RIMS Kôkyûroku Bessatsu
B80 (2020), 81–94

Global existence and monotonicity formula for volume preserving mean curvature flow

By

Keisuke TAKASAO*

Abstract

In this note, the global existence of the weak solution for the volume preserving mean curvature flow is considered. We compare two phase field methods for the volume preserving mean curvature flow from the viewpoint of L^2-estimate of the mean curvature, which is the key estimate of the existence theorem. We also study the monotonicity formula for the volume preserving mean curvature flow.

§ 1. Introduction

This note is a survey about the existence of weak solutions of the volume preserving mean curvature flow. Let $T > 0$ and $d \geq 2$ be an integer. For any $t \in [0, T)$, let $U_t \subset \mathbb{R}^d$ be a bounded open set with a smooth boundary M_t. A family of the hypersurfaces $\{M_t\}_{t\in[0,T)}$ is called a volume preserving mean curvature flow if the normal velocity vector v of M_t is given by

$$v = h - \langle h \cdot \nu \rangle \nu \quad \text{on } M_t, \tag{1.1}$$

where h and ν are the mean curvature vector of M_t and the inner unit normal vector of M_t respectively, and

$$\langle h \cdot \nu \rangle = \frac{1}{\mathcal{H}^{d-1}(M_t)} \int_{M_t} h \cdot \nu \, d\mathcal{H}^{d-1}.$$

Here $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. If $\{M_t\}_{t\in[0,T)}$ is the solution of (1.1), then

$$\frac{d}{dt}\mathcal{L}^d(U_t) = -\int_{M_t} v \cdot \nu \, d\mathcal{H}^{d-1} = 0, \quad t \in (0, T). \tag{1.2}$$

Received January 16, 2017. Revised November 22, 2018.
2010 Mathematics Subject Classification(s): Primary 35K93, Secondary 53C44
Key Words: mean curvature flow, Allen-Cahn equation, phase field method.
This work was supported by JSPS KAKENHI Grant Number JP16K17622.
*Department of Mathematics/Hakubi Center, Kyoto University, Kitashirakawa-Oiwakecho Sakyo Kyoto 606-8502, Japan. e-mail: k.takasao@math.kyoto-u.ac.jp

Here $\mathcal{L}^d$ is the d-dimensional Lebesgue measure. The formula (1.2) is called the volume preserving property. By (1.2) we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{H}^{d-1}(M_t) = -\int_{M_t} h\cdot v\, d\mathcal{H}^{d-1} = -\int_{M_t} (v + \langle h\cdot \nu\rangle \nu)\cdot v\, d\mathcal{H}^{d-1} \\
&= -\int_{M_t} |v|^2\, d\mathcal{H}^{d-1} - \langle h\cdot\nu\rangle \int_{M_t} v\cdot \nu\, d\mathcal{H}^{d-1} = -\int_{M_t} |v|^2\, d\mathcal{H}^{d-1}.
\end{aligned} \tag{1.3}
$$

In this note, we consider the global existence for the weak solution to (1.1) by using the phase field method. The time global existence for the solution to (1.1) for convex U_0 proved by Gage [9] ($d = 2$) and Huisken [11] ($d \geq 2$). Escher and Simonett [6] showed that if M_0 is sufficiently close to a Euclidean sphere, then there exists a time global solution to (1.1). For general initial data M_0, Mugnai, Seis and Spadaro [16] proved the time global existence for the weak solution to (1.1) via the variational approach. Takasao [20] showed the time global existence for the weak solution to (1.1) for $d = 2, 3$ by using the phase field method.

§ 2. Phase field method

Let $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ and $U_0 \subset \Omega$ be an open set. We consider the following Allen-Cahn equation:

$$
\begin{cases}
\varepsilon\varphi^\varepsilon_t = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega\times(0,\infty), \\
\varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x\in\Omega.
\end{cases} \tag{2.1}
$$

Here $\varepsilon \in (0,1)$ and $W(s) = \dfrac{(1-s^2)^2}{2}$. We set initial data φ^ε_0 so that

$$
\varphi^\varepsilon_0(x) \approx \begin{cases} +1, & x\in U_0 \\ -1, & x \in \Omega\setminus U_0. \end{cases}
$$

For sufficiently small $\varepsilon > 0$, Ω is divided into $\{x\in\Omega : \varphi^\varepsilon(x,t)\approx 1\}$ and $\{x\in\Omega : \varphi^\varepsilon(x,t) \approx -1\}$, and $M^\varepsilon_t := \{x\in\Omega : \varphi^\varepsilon(x,t) = 0\}$ converges to the solution to the mean curvature flow as $\varepsilon\to 0$ formally [7, 13]. Similarly, the phase field methods for (1.1) by using (2.1) with non-local term are also well known [2, 4, 5, 10, 18]. Rubinstein and Sternberg [18] considered the following Allen-Cahn equation with non-local term:

$$
\begin{cases}
\varepsilon\varphi^\varepsilon_t = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda^\varepsilon_1, & (x,t) \in \Omega\times(0,\infty), \\
\varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x\in\Omega.
\end{cases} \tag{2.2}
$$

Here $\lambda_1^\varepsilon(t) := \frac{1}{|\Omega|}\int_\Omega \frac{W'(\varphi^\varepsilon(x,t))}{\varepsilon}\,dx = \int_{[0,1)^d} \frac{W'(\varphi^\varepsilon(x,t))}{\varepsilon}\,dx$. The solution φ^ε of (2.2) satisfies the volume preserving property, that is

$$\frac{d}{dt}\int_\Omega \varphi^\varepsilon(x,t)\,dx = 0 \quad \text{for any } t \in (0,\infty). \tag{2.3}$$

