_0)) C(\beta_1(t_0), \beta_2(t_0))}}{2}.
\end{aligned}$$

From the arbitrariness of t_1, we obtain

$$\begin{aligned}
&\sup_{(\xi,t)\in M\times[0,T_1)} \widehat{v}(\xi,t) \\
&\le \frac{K_1(\beta_1(t_0), \beta_2(t_0)) + \sqrt{K_1(\beta_1(t_0), \beta_2(t_0))^2 + 4K_2(\beta_1(t_0)) C(\beta_1(t_0), \beta_2(t_0))}}{2},
\end{aligned}$$

which implies that M_{T_1} is a tube over B and that, furthermore, so are also M_t's ($t \in [T_1, T_1+\varepsilon)$) for a sufficiently small positive number ε (smaller than $t_0 - T_1$). This contradicts the definition of T_1. Therefore we obtain $T_1 = T$. That is, M_t ($t \in [0,T)$) remain to be tubes over B. q.e.d.

§ 8. Proofs of Theorem B and C

We use the notations in Introduction and Sections 1-7. Assume that $\overline{M}$ is a rank one symmetric space of non-compact type other than a (real) hyperbolic space and r_0 is radial (hence so are lso r_t's ($0 \le t < T$)). Set $r_{\inf} := \inf_{(x,t)\in B\times[0,T)} r_t(x)$ and $r_{\sup} := \sup_{(x,t)\in B\times[0,T)} r_t(x)$. In this section, we shall prove Theorems B and C. Since $\overline{M}$ is a symmetric space, we have $\overline{\nabla}\,\overline{R} = 0$. Hence, according to (6.1) in [CM2], we obtain the following evolution equation for $||A_t||_t^2$.

Lemma 8.1([CM2]). *The family* $\{||A_t||_t^2\}_{t\in[0,T)}$ *satisfies*

$$\begin{aligned}
&\frac{\partial ||A_t||_t^2}{\partial t} - \triangle_t ||A_t||_t^2\\
&= -2||\nabla^t A_t||_t^2 + 2||A_t||_t^2(||A_t||_t^2 + \overline{\mathrm{Ric}}(N_t, N_t))\\
&\quad -2\overline{H}_t\left(\mathrm{Tr}\, A_t^3 + Tr^{\bullet}_{g_t}\overline{R}(A_t(\bullet), N_t, N_t, \bullet)\right)\\
&\quad -4\mathrm{Tr}^{\bullet}_{g_t}\mathrm{Tr}^{\cdot}_{g_t}\overline{R}(\cdot, A_t(\bullet), A_t(\bullet), \cdot) + 4\mathrm{Tr}^{\bullet}_{g_t}\mathrm{Tr}^{\cdot}_{g_t}\overline{R}(\cdot, A_t(\bullet), \bullet, A_t(\cdot)).
\end{aligned}$$

Set $\kappa := \frac{1}{2\sup_{(\xi,t)\in M\times[0,T)} \widehat{v}(\xi,t)^2}$. According to Theorem A, we have $\kappa < \infty$. Define a function φ over $[0, 1/\sqrt{\kappa})$ by $\varphi(s) := \dfrac{s^2}{1-\kappa s^2}$ and a function Φ_t over M by

$$\Phi_t(\xi) := (\varphi\circ\widehat{v}_t)(\xi)||(A_t)_\xi||_t^2 \quad (\xi \in M).$$

Also, define a function Φ over $M\times[0,T)$ by $\Phi(\xi,t) := \Phi_t(\xi)$ ($(\xi,t)\in M\times[0,T)$). By a simple calculation, we can derive the following evolution equation for Φ_t's ($t\in[0,T)$).

Lemma 8.2. *The family* $\{\Phi_t\}_{t\in[0,T)}$ *satisfies*

$$\begin{aligned}
&\frac{\partial\,\Phi}{\partial t} - \triangle_t\Phi_t\\
&= (\varphi'\circ\widehat{v}_t)||A_t||_t^2\left(\frac{\partial\,\widehat{v}}{\partial t} - \triangle_t\widehat{v}_t\right) + (\varphi\circ\widehat{v}_t)\left(\frac{\partial\,||A_t||_t^2}{\partial t} - \triangle_t||A_t||_t^2\right)\\
&\quad -(\varphi''\circ\widehat{v}_t)||A_t||_t^2||\mathrm{grad}_t\widehat{v}||_t^2 - 2g_t(\mathrm{grad}_t((\varphi\circ\widehat{v}_t), \mathrm{grad}_t||A_t||_t^2).
\end{aligned}$$

By using these lemmas, (5.3), Propositions 6.1, 6.2 and 6.3, we shall prove Theorem B.

