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On the Chorin method for thermal convection equations for viscous incompressible fluids

By

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Abstract

Stability of stationary solutions of the Oberbeck Boussinesq system and the corresponding artificial compressible system is considered. The former system is obtained by a singular limit of zero artificial Mach number in the latter system. This paper overviews recent results on a relation of the stability of stationary solutions between both systems from the viewpoint of the singular perturbation.

§ 1. Introduction

In this paper we consider the stability of stationary solutions of thermal convection equations, called the Oberbeck-Boussinesq equations,

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\mathrm{Pr}^{-1} (\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \Delta \boldsymbol{v} + \nabla p - \sqrt{\mathrm{Ra}}\, \theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{1.2}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta - \Delta \theta - \sqrt{\mathrm{Ra}}\, \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0, \tag{1.3}$$

and the artificial compressible system for (1.1)–(1.3):

$$\varepsilon^2 \partial_t p + \operatorname{div} \boldsymbol{v} = 0, \tag{1.4}$$

$$\mathrm{Pr}^{-1} (\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \Delta \boldsymbol{v} + \nabla p - \sqrt{\mathrm{Ra}}\, \theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{1.5}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta - \Delta \theta - \sqrt{\mathrm{Ra}}\, \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0. \tag{1.6}$$

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Here $\boldsymbol{v} = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, $p = p(x,t)$ and $\theta = \theta(x,t)$ denote the unknown velocity field, pressure and temperature deviation from the motionless state, respectively, at time $t > 0$ and position $x \in \mathbb{R}^3$; $\boldsymbol{e}_3 = {}^\top(0,0,1) \in \mathbb{R}^3$; $\mathrm{Pr} > 0$ and $\mathrm{Ra} > 0$ are non-dimensional parameters, called Prandtl and Rayleigh numbers, respectively; and $\varepsilon > 0$ is a small parameter, called artificial Mach number. Here and in what follows, the superscript ${}^\top\cdot$ stands for the transposition. The systems (1.1)–(1.3) and (1.4)–(1.6) are considered in the infinite layer Ω:

$$\Omega = \{x = (x', x_3); x' = (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\}.$$

If $\varepsilon \to 0$ in the artificial compressible system (1.4)–(1.6), one obtains the incompressible system (1.1)–(1.3). One could therefore expect that solutions of (1.1)–(1.3) would be approximated by solutions of (1.4)–(1.6) when $\varepsilon \ll 1$. But this is a singular limit, and hence, it is not straightforward to conclude that the artificial compressible system (1.4)–(1.6) gives a good approximation of the incompressible system (1.1)–(1.3). In [12, 13] the question whether (1.4)–(1.6) gives a good approximation of (1.1)–(1.3) was investigated from the viewpoint of the stability of stationary solutions. The purpose of this paper is to give an overview of the results in [12, 13] and append some remarks to [12, 13].

The Oberbeck-Boussinesq equations (1.1)–(1.3) are a system of equations for a convection phenomena (Bénard convection) of viscous incompressible fluid occupying Ω heated from below under the gravitational force. It is known (see, e.g., [1, 10, 11, 16]) that under the boundary condition

$$\boldsymbol{v}|_{x_3=0,1} = \mathbf{0}, \quad \theta|_{x_3=0,1} = 0, \tag{1.7}$$

there exists a critical number $\mathrm{Ra}_c > 0$ such that when $\mathrm{Ra} < \mathrm{Ra}_c$, the motionless state $\boldsymbol{v} = \mathbf{0}$, $\theta = 0$ is stable, while, when $\mathrm{Ra} > \mathrm{Ra}_c$, the motionless state is unstable and spatially periodic convective stationary solutions, such as roll pattern, hexagonal pattern and etc., bifurcate from the motionless state.

The artificial compressible system such as (1.4)–(1.6) was proposed by A. Chorin ([2, 3, 4]) for the purpose of finding stationary solutions of equations for viscous incompressible fluid numerically. If a solution of the artificial compressible system (1.4)–(1.6) converges to a function $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ as $t \to \infty$, then the limit function u_s is a stationary solution of (1.4)–(1.6). Since the sets of stationary solutions of (1.1)–(1.3) and (1.4)–(1.6) coincide, one consequently obtains a stationary solution of (1.1)–(1.3). With this method, Chorin numerically obtained periodic convective stationary patterns of (1.1)–(1.3).