By (2.3), the volume of $U_t^\varepsilon := \{x \in \Omega : \varphi^\varepsilon(x,t) \approx 1\}$ is almost constant. Chen, Hilhorst and Logak [5] proved that if $\{M_t\}_{t\in[0,T)}$ is the classical solution for (1.1), then there exists a family of the solution $\{\varphi^\varepsilon\}_{\varepsilon\in(0,1)}$ such that $M_t^\varepsilon := \{x \in \Omega : \varphi^\varepsilon(x,t) = 0\}$ converges to M_t as $\varepsilon \to 0$. Let φ^ε be a solution for (2.2). By (2.3) and the integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx = \int_\Omega \Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi_t^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon}\varphi_t^\varepsilon\Big)\,dx\\
&= \int_\Omega \varepsilon\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\varphi_t^\varepsilon\,dx = \int_\Omega \varepsilon\Big(-\varphi_t^\varepsilon + \frac{\lambda_1^\varepsilon}{\varepsilon}\Big)\varphi_t^\varepsilon\,dx\\
&= -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx + \lambda_1^\varepsilon\int_\Omega \varphi_t^\varepsilon\,dx = -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx.
\end{aligned} \tag{2.4}$$

We remark that (2.4) corresponds to (1.3). Define

$$E(\varphi^\varepsilon) := \int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}dx \quad \text{and} \quad F(\varphi^\varepsilon) := \int_\Omega \varphi^\varepsilon\,dx.$$

The values of $E(\varphi^\varepsilon)$ and $F(\varphi^\varepsilon)$ correspond to $\mathcal{H}^{d-1}(M_t^\varepsilon)$ and $2\mathcal{L}^d(U_t^\varepsilon) - 1$, respectively. Set $\alpha \in (0,1)$. If φ^ε is the minimizer of E subject to $F = \alpha$, then we have

$$0 = \delta E - \lambda\delta F = -\varepsilon\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon} - \lambda \tag{2.5}$$

for some λ. Therefore we can regard (2.2) as a gradient flow of E subject to $F = \alpha$. Ilmanen proved the global existence of the weak solution for the mean curvature flow via the phase field method [13]. However, the existence theorem for the weak solution for (1.1) via (2.2) is not known. The reason is the difficulty of the estimates of λ_1^ε (see Remark 2 below). Next, we consider the following Allen-Cahn equation with non-local term studied by Golovaty [10]:

$$\begin{cases} \varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \dfrac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda^\varepsilon\sqrt{2W(\varphi^\varepsilon)}, & (x,t) \in \Omega\times(0,\infty),\\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega. \end{cases} \tag{2.6}$$

Here

$$\lambda^\varepsilon(t) := \frac{-\int_\Omega \sqrt{2W(\varphi^\varepsilon)}\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx}{2\int_\Omega W(\varphi^\varepsilon)\,dx.} \tag{2.7}$$

Define $k(s) := \int_0^s \sqrt{2W(\tau)}d\tau = s - \frac{1}{3}s^3$. By the definition of λ^ε we obtain

$$\frac{d}{dt}\int_\Omega k(\varphi^\varepsilon)\,dx = \int_\Omega \sqrt{2W(\varphi^\varepsilon)}\varphi_t^\varepsilon\,dx = 0. \tag{2.8}$$

Note that if $\varphi^\varepsilon \approx \pm 1$, then we have $k(\varphi^\varepsilon) \approx \frac{2}{3}\varphi^\varepsilon$. Thus we can regard (2.8) as the volume preserving property. By (2.8) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx = \int_\Omega \Big(\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi_t^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon}\varphi_t^\varepsilon\Big)\,dx\\
&= \int_\Omega \varepsilon\Big(-\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)\varphi_t^\varepsilon\,dx = \int_\Omega \varepsilon\Big(-\varphi_t^\varepsilon + \lambda^\varepsilon\frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\Big)\varphi_t^\varepsilon\,dx\\
&= -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx + \lambda^\varepsilon\int_\Omega \varphi_t^\varepsilon\sqrt{2W(\varphi^\varepsilon)}\,dx = -\int_\Omega \varepsilon(\varphi_t^\varepsilon)^2\,dx.
\end{aligned} \tag{2.9}$$

Hence (2.9) also corresponds to (1.3). Set

$$\tilde F(\varphi^\varepsilon) := \int_\Omega k(\varphi^\varepsilon)\,dx.$$

By an argument similar to (2.5), if φ^ε is the minimizer of E subject to $\tilde F = \alpha \in (0, \frac{2}{3})$, then we have

$$0 = \delta E - \lambda\delta\tilde F = -\varepsilon\Delta\varphi^\varepsilon + \frac{W'(\varphi^\varepsilon)}{\varepsilon} - \lambda\sqrt{2W(\varphi^\varepsilon)}$$

for some λ. Therefore we can regard (2.6) as a gradient flow of E subject to $\tilde F = \alpha$.