Proof of Theorem B. Suppose that $\inf_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)>0$ and $T<\infty$. Then we suffice to show that $T=\infty$ and that M_t converges to a tube of constant mean curvature over B (in C^∞-topology) as $t\to\infty$. Set $a_1:=\inf_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)$ and $a_2:=\sup_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)$. According to Proposition 6.1, we have

$$a_2\leq\delta_2^{-1}\left(\delta_2(\widehat{r}_1)+\frac{\mathrm{Vol}(M_0)}{a_{r_B}v_{m^V}v_{m^H-1}}\right).$$

According to Propositions 6.2 and 6.3, we have

$$0<\widehat{C}(a_1)\leq\overline{H}_t\leq C(a_1,a_2)<\infty\qquad(t\in[0,T)).\tag{8.1}$$

According to the first inequality in Page 555 in [EH] (see also (6.4) and (6.5) of P502 in [CM2]), we have

$$\begin{aligned}&-2g_t(\mathrm{grad}_t(\varphi\circ\widehat{v}_t),\,\mathrm{grad}_t(||A_t||_t^2))\\&\leq-\frac{1}{\varphi\circ\widehat{v}_t}g_t(\mathrm{grad}_t\Phi_t,\mathrm{grad}_t(\varphi\circ\widehat{v}_t))+2(\varphi\circ\widehat{v}_t)||\nabla^tA_t||_t^2\\&\quad+\frac{3}{2(\varphi\circ\widehat{v}_t)}||A_t||_t^2||\mathrm{grad}_t(\varphi\circ\widehat{v}_t)||_t^2.\end{aligned}\tag{8.2}$$

Also, according to Kato's inequality, we have

$$||\mathrm{grad}_t||A_t||_t\,||_t^2\leq||\nabla^tA_t||_t^2.\tag{8.3}$$

By using these relations, (5.3), Lemmas 8.1 and 8.2, we can derive

$$\begin{aligned}&\frac{\partial\,\Phi}{\partial t}-\triangle_t\Phi_t\\&\leq(\varphi'\circ\widehat{v}_t)||A_t||_t^2\left\{K_1(a_1,a_2)\widehat{v}_t-K_2(a_1)\widehat{v}_t^2-\frac{2}{\widehat{v}_t}||\mathrm{grad}_t\widehat{v}_t||_t^2\right.\\&\qquad\qquad\left.-\widehat{v}_t\left(\lambda_t+\frac{m^Hb\tanh(b\widehat{r}_t)}{\widehat{v}_t}\right)^2\right\}\\&\quad+2(\varphi\circ\widehat{v}_t)||A_t||_t^4+2(\varphi\circ\widehat{v}_t)||A_t||_t^2\overline{\mathrm{Ric}}(N_t,N_t)\\&\quad-2(\varphi\circ\widehat{v}_t)\overline{H}_t\cdot\mathrm{Tr}\,A_t^3-2(\varphi\circ\widehat{v}_t)\overline{H}_t\mathrm{Tr}^\bullet_{g_t}\overline{R}(A_t(\bullet),N_t,N_t,\bullet)\\&\quad-4(\varphi\circ\widehat{v}_t)\mathrm{Tr}^\bullet_{g_t}\mathrm{Tr}^\cdot_{g_t}\overline{R}(\cdot,A_t(\bullet),A_t(\bullet),\cdot)\\&\quad+4(\varphi\circ\widehat{v}_t)\mathrm{Tr}^\bullet_{g_t}\mathrm{Tr}^\cdot_{g_t}\overline{R}(\cdot,A_t(\bullet),\bullet,A_t(\cdot))\\&\quad-(\varphi''\circ\widehat{v}_t)||A_t||_t^2\cdot||\mathrm{grad}_t\widehat{v}_t||_t^2-\frac{1}{\varphi\circ\widehat{v}_t}g_t(\mathrm{grad}_t\Phi_t,\mathrm{grad}_t(\varphi\circ\widehat{v}_t))\\&\quad+\frac{3}{2(\varphi\circ\widehat{v}_t)}||A_t||_t^2\cdot||\mathrm{grad}_t(\phi\circ\widehat{v}_t)||_t^2\end{aligned}\tag{8.4}$$