The limit function u_s obtained by Chorin's method is a large time limit of solution of (1.4)–(1.6), and therefore, u_s is stable as a solution of (1.4)–(1.6). It is of interest to consider the following questions:

(i) whether u_s is stable as a solution of (1.1)–(1.3), in other words, whether u_s represents an observable stationary flow in the real world ?
(ii) conversely, what kind of stationary flows can be computed by the Chorin method ?

These questions were considered in [12, 13]. To address the stability questions of stationary solutions, in [12, 13], the spectra of the linearized operators around a stationary solution of (1.1)–(1.3) and (1.4)–(1.6) for $0 < \varepsilon \ll 1$ were considered under the additional boundary condition

$$p,\ \boldsymbol{v} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2), \tag{1.8}$$

where

$$\mathcal{Q} = [-\pi/\alpha_1, \pi/\alpha_1) \times [-\pi/\alpha_2, \pi/\alpha_2).$$

Here α_j, $j = 1, 2$, are given positive constants. We will denote the basic period domain by

$$\Omega_{per} = \mathcal{Q} \times (0, 1).$$

We briefly summarize the main results of [12, 13]. It was shown in [12] that if a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ of (1.4)–(1.6) is asymptotically stable for sufficiently small ε, then so is u_s as a stationary solution of (1.1)–(1.3). Furthermore, an instability result was obtained; if u_s is unstable as a stationary solution of (1.1)–(1.3), then so is u_s as a stationary solution of (1.4)–(1.6) for $0 < \varepsilon \ll 1$. This shows that unstable stationary solutions of (1.1)–(1.3) cannot be obtained by Chorin's method with $0 < \varepsilon \ll 1$. As for the converse question, it was proved in [12] that if a stationary solution $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ of (1.1)–(1.3) is asymptotically stable, then so is u_s as a stationary solution of (1.4)–(1.6) for $0 < \varepsilon \ll 1$, provided that

$$\inf_{\boldsymbol{w} \in (H^1_{0,per})^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\mathrm{Re}\,(\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \boldsymbol{w})_{L^2}}{\|\nabla \boldsymbol{w}\|^2_{L^2}} \geq -\delta_0 \tag{1.9}$$

for some positive constant δ_0. This gives a sufficient condition for u_s to be computed by Chorin's method with $0 < \varepsilon \ll 1$. We note that no condition for the temperature θ_s of u_s is required. These results are applicable to stable bifurcating periodic convective patterns such as roll pattern, hexagonal pattern and etc., when $\mathrm{Ra} \sim \mathrm{Ra}_c$. In fact, the velocity fields of bifurcating convective patterns are small when $\mathrm{Ra} \sim \mathrm{Ra}_c$ since they bifurcate from $\boldsymbol{v} = \boldsymbol{0}$, $\theta = 0$ when Ra crosses Ra_c, and hence, the condition (1.9) is satisfied. However, the condition (1.9) seems to be somewhat strong since most of applications might be limited to stationary flows whose velocity fields $\boldsymbol{v}_s$ are small enough.

The condition (1.9) was later improved in [13]. It was proved in [13] that the condition (1.9) can be replaced by

$$\inf_{\boldsymbol{w} \in (H^1_{0,per})^3, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\mathrm{Re}\,(\mathbb{Q}\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \mathbb{Q}\boldsymbol{w})_{L^2}}{\|\nabla \mathbb{Q}\boldsymbol{w}\|^2_{L^2}} \geq -\delta_0. \tag{1.10}$$

Here $\mathbb{Q}$ is the orthogonal projection from the space of L^2 vector fields on Ω_{per} to the subspace $\{\nabla\varphi \in L^2(\Omega_{per})^3; \varphi \in H^1_{per}(\Omega_{per})\}$.

Due to this improvement, one can also consider the Taylor problem, namely, a flow between two concentric infinite cylinders, whose inner cylinder rotates with a uniform speed and outer one is at rest. As is the case of the Bénard convection problem, the Taylor problem has also been widely studied as a good subject of the pattern formation problem. When the rotation speed is sufficiently small, a laminar flow, called the Couette flow, is stable. When the rotation speed increases, the Couette flow becomes unstable beyond a certain value of the rotation speed, and a vortex pattern, called the Taylor vortex, is observed. The Taylor vortex pattern is periodic in the direction of the axis of the cylinders. Mathematically, this phenomenon is formulated as a bifurcation problem for the incompressible system (see [5, 10, 11, 14, 21]). The velocity field near the bifurcation point of the Taylor vortex is not necessarily small, but one can show that the condition (1.10) is satisfied with $\boldsymbol{v}_s$ being the Taylor vortex under *axi-symmetric perturbations* (i.e., $\boldsymbol{w}$ in (1.10) are *axi-symmetric*). This implies that one can compute the Taylor vortex by using Chorin's method. (See [13, Section 5].)