For the solution φ^ε of (2.6), we define a Radon measure μ_t^ε on Ω by

$$\mu_t^\varepsilon(\phi) := \frac{1}{\sigma}\int_\Omega \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}\Big)dx \quad \text{for any } \phi \in C_c(\Omega). \tag{2.10}$$

Here $\sigma := \int_{-1}^1 \sqrt{2W(s)}\,ds$. Formally, μ_t^ε is an approximation of $\mathcal{H}^{d-1}\lfloor_{M_t}$, where $M_t := \{x \in \mathbb{R}^d : \varphi^\varepsilon(x,t) = 0\}$. For $d = 2, 3$, Takasao [20] proved the global existence of the weak solution for (1.1) via (2.6). In the proof, the following estimate is important:

Lemma 2.1 ([20]). *Let $d \geq 2$. Assume that there exist $\omega > 0$ and $D > 0$ such that $\mu_0^\varepsilon(\Omega) \leq D$ and $\left|\int_\Omega k(\varphi_0^\varepsilon)\,dx\right| \leq \frac{2}{3} - \omega$ for any $\varepsilon \in (0,1)$. Then there exist $c_1 = c_1(d,\omega,D) > 0$ and $\epsilon_1 = \epsilon_1(d,\omega,D) \in (0,1)$ such that*

$$\sup_{\varepsilon\in(0,\epsilon_1)} \int_{t_1}^{t_2} |\lambda^\varepsilon(t)|^2\,dt \leq c_1(1 + t_2 - t_1) \tag{2.11}$$

for any $0 \leq t_1 < t_2$.

Remark. Let $U_0 \subset \Omega$ be an open set and $M_0 := \partial U_0$. If $\mu_0^\varepsilon \approx \mathcal{H}^{d-1}\lfloor_{M_0}$, then the assumptions $\mu_0^\varepsilon(\Omega) \leq D$ and $\left|\int_\Omega k(\varphi_0^\varepsilon)\,dx\right| \leq \frac{2}{3} - \omega$ correspond to $\mathcal{H}^{d-1}(M_0) \leq D$ and $||U_0| - |\Omega \setminus U_0|| \leq 1 - \frac{3}{2}\omega$, respectively. Moreover, if $U_0 = \emptyset$, then by $|\Omega| = 1$, we have $||U_0| - |\Omega \setminus U_0|| = 1$. Thus, the assumptions are natural.

Remark. To obtain the weak solution for the volume preserving mean curvature flow, the L^2-estimates of the mean curvature is important (see [14, 15]). For the solution of (2.6), we can obtain the estimate via Lemma 2.1. Let $M_t \approx \{x \in \Omega : \varphi^\varepsilon(x,t) = 0\}$ and h be the mean curvature vector of M_t, then we have

$$\int_0^T \int_{M_t} |h|^2 \, d\mathcal{H}^{d-1} dt \approx \frac{1}{\sigma} \int_0^T \int_\Omega \varepsilon \Big(\Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \Big)^2 \, dxdt.$$

From (2.9), we have

$$\mu_T^\varepsilon(\Omega) + \int_0^T \int_\Omega \varepsilon (\varphi_t^\varepsilon)^2 \, dxdt \le \mu_0^\varepsilon(\Omega) \le D \quad \text{and} \quad \sup_{t\in[0,T]} \int_\Omega \frac{2W(\varphi^\varepsilon(x,t))}{\varepsilon} \, dx \le 2\sigma D$$

for any $T > 0$. Thus

$$\begin{aligned} &\int_0^T \int_\Omega \varepsilon \Big(\Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \Big)^2 \, dxdt \\ \le& 2 \int_0^T \int_\Omega \varepsilon (\varphi_t^\varepsilon)^2 \, dxdt + 2 \int_0^T \int_\Omega \varepsilon \Big(\lambda^\varepsilon \frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon} \Big)^2 \, dxdt \\ \le& 2\sigma D + 2 \int_0^T (\lambda^\varepsilon)^2 \int_\Omega \frac{2W(\varphi^\varepsilon)}{\varepsilon} \, dxdt \le 2\sigma D + 4\sigma D \int_0^T (\lambda^\varepsilon)^2 dt \\ \le& 2\sigma D(1 + 2c_1(1+T)). \end{aligned} \tag{2.12}$$

Hence $\int_0^T \int_{M_t} |h|^2 \, d\mathcal{H}^{d-1} dt$ is bounded, formally. For the solution of (2.2), the boundedness of $\sup_\varepsilon \int |\lambda_1^\varepsilon(t)|^2 \, dt$ is also known [2]. However, in order to obtain the L^2-estimate of the mean curvature by using an argument similar to (2.12), the boundedness of $\sup_\varepsilon \int |\lambda_1^\varepsilon(t)|^2 \, dt$ is not enough.

§ 3. Existence of the weak solution

The main results of this note are the partial extension of the results of [20] for $d \ge 2$. To state the main results, we recall several definitions from the geometric measure theory and refer to [1, 8, 19] for more details. We define $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ for $r > 0$ and $x \in \mathbb{R}^d$. For $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$, we write $a \otimes b := (a_i b_j)$. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{d\times d}$, we denote $A : B := \sum_{i,j=1}^d a_{ij} b_{ij}$. We write the Grassmann manifold of unoriented k-dimensional subspaces in $\mathbb{R}^d$ by $G_k(\mathbb{R}^d)$. For $S \in G_k(\mathbb{R}^d)$, we also use S to define the d by d matrix representing the orthogonal projection $\mathbb{R}^d \to S$. We remark that if $k = d-1$ then the projection for $S \in G_{d-1}(\mathbb{R}^d)$ is given by $S = I - \nu \otimes \nu$, where I is the identity matrix and ν is the unit normal vector of S. For a subset $A \subset \mathbb{R}^d$, we write the reduced boundary of A by $\partial^* A$.