According to (8.1), $\{\tanh(b\widehat{r}_t)\}_{t\in[0,T)}$ and $\{\overline{\mathrm{Ric}}(N_t,N_t)\}_{t\in[0,T)}$ are uniform bounded. Hence, by the same discussion as in the second-half part in the proof of Theorem A of

[CM2] (see Line 15 of Page 502-Line 14 of Page 503), we can derive

$$\begin{aligned}\frac{\partial\Phi}{\partial t}-\triangle_t\Phi_t\leq&-\frac{\kappa}{2}\Phi_t^2+K_3\Phi_t+K_4\sqrt{\Phi_t}\\&-\frac{1}{\varphi\circ\widehat{v}_t}g_t(\mathrm{grad}_t\Phi_t,\,\mathrm{grad}_t(\varphi\circ\widehat{v}_t))\end{aligned}\tag{8.5}$$

for some positive constants K_3 and K_4. Take any $t_0\in[0,T)$. Let (ξ_1,t_1) be a point attainning $\max_{(\xi,t)\in M\times[0,t_0]}\Phi_t(\xi)$. We consider the case where ξ_1 belongs to the boundary of M. Define a radial function $\widetilde{r}_t$ over the geodesic ball $\widetilde{B}$ of radius $2r_B$ centered at x_* in F by

$$\widetilde{r}_t(\gamma_X(z)):=\begin{cases}r_t^\circ(z) & (0\leq z\leq r_B)\\ r_t^\circ(2r_B-z) & (r_B\leq z\leq 2r_B)\end{cases}$$

Set $\widetilde{M}:=t_{\widetilde{r}_0}(\widetilde{B})$, which includes M. Then it is easy to show that the volume-preserving mean curvature flow $\widetilde{f}_t$ $(t\in[0,T))$ starting from $\widetilde{M}$ satisfies $\widetilde{f}_t(\widetilde{M})=\exp^\perp(t_{\widetilde{r}_t}(\widetilde{B}))$. Also, it is clear that (ξ_1,t_1) is a point attainning $\max_{(\xi,t)\in\widetilde{M}\times[0,t_0]}\widetilde{\Phi}_t(\xi)$ and that ξ_1 belongs to the interior of $\widetilde{M}$, where $\widetilde{\Phi}_t$ is the function defined for $\widetilde{f}_t$ similar to Φ_t. Thus we may assume that ξ_1 belongs to the interior of M without loss of generality. Set $b_1:=\Phi_{t_1}(\xi_1)$. Assume that $b_1>1$. Then, it follows from (8.5) that

$$0\leq\frac{\partial\Phi}{\partial t}(\xi_1,t_1)-(\triangle_{t_1}\Phi_{t_1})(\xi_1)\leq-\frac{\kappa b_1^2}{2}+(K_3+K_4)b_1,$$

that is, $b_1\leq\frac{2}{\kappa}(K_3+K_4)$. Hence we obtain

$$\max_{(\xi,t)\in M\times[0,t_0]}\Phi_t(\xi)=b_1\leq\max\left\{1,\,\frac{2}{\kappa}(K_3+K_4)\right\}.$$

Furthermore, by the arbitariness of t_0, we obtain

$$\sup_{(\xi,t)\in M\times[0,T)}\Phi_t(\xi)\leq\max\left\{1,\,\frac{2}{\kappa}(K_3+K_4)\right\}.$$