We close this section with mentioning related results on the artificial compressible system. The convergence of solutions as $\varepsilon \to 0$ was discussed in [17, 18, 19] for the artificial system with the additional stabilizing nonlinear term $+\frac{1}{2}(\operatorname{div}\boldsymbol{v})\boldsymbol{v}$:

$$\begin{aligned} \varepsilon^2\partial_t p + \operatorname{div}\boldsymbol{v} &= 0, \\ \partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \tfrac{1}{2}(\operatorname{div}\boldsymbol{v})\boldsymbol{v} + \nabla p &= \boldsymbol{g} \end{aligned}$$

on a bounded domain. Here $\boldsymbol{g}$ is a given external force. It was shown that there exists a weak solution ${}^\top(p_\varepsilon, \boldsymbol{v}_\varepsilon)$ for each $\varepsilon > 0$ such that $\boldsymbol{v}_{\varepsilon'} \to \boldsymbol{v}$ in $L^2(0,T;L^2(\Omega)^3)$ and $\nabla p_{\varepsilon'} \to \nabla p$ weakly in $H^{-1}(\Omega\times(0,T))$ for all $T > 0$ along a sequence $\varepsilon' \to 0$, where ${}^\top(p,\boldsymbol{v})$ is a weak solution of the corresponding incompressible Navier-Stokes equations:

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0, \\ \partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p &= \boldsymbol{g}. \end{aligned}$$

We also mention the works by Donatelli [6, 7] and Donatelli and Marcati [8, 9] where similar convergence results were obtained in the case of unbounded domains by using the wave equation structure of the pressure and the dispersive estimates.

This paper is organized as follows. In section 2 we introduce notations used in this paper. In section 3 we state the precise statement of the stability results in [12, 13] mentioned above. In section 4 we list some properties of the null spaces of the linearized operators. In section 5 we give an outline of a proof of the result that the condition (1.10) is a sufficient condition for a stationary solution to be stable as a solution of the artificial compressible system.

§ 2. Preliminaries

In this section we introduce notation used in this paper. We denote by C^∞_{per} the space of restrictions of functions in $C^\infty(\overline{\Omega})$ which are $\mathcal{Q}$-periodic in $x' = (x_1, x_2)$. We also denote by $C^\infty_{0,per}$ the space of restrictions of functions in C^∞ which are $\mathcal{Q}$-periodic in $x' = (x_1, x_2)$ and vanish near $x_3 = 0, 1$. For $1 \le r \le \infty$ we denote by $L^r(\Omega_{per})$ the usual Lebesgue space over Ω_{per}, and its norm is denoted by $\| \cdot \|_r$. The k th order L^2 Sobolev space over Ω_{per} is denoted by $H^k(\Omega_{per})$, and its norm is denoted by $\| \cdot \|_{H^k}$.

We set

$$L^2_{per} = \text{the } L^2(\Omega_{per})\text{-closure of } C^\infty_{0,per},$$

$$H^k_{per} = \text{the } H^k(\Omega_{per})\text{-closure of } C^\infty_{per},$$

$$H^1_{0,per} = \text{the } H^1(\Omega_{per})\text{-closure of } C^\infty_{0,per}.$$

We note that if $f \in H^1_{0,per}$, then $f|_{x_j=-\pi/\alpha_j} = f|_{x_j=\pi/\alpha_j}$ and $f|_{x_3=0,1} = 0$. The inner product of $f_j \in L^2_{per}$ $(j = 1, 2)$ is denoted by

$$(f_1, f_2) = \int_{\Omega_{per}} f_1(x)\overline{f_2(x)}\, dx,$$

where $\overline{z}$ denotes the complex conjugate of z.