Definition 3.1. A set $M \subset \mathbb{R}^d$ is called countably k-rectifiable set if M is $\mathcal{H}^k$-measurable and there exists a family of C^1 k-dimensional embedded submanifolds $\{M_i\}_{i=1}^{\infty}$ such that $\mathcal{H}^k(M \setminus \cup_{i=1}^{\infty} M_i) = 0$.

We call a Radon measure on $\mathbb{R}^d \times G_k(\mathbb{R}^d)$ a general k-varifold in $\mathbb{R}^d$. We write the set of all general k-varifolds by $\mathbf{V}_k(\mathbb{R}^d)$. For $V \in \mathbf{V}_k(\mathbb{R}^d)$, we denote a mass measure of V by

$$\|V\|(A) := V((\mathbb{R}^d \cap A) \times G_k(\mathbb{R}^d))$$

for any Borel set $A \subset \mathbb{R}^d$. We also define

$$\|V\|(\phi) := \int_{\mathbb{R}^d \times G_k(\mathbb{R}^d)} \phi(x)\, dV(x,S) \quad \text{for} \quad \phi \in C_c(\mathbb{R}^d).$$

The first variation $\delta V : C_c^1(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R}$ of $V \in \mathbf{V}_k(\mathbb{R}^d)$ is denoted by

$$\delta V(g) := \int_{\mathbb{R}^d \times G_k(\mathbb{R}^d)} \nabla g(x) : S\, dV(x,S) \quad \text{for} \quad g \in C_c^1(\mathbb{R}^d;\mathbb{R}^d).$$

We define a total variation $\|\delta V\|$ by

$$\|\delta V\|(G) := \sup\{\delta V(g) : g \in C_c^1(G;\mathbb{R}^d),\ |g| \le 1\}$$

for any open set $G \subset \mathbb{R}^d$. If $\|\delta V\|$ is locally bounded and absolutely continuous with respect to $\|V\|$, then by the Radon-Nikodym theorem, there exists a $\|V\|$-measurable function $h(x) : \operatorname{spt}\|V\| \to \mathbb{R}^d$ such that

$$\delta V(g) = -\int_{\mathbb{R}^d} h(x) \cdot g(x)\, d\|V\|(x) \quad \text{for} \quad g \in C_c(\mathbb{R}^d;\mathbb{R}^d).$$

Moreover h is called the generalized mean curvature vector of V. We call a Radon measure μ k-rectifiable if μ is given by $\mu = \theta \mathcal{H}^k\lfloor_M$, that is, there exist a countably k-rectifiable set M and $\theta \in L^1_{loc}(\mathcal{H}^k\lfloor_M; \mathbb{R}_{\ge 0})$ such that $\mu(\phi) := \int_{\mathbb{R}^d} \phi\, d\mu = \int_M \phi\theta\, d\mathcal{H}^k$ for any $\phi \in C_c(\mathbb{R}^d)$. Moreover if θ is integer-valued $\mathcal{H}^k$-a.e. on M then we call μ k-integral. For a k-rectifiable Radon measure $\mu = \theta\mathcal{H}^k\lfloor_M$ we define a unique rectifiable k-varifold V by

$$\int_{\mathbb{R}^d \times G_k(\mathbb{R}^d)} \phi(x,S)\, dV(x,S) := \int_{\mathbb{R}^d} \phi(x, T_xM)\theta(x)\, d\mathcal{H}^k(x) \qquad \text{for } \phi \in C_c(\mathbb{R}^d \times G_k(\mathbb{R}^d)),$$

where T_xM is the approximate tangent space of M at x. We remark that T_xM exists $\mathcal{H}^k$-a.e. on M and $\mu = \|V\|$ in this case.

The following definition is similar to the formulation of Brakke's mean curvature flow [3]:

Definition 3.2. Let $T > 0$ and $\{\mu_t\}_{t\in(0,T)}$ be a family of Radon measures on Ω. Set $d\mu := d\mu_t dt$. We call $\{\mu_t\}_{t\in(0,T)}$ rectifiable L^2-flow with the generalized velocity vector v if the following hold:

1. $v \in L^2(0,T;(L^2(\mu_t))^d)$ and μ_t is $(d-1)$-rectifiable and has a generalized mean curvature vector $h \in L^2(\mu_t;\mathbb{R}^d)$ a.e. $t \in (0,T)$.

2. v and μ_t satisfy

$$v(x,t) \perp T_x\mu_t \quad \text{for } \mu\text{-a.e. } (x,t) \in \Omega \times (0,T). \tag{3.1}$$

Here $T_x\mu_t$ is the approximate tangent space of μ_t at x.

3. There exists $C_T > 0$ such that

$$\Big| \int_0^T \int_\Omega (\eta_t + \nabla\eta \cdot v)\, d\mu_t dt \Big| \le C_T \|\eta\|_{C^0(\Omega\times(0,T))} \tag{3.2}$$

for any $\eta \in C_c^1(\Omega \times (0,T))$.

Remark. In [14], the original definition of L^2-flow requires that μ_t is integral a.e. $t \in (0,T)$, in addition.

The main results of this note are following:

Theorem 3.3. *Let $d \ge 2$ and $U_0 \subset \Omega$ be an open set with C^1 boundary M_0. Then there exist a positive sequence $\{\varepsilon_i\}_{i=1}^\infty$ with $\varepsilon_i \to 0$ and $\{\varphi_0^{\varepsilon_i}\}_{i=1}^\infty$ such that the following hold:*

(a) Let φ^{ε_i} be a solution of (2.6) with the initial data $\varphi_0^{\varepsilon_i}$. Then there exists $\psi \in BV_{loc}(\Omega \times [0,\infty)) \cap C^{\frac{1}{2}}_{loc}([0,\infty);L^1(\Omega))$ such that

(a1) $\psi(\cdot,0) = \chi_{U_0}$ a.e. on Ω and $\varphi^{\varepsilon_i} \to 2\psi - 1$ in $L^1_{loc}(\Omega \times [0,\infty))$.