Thus $\{\Phi_t\}_{t\in[0,T)}$ is uniform bounded. On the other hand, we have $\varphi\circ\widehat{v}_t\geq 2\widehat{v}_t^2\geq 2$. From these facts, it follows that $\{||A_t||_t^2\}_{t\in[0,T)}$ is uniform bounded. Hence, by the discussion in [Hu1,2], it is shown that $\{||(\nabla^t)^jA_t||_t^2\}_{t\in[0,T)}$ also is uniform bounded for any positive integar j, where we use $T<\infty$ and $g_t(\mathrm{grad}_t||(\nabla^t)^jA_t||_t^2,\,N_t)$
$=0$ along ∂B (This fact holds because $\mathrm{grad}_tr_t=0$ along ∂B by the assumption). Hence, since $\{f_t\}_{t\in[0,T)}$ is equicontinuous in C^∞-norm, there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $\lim\limits_{k\to\infty}t_k=T$ and that $\{f_{t_k}\}_{k=1}^\infty$ converges to some C^∞-embedding f_T in the C^∞-topology by Arzelá-Ascoli's theorem. By the standard discussion as in [CM1, P2075-2077], it is shown that $f_T(M)$ is of constant mean curvature. On the other hand,

according to (4.17), we have

$$\begin{aligned}\frac{\partial r}{\partial t} &= \frac{\triangle_F r_t}{\cosh^2(br_t)+\|\mathrm{grad}\, r_t\|^2} + \frac{\sqrt{\cosh^2(br_t)+\|\mathrm{grad}\, r_t\|^2}}{\cosh(br_t)}(\overline{H}_t-\rho_{r_t})\\ &= \frac{\sqrt{\cosh^2(br_t)+\|\mathrm{grad}\, r_t\|^2}}{\cosh(br_t)}(\overline{H}_t-H_t).\end{aligned}$$

Hence, for any τ_1 and τ_2 of $[0,T)$ ($\tau_1<\tau_2$), we have

$$\begin{aligned}&\max_{x\in B}|r_{\tau_1}(x)-r_{\tau_2}(x)| \le \max_{x\in M}\int_{\tau_1}^{\tau_2}\left|\frac{\partial r}{\partial t}\right| dt\\ &\le \max_{x\in M}\int_{\tau_1}^{\tau_2}|\overline{H}_t-\rho_{r_t}|\sqrt{\cosh^2(br_t)+\|\mathrm{grad}\, r_t\|^2}\, dt\\ &\le C\varepsilon(t)(\tau_2-\tau_1),\end{aligned}$$

where t is an element of (τ_1,τ_2) and $\varepsilon(t)\to 0$ as $t\to T$. From this fact, we can show that r_t converges to r_T in the C^∞-topology by the discussion as in [Hu3], where r_T is the radius function of f_T. Thus f_t converges to f_T in the C^∞-topology as $t\to T$. It is clear that $f_T(M)$ also is a tube over B. Hence, according to Proposition 4.2, there exists the volume-preserving mean curvature flow starting from f_T in short time. Hence the volume-preserving mean curvature flow f_t starting from M is continued after T. This contradicts the definition of T. Therefore $T=\infty$ or $\inf_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)=0$ holds. Furthermore, in the case of $T=\infty$, we can show that f_t converges to a tube of constant mean curvature over B in C^∞-topology as $t\to\infty$ by imitating the discussion in P2075-2080 of [CM1]. q.e.d.

Next we shall prove Theorem C.

Proof of Theorem C. Suppose that $\inf_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)=0$. Then there exists a sequence $\{(\xi_k,t_k)\}_{k=1}^{\infty}$ such that $\widehat{r}(\xi_k,t_k)<\frac{1}{k}$ ($k\in\mathbb{N}$). By using (6.2) and $(r_t)_{\min}\le \widehat{r}_1\le (r_t)_{\max}$, we have

$$\begin{aligned}\mathrm{Vol}(M_0)\ge\mathrm{Vol}(M_{t_k}) &\ge a_{r_B}v_{m^V}v_{m^H-1}\,(\delta_2((r_{t_k})_{\max})-\delta_2((r_{t_k})_{\min}))\\ &> v_{m^V}v_{m^H-1}\,(\delta_2(\widehat{r}_1)-\delta_2(1/k))\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Vol}(M_0) &\ge \lim_{k\to\infty}v_{m^V}v_{m^H-1}\,(\delta_2(\widehat{r}_1)-\delta_2(1/k)) = v_{m^V}v_{m^H-1}\delta_2(\widehat{r}_1)\\ &= v_{m^V}v_{m^H-1}(\delta_2\circ\delta_1^{-1})\left(\frac{\mathrm{Vol}(D)}{v_{m^V}\mathrm{Vol}(B)}\right),\end{aligned}$$

where we note that $a_{r_B}=1$ because $\overline{M}$ is of non-compact type. This contradicts the assumption. Hence we obtain $\inf_{(\xi,t)\in M\times[0,T)}\widehat{r}(\xi,t)>0$. Therefore, by using Theorem