(a2)

$$\int_\Omega \psi(\cdot,t)\, dx = \mathcal{L}^d(U_0) \quad \text{for any } t \in [0,\infty).$$

(b) There exists a family of $(d-1)$-rectifiable Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ such that $\mu_0 = \mathcal{H}^{d-1}\lfloor_{M_0}$ and $\mu_t^{\varepsilon_i} \to \mu_t$ as Radon measures on Ω for any $t \in [0,\infty)$.

(c) There exists a function $\lambda \in L^2_{loc}(0,\infty)$ such that

$$\lambda^{\varepsilon_i} \to \lambda \quad \text{weakly in } L^2(0,T) \quad \text{for any } T > 0.$$

(d) *There exists* $g \in L^2_{loc}(0,T;(L^2(\mu_t))^d)$ *such that* $\{\mu_t\}_{t\in(0,\infty)}$ *is rectifiable* L^2*-flow with a generalized velocity vector*

$$v = h + g,$$

and v *satisfies*

$$\lim_{i\to\infty}\int_{\{|\nabla\varphi^{\varepsilon_i}(\cdot,t)|\neq 0\}\times(0,\infty)} \frac{-\varphi_t^{\varepsilon_i}}{|\nabla\varphi^{\varepsilon_i}|}\frac{\nabla\varphi^{\varepsilon_i}}{|\nabla\varphi^{\varepsilon_i}|}\cdot\Phi\, d\mu_t^{\varepsilon_i}dt = \int_{\Omega\times(0,\infty)} v\cdot\Phi\, d\mu_t dt$$

for any $\Phi \in C_c(\Omega\times[0,\infty);\mathbb{R}^d)$. *Moreover there exists* $\theta : \partial^*\{(x,t) : \psi(x,t) = 1\} \to (0,\infty)$ *such that*

$$v = h - \frac{\lambda}{\theta}\nu \qquad \text{for } \mathcal{H}^d\text{-a.e. on } \partial^*\{\psi = 1\}.$$

Here $\nu = \nu(x,t)$ *is the inner unit normal vector of* $\partial^*\{x : \psi(x,t) = 1\}$.

(e)

$$\int_\Omega v\cdot\nu\, d\|\nabla\psi(\cdot,t)\| = 0 \quad \text{for a.e. } t\in(0,\infty).$$

Here $\|\nabla\psi(\cdot,t)\|$ *is the total variation measure of* $\nabla\psi(\cdot,t)$.

Remark. The author showed that if $d = 2,3$, then $\{\mu_t\}_{t\in(0,\infty)}$ is L^2-flow, namely, μ_t is $(d-1)$-integral for a.e. $t \geq 0$ and $\theta\in\mathbb{N}$ for $\mathcal{H}^d$-a.e. on $\partial^*\{\psi = 1\}$ in addition (see [20, Theorem 2.5]). Thus Theorem 3.3 is the partial extension of the results of [20] for $d \geq 4$. The integral property of μ_t for $d \geq 4$ is an open problem.

§ 4. The monotonicity formula

The monotonicity formula for the mean curvature flow is proved by Huisken [12]. Ilmanen [13] proved the monotonicity formula for the Allen-Cahn equation which corresponds to the mean curvature flow. In this section we consider the monotonicity formula for (2.6). Define

$$\rho_{y,s}(x,t) := \frac{1}{(4\pi(s-t))^{\frac{d-1}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}}, \qquad t < s,\ x,y\in\mathbb{R}^d.$$

Let the measure μ_t^ε be extended periodically to $\mathbb{R}^d$. Then we have the following:

Proposition 4.1 (Monotonicity formula [20]). *Let* φ^ε *be a solution of* (2.6) *and satisfy the assumptions of Lemma 2.1 and*

$$|\varphi_0^\varepsilon| < 1 \quad \text{and} \quad \frac{\varepsilon|\nabla\varphi_0^\varepsilon|^2}{2} \leq \frac{W(\varphi_0^\varepsilon)}{\varepsilon} \quad \text{on } \Omega.$$

Then we have

$$\int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_t^\varepsilon\Big|_{t=t_2} \le \int_{\mathbb{R}^n} \rho_{y,s}(x,t)\, d\mu_t^\varepsilon\Big|_{t=t_1} \exp\Big(\frac{c_1}{2}(1+t_2-t_1)\Big) \tag{4.1}$$

for any $y \in \mathbb{R}^d$, $0 \le t_1 < t_2 < s$ *and* $\varepsilon \in (0, \epsilon_1)$.