B, we can derive that $T = \infty$ and that f_t converges to a tube of constant mean curvature over B in the C^∞-topology as $t \to \infty$. q.e.d.

References

[A1] M. Athanassenas, Volume-preserving mean curvature flow of rotationally symmetric surfaces, Comment. Math. Helv. **72** (1997) 52-66.

[A2] M. Athanassenas, Behaviour of singularities of the rotationally symmetric, volume-preserving mean curvature flow, Calc. Var. **17** (2003) 1-16.

[BV] J. Berndt and L. Vanhecke, Curvature-adapted submanifolds, Nihonkai Math. J. **3** (1992) 177-185.

[BT] J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit, Tohoku Math. J. (2) **56** (2004), 163–177.

[CM1] E. Cabezas-Rivas and V. Miquel, Volume preserving mean curvature flow in the hyperbolic space, Indiana Univ. Math. J. **56** (2007) 2061-2086.

[CM2] E. Cabezas-Rivas and V. Miquel, Volume-preserving mean curvature flow of revolution hypersurfaces in a rotationally symmetric space, Math. Z. **261** (2009) 489-510.

[CM3] E. Cabezas-Rivas and V. Miquel, Volume-preserving mean curvature flow of revolution hypersurfaces between two equidistants, Calc. Var. **43** (2012) 185-210.

[CN] B. Y. Chen and T. Nagano, Totally geodesic submanidfold in symmetric spaces II, Duke Math. J. **45** (1978) 405-425.

[EH] K. Ecker and G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, Invent. Math. **105** (1991) 547-569.

[He] S. Helgason, Differential geometry, Lie groups and symmetric spaces, Academic Press, New York, 1978.

[Hu1] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. **20** (1984) 237-266.

[Hu2] G. Huisken, Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, Invent. math. **84** (1986) 463-480.

[Hu3] G. Huisken, Parabolic Monge-Ampere equations on Riemannian manifolds, J. Funct. Anal. **147** (1997) 140-163.

[KT] T. Kimura and M. S. Tanaka, Stability of certain minimal submanifolds in compact symmetric spaces of rank two, Differential Geom. Appl. **27** (2009) 23–33.

[Ko1] N. Koike, Tubes of non-constant radius in symmetric spaces, Kyushu J. Math. **56** (2002) 267-291.

[Ko2] N. Koike, Collapse of the mean curvature flow for equifocal submanifolds, Asian J. Math. **15** (2011) 101-128.

[Ko3] N. Koike, Collapse of the mean curvature flow for isoparametric submanifolds in a symmetric space of non-compact type, Kodai Math. J. **37** (2014) 355-382.

[Ko4] N. Koike, Volume-preserving mean curvature flow for tubes in rank one symmetric spaces of non-compact type, Calc. Var. Partial Differ. Equ. **56** (2017) 51 pages.

[L1] D. S. P. Leung, Errata: "On the classification of reflective submanifolds of Riemannian symmetric spaces", (Indiana Univ. Math. J. **24** (1974) 327-339), Indiana Univ. Math. J. **24** (1975) 1199.

[L2] D. S. P. Leung, Reflective submanifolds. III. Congruenccy of isometric reflective submanifolds and corrigenda to the classification of reflective submanifolds, J. Differential Geom. **14** (1979) 167-177.

[M] J.A. McCoy, Mixed volume-preserving curvature flows, Calc. Var. Partial Differ. Equ. **24** (2005) 131-154.

[PT] R.S. Palais and C.L. Terng, Critical point theory and submanifold geometry, Lecture Notes in Math. **1353**, Springer, Berlin, 1988.