Proof. By an argument similar to that in [21, p.870], we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho\, d\mu_t^\varepsilon \le \frac{1}{2(s-t)}\int_{\mathbb{R}^d} \rho\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\, dx + \frac{1}{2}(\lambda^\varepsilon)^2 \int_{\mathbb{R}^d} \rho\, d\mu_t^\varepsilon \tag{4.2}$$

for any $t < s$. By (2.6) and Lemma 4.2 which we show later, we obtain

$$\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} \le \frac{W(\varphi^\varepsilon)}{\varepsilon} \quad \text{in} \quad \mathbb{R}^d \times [0,\infty). \tag{4.3}$$

Therefore, by (2.11), (4.2), (4.3) and Gronwall's inequality we have (4.1). □

Lemma 4.2. *Let* $u = u(t) \in C^1((0,\infty))$ *and* $\varphi^\varepsilon \in C^2(\mathbb{R}^d \times (0,\infty))$ *be a solution of*

$$\varepsilon\varphi_t^\varepsilon = \varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} + u\sqrt{2W(\varphi^\varepsilon)}. \tag{4.4}$$

Moreover we assume that

$$|\varphi^\varepsilon|\Big|_{t=0} < 1 \quad \textit{and} \quad \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big|_{t=0} \le 0 \quad \textit{on} \quad \mathbb{R}^d.$$

Then we have

$$\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon} \le 0 \quad \textit{in} \quad \mathbb{R}^d \times [0,\infty). \tag{4.5}$$

Proof. Set $q^\varepsilon(r) := \tanh(r/\varepsilon)$. We define $r : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ by

$$\varphi^\varepsilon(x,t) = q^\varepsilon(r(x,t)).$$

Then, by $\dfrac{\varepsilon|q_r^\varepsilon|^2}{2} = \dfrac{W(q^\varepsilon)}{\varepsilon}$ we obtain

$$\frac{\varepsilon|\nabla\varphi^\varepsilon|^2/2}{W(\varphi^\varepsilon)/\varepsilon} \le |\nabla r|^2 \quad \text{on} \quad \mathbb{R}^d \times [0,\infty).$$

If $|\nabla r| \le 1$ then by $\dfrac{\varepsilon|\nabla\varphi^\varepsilon|^2/2}{W(\varphi^\varepsilon)/\varepsilon} \le 1$ we obtain (4.5). Therefore we only need to prove that $|\nabla r| \le 1$ on $\mathbb{R}^d \times [0,\infty)$.

Define $g(q) := k'(q) = \sqrt{2W(q)}$. By the property of q^ε we have

$$q_r^\varepsilon = \frac{g(q^\varepsilon)}{\varepsilon} \quad \text{and} \quad q_{rr}^\varepsilon = \frac{(g(q^\varepsilon))_r}{\varepsilon} = \frac{g_q(q^\varepsilon)}{\varepsilon} q_r^\varepsilon. \tag{4.6}$$

By (4.4) and (4.6) we obtain

$$\begin{aligned} q_r^\varepsilon r_t &= q_r^\varepsilon \Delta r + q_{rr}^\varepsilon |\nabla r|^2 - q_{rr}^\varepsilon + u q_r^\varepsilon \\ &= q_r^\varepsilon \Delta r + q_r^\varepsilon \frac{g_q}{\varepsilon}(|\nabla r|^2 - 1) + u q_r^\varepsilon. \end{aligned}$$

Thus we have

$$r_t = \Delta r + \frac{g_q}{\varepsilon}(|\nabla r|^2 - 1) + u$$

and

$$\partial_t |\nabla r|^2 = \Delta |\nabla r|^2 - 2|\nabla^2 r|^2 + \frac{2}{\varepsilon} \nabla r \cdot \nabla g_q (|\nabla r|^2 - 1) + \frac{2g_q}{\varepsilon} \nabla r \cdot \nabla |\nabla r|^2, \tag{4.7}$$

where $\nabla u = 0$ is used. By $|\nabla r(\cdot, 0)| \leq 1$ on $\mathbb{R}^d$ and applying the maximum principle to (4.7), we obtain $|\nabla r| \leq 1$ in $\mathbb{R}^d \times [0, \infty)$. □

Remark. For the solution of (2.2), we can not obtain (4.5) by an argument similar to that in the proof of Lemma 4.2. The estimate (4.5) is important to prove the existence theorem.

Define $d\xi_t^\varepsilon(x) := \sigma^{-1}\left(\frac{\varepsilon|\nabla\varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon}\right) dx$. The signed measure ξ_t^ε is called the discrepancy measure. By using the monotonicity formula and arguments similar to that in [13, Section 8] and [21, Section 6.2], we obtain the following:

Proposition 4.3. *Let $\{\varepsilon_i\}_{i=1}^\infty$ be a positive sequence with $\varepsilon_i \to 0$ as $i \to \infty$. Let φ^{ε_i} satisfy the assumptions of Lemma 2.1 and Proposition 4.1. Moreover let $\mu_t^{\varepsilon_i}$ converge to μ_t as Radon measures for any $t \geq 0$. Then there exists a subsequence $\{\varepsilon_{i_j}\}_{j=1}^\infty$ such that*

$$\lim_{j\to\infty} |\xi_t^{\varepsilon_{i_j}}| = 0 \quad a.e.\ t \geq 0. \tag{4.8}$$

Remark. (4.8) means the balance of the Dirichlet energy and the potential energy.

§ 5. Rectifiability of μ_t

In this section we study the rectifiability of μ_t. There are methods of Ilmanen [13] and Röger and Schätzle [17] ($d = 2, 3$) as proof of the rectifiability of μ_t. First we consider the method of [17].

Theorem 5.1 ([17]). *Let $d = 2, 3$ and $U \subset \mathbb{R}^d$ be an open set. For a function $\varphi^\varepsilon \in C^2(U)$, we define $\mu^\varepsilon(\phi) := \frac{1}{\sigma}\int_U \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx$. Assume that*

$$\sup_{\varepsilon>0}\mu^\varepsilon(U) < \infty, \quad \sup_{\varepsilon>0}\int_U \varepsilon\Big(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)^2\,dx < \infty$$

and

$$\mu^\varepsilon \to \mu \quad \text{as Radon measures.}$$

Then μ is $(d-1)$-rectifiable and integral. Moreover, for the generalized mean curvature vector h of μ and the signed measure $\xi^\varepsilon(\phi) := \frac{1}{\sigma}\int_U \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx$ we have

$$\int_U |h|^2\,d\mu \le \frac{1}{\sigma}\liminf_{\varepsilon>0}\int_U \varepsilon\Big(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\Big)^2\,dx \quad \text{and} \quad |\xi^\varepsilon| \to 0.$$

Takasao [20] proved the existence of L^2-flow of (1.1) for $d = 2, 3$, by using Theorem 5.1. In order to obtain the rectifiability for any dimension, we consider the method of [13]. The following theorem is important to obtain the rectifiability:

Theorem 5.2 (Allard's rectifiability theorem [1]). *Let $k \le d-1$. Assume $U \subset \mathbb{R}^d$ is an open set. Suppose $V \in \mathbf{V}_k(U)$ and $\|\delta V\|$ is a Radon measure on U. In order that V be rectifiable it is necessary and sufficient that there exist subsets $A_1, A_2, \dots$ of U such that $\mathcal{H}^k(A_i) < \infty$ for any $i \in \mathbb{N}$ and $\|V\|(U \setminus \cup_{i=1}^\infty A_i) = 0$.*

In this section, we suppose that $d \ge 2$, φ^ε is a solution of (2.6) and satisfies the assumption of Proposition 4.1. Moreover we assume that there exists μ_t such that $\mu_t = \lim_{\varepsilon\to 0}\mu_t^\varepsilon$ for any $t \ge 0$ and

$$\sup_{x\in\mathbb{R}^d, r\in(0,1)} \frac{\mu_t(B_r(x))}{r^{d-1}} \le D \quad \text{for any } t \ge 0.$$

By using the monotonicity formula (4.1) and an argument similar to that in [13, Corollary 6.3], we obtain

Lemma 5.3. *There exists $c_1 = c_1(d, D) > 0$ such that*

$$\mathcal{H}^{d-1}(\operatorname{spt}\mu_t) \le c_1 \liminf_{\tau\uparrow t}\mu_\tau(\Omega) \quad \text{for} \quad t > 0. \tag{5.1}$$

Next we consider the first variation of the varifold which is naturally associated with μ_t^ε.

Definition 5.4. Let φ^ε be a solution of (2.6). Define $a^\varepsilon(x,t) := \frac{\nabla\varphi^\varepsilon(x,t)}{|\nabla\varphi^\varepsilon(x,t)|}$ for (x,t) with $|\nabla\varphi^\varepsilon(x,t)| \ne 0$. We consider the following varifold:

$$V_t^\varepsilon(\phi) = \int_{\Omega\cap\{|\nabla\varphi^\varepsilon|\ne 0\}} \phi(x, I - a^\varepsilon \otimes a^\varepsilon)\,d\mu_t^\varepsilon, \quad \phi \in C_c(\Omega \times G_{d-1}(\mathbb{R}^d)).$$

The first variation of V_t^ε is given by

$$\delta V_t^\varepsilon(g) = \int_{\Omega\cap\{|\nabla\varphi^\varepsilon|\neq 0\}} \nabla g\cdot(I - a^\varepsilon\otimes a^\varepsilon)\,d\mu_t^\varepsilon, \quad g\in C_c(\Omega;\mathbb{R}^d).$$

By the integration by parts, we have

Proposition 5.5.

(5.2)

$$\begin{aligned}\delta V_t^\varepsilon(g) =&\sigma^{-1}\int_\Omega (g\cdot\nabla\varphi^\varepsilon)\Big(\varepsilon\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\Big)\,dx + \int_{\Omega\cap\{|\nabla\varphi^\varepsilon|\neq 0\}}\nabla g\cdot(a^\varepsilon\otimes a^\varepsilon)\,d\xi_t^\varepsilon\\ &-\sigma^{-1}\int_{\Omega\cap\{|\nabla\varphi^\varepsilon|=0\}}\nabla g\cdot I\frac{W(\varphi^\varepsilon)}{\varepsilon}\,dx.\end{aligned}$$

Finally we show the rectifiability of μ_t.

Proposition 5.6. *μ_t is $(d-1)$-rectifiable a.e. $t\geq 0$.*

Proof. Let $\{\varepsilon_i\}_{i=1}^\infty$ be a positive sequence with $\varepsilon_i\to 0$ as $i\to\infty$. By $\sup_{i\geq 1}\mu_t^{\varepsilon_i}(\Omega)\leq D$, (2.12) and Fatou's lemma, we have

$$\liminf_{i\to\infty}\Big\{\int_\Omega |\nabla\varphi^{\varepsilon_i}|\cdot|\varepsilon_i\Delta\varphi^{\varepsilon_i} - W'(\varphi^{\varepsilon_i})/\varepsilon_i|\,dx\Big\}<\infty \quad \text{for a.e. } t\geq 0. \tag{5.3}$$

Fix $t\geq 0$ such that (4.8) and (5.3) hold. By the compactness of Radon measures, there exist a subsequence $\{V_t^{\varepsilon_{i_j}}\}_{j=1}^\infty$ and a varifold $\tilde V_t\in\mathbf{V}_{d-1}(\Omega)$ such that

$$V_t^{\varepsilon_{i_j}}\to\tilde V_t \quad \text{as varifolds.} \tag{5.4}$$

By (4.8), (5.2) and (5.4), we have

$$\begin{aligned}\delta\tilde V_t(g) &= \int\nabla g(x):S\,d\tilde V_t(x,S)\\ &=\lim_{j\to\infty}\Big\{\int\nabla g:(I - a^{\varepsilon_{i_j}}\otimes a^{\varepsilon_{i_j}})\,d\mu_t^{\varepsilon_{i_j}}\\ &\qquad + \int_{\Omega\cap\{|\nabla\varphi^{\varepsilon_{i_j}}|\neq 0\}}\nabla g\cdot(a^{\varepsilon_{i_j}}\otimes a^{\varepsilon_{i_j}})\,d\xi_t^{\varepsilon_{i_j}}\\ &\qquad -\sigma^{-1}\int_{\Omega\cap\{|\nabla\varphi^{\varepsilon_{i_j}}|=0\}}\nabla g\cdot I\frac{W(\varphi^{\varepsilon_{i_j}})}{\varepsilon_{i_j}}\,dx\Big\}\\ &=\lim_{j\to\infty}\sigma^{-1}\int_\Omega g\cdot\nabla\varphi^{\varepsilon_{i_j}}(\varepsilon_{i_j}\Delta\varphi^{\varepsilon_{i_j}} - W'(\varphi^{\varepsilon_{i_j}})/\varepsilon_{i_j})\,dx\end{aligned} \tag{5.5}$$

for any $g\in(C^1(\Omega))^d$. By (5.3) and (5.5), there exists $C_1>0$ such that

$$\begin{aligned}|\delta\tilde V_t(g)|\leq&\|g\|_\infty\liminf_{j\to\infty}\sigma^{-1}\int_\Omega|\nabla\varphi^{\varepsilon_{i_j}}|\cdot|\varepsilon_{i_j}\Delta\varphi^{\varepsilon_{i_j}} - W'(\varphi^{\varepsilon_{i_j}})/\varepsilon_{i_j}|\,dx\\ \leq& C_1\|g\|_\infty\end{aligned} \tag{5.6}$$

for any $g \in (C^1(\Omega))^d$. Thus $|\delta \tilde{V}_t|$ is a Radon measure. Moreover, $\mathcal{H}^{d-1}(\operatorname{spt}\mu_t) < \infty$ by (5.1). By Theorem 5.2, $\tilde{V}_t$ is $(d-1)$-rectifiable. Moreover, by $\|V_t^{\varepsilon_{i_j}}\| = \mu_t^{\varepsilon_{i_j}}$ we have $\|\tilde{V}_t\| = \mu_t$. Hence $\tilde{V}_t$ is uniquely determined by μ_t. Thus μ_t is $(d-1)$-rectifiable. □

References

[1] Allard, W., *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.

[2] Bronsard, L. and Stoth, B., *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation*, SIAM J. Math. Anal., **28** (1997), 769–807.

[3] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

[4] Brassel, M. and Bretin, E., *A modified phase field approximation for mean curvature flow with conservation of the volume*, Math. Methods Appl. Sci., **34** (2011), 1157–1180.

[5] Chen, X., Hilhorst, D. and Logak, E., *Mass conserving Allen-Cahn equation and volume preserving mean curvature flow*, Interfaces Free Bound., **12** (2010), 527–549.

[6] Escher, J. and Simonett, G., *The volume preserving mean curvature flow near spheres*, Proc. Amer. Math. Soc., **126** (1998), 2789–2796.

[7] Evans, L. C. and Soner, H. M. and Souganidis, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., **45** (1992), 1097–1123.

[8] Federer, H., *Geometric Measure Theory*, Springer-Verlag, New York, (1969).

[9] Gage, M., *On an area-preserving evolution equation for plane curves*, Nonlinear problems in geometry (Mobile, Ala., 1985), Contemp. Math., **51** (1986), 51–62.

[10] Golovaty, D., *The volume-preserving motion by mean curvature as an asymptotic limit of reaction-diffusion equations*, Quart. Appl. Math., **55** (1997), 243–298.

[11] Huisken, G., *The volume preserving mean curvature flow*, J. Reine Angew. Math., **382** (1987), 35–48.

[12] Huisken, G., *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom., **31** (1990), 285–299.

[13] Ilmanen, T., *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., **38** (1993), 417–461.

[14] Mugnai, L. and Röger, M., *The Allen-Cahn action functional in higher dimensions*, Interfaces Free Bound., **10** (2008), 45–78.

[15] Mugnai, L. and Röger, M., *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, Indiana Univ. Math. J., **60** (2011), 41–75.

[16] Mugnai, L., Seis, C. and Spadaro, E., *Global solutions to the volume-preserving mean-curvature flow*, Calc. Var. Partial Differential Equations, **55** (2016), 55:18.

[17] Röger, M. and Schätzle, R., *On a modified conjecture of De Giorgi*, Math. Z., **254** (2006), 675–714.

[18] Rubinstein, J. and Sternberg, P., *Nonlocal reaction-diffusion equations and nucleation*, IMA Journal of Applied Mathematics, **48** (1992), 249–264.

[19] Simon, L., *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[20] Takasao, K., *Existence of weak solution for volume preserving mean curvature flow via phase field method*, Indiana Univ. Math. J. **66** (2017), 2015–2035.

[21] Takasao, K. and Tonegawa, Y., *Existence and regularity of mean curvature flow with transport term in higher dimensions*, Math. Ann., **364** (2016), 857–935.