The mean value of a function $\phi(x)$ over Ω_{per} is denoted by $\langle \phi \rangle$:

$$\langle \phi \rangle = \frac{1}{|\Omega_{per}|} \int_{\Omega_{per}} \phi(x)\, dx.$$

The set of all $\phi \in L^2_{per}$ with $\langle \phi \rangle = 0$ is denoted by $L^2_{per,*}$, i.e.,

$$L^2_{per,*} = \{\phi \in L^2_{per} : \langle \phi \rangle = 0\}.$$

Furthermore, we set

$$H^k_{per,*} = H^k_{per} \cap L^2_{per,*}.$$

We denote by $C^\infty_{0,per,\sigma}$ the set of all vector fields $\boldsymbol{v}$ in $(C^\infty_{0,per})^3$ with $\operatorname{div} \boldsymbol{v} = 0$. We set

$$L^2_{per,\sigma} = \text{the } L^2(\Omega_{per})^3\text{-closure of } C^\infty_{0,per,\sigma}.$$

It is known that $(L^2_{per})^3 = L^2_{per,\sigma} \oplus G^2_{per}$, where $G^2_{per} = \{\nabla p;\, p \in H^1_{per,*}\}$ is the orthogonal complement of $L^2_{per,\sigma}$. The orthogonal projection $\mathbb{P}$ on $L^2_{per,\sigma}$ is called the Helmholtz projection. We define the projection $\boldsymbol{P}$ from $(L^2_{per})^3 \times L^2_{per}$ onto $L^2_{per,\sigma} \times L^2_{per}$ by

$$\boldsymbol{P} = \begin{pmatrix} \mathbb{P} & 0 \\ 0 & I \end{pmatrix}.$$

The orthogonal projection to G^2_{per} is denoted by $\mathbb{Q}$, i.e.,

$$\mathbb{Q} = I - \mathbb{P}.$$

For simplicity the set of all vector fields in $(L^2_{per})^3$ (resp. $(H^1_{0,per})^3$, $(H^k_{per})^3$) are frequently denoted by L^2_{per} (resp. $H^1_{0,per}$, H^k_{per}) if no confusion will occur.

We also use notation L^2_{per} for the set of all $u = {}^\top(p, \boldsymbol{w}, \theta)$ with $p \in L^2_{per}$, $\boldsymbol{w} = {}^\top(w^1, w^2, w^3) \in L^2_{per}$ and $\theta \in L^2_{per}$ if no confusion will occur.

Let ε be a positive number. We introduce an inner product $\langle\langle u_1, u_2 \rangle\rangle_\varepsilon$ for $u_j = {}^\top(p_j, \boldsymbol{w}_j, \theta_j)$ $(j = 1, 2)$ defined by

$$\langle\langle u_1, u_2 \rangle\rangle_\varepsilon = \varepsilon^2 (p_1, p_2) + \mathrm{Pr}^{-1}(\boldsymbol{w}_1, \boldsymbol{w}_2) + (\theta_1, \theta_2).$$

We also define the inner product $\langle \boldsymbol{U}_1, \boldsymbol{U}_2 \rangle$ for $\boldsymbol{U}_j = {}^\top(\boldsymbol{w}_j, \theta_j)$ $(j = 1, 2)$ by

$$\langle \boldsymbol{U}_1, \boldsymbol{U}_2 \rangle = \mathrm{Pr}^{-1}(\boldsymbol{w}_1, \boldsymbol{w}_2) + (\theta_1, \theta_2).$$

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. The null space and the range of A are denoted by $\mathrm{Ker}\,(A)$ and $R(A)$, respectively.

§ 3. Stability Results

In this section we state the stability results obtained in [12, 13].

Suppose that $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ is a stationary solution of (1.1)–(1.3) under the boundary conditions (1.7) and (1.8) with vanishing mean value condition $\int_{\Omega_{per}} p_s(x)\,dx = 0$. The linearized problem for the Oberbeck-Boussinesq system (1.1)–(1.3) around $u_s = {}^\top(p_s, \boldsymbol{v}_s, \theta_s)$ is written as

$$\mathrm{div}\,\boldsymbol{w} = 0, \tag{3.1}$$

$$\mathrm{Pr}^{-1}\partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \sqrt{\mathrm{Ra}}\,\theta \boldsymbol{e}_3 = \boldsymbol{0}, \tag{3.2}$$

$$\partial_t \theta - \Delta \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \sqrt{\mathrm{Ra}}\,\boldsymbol{w} \cdot \boldsymbol{e}_3 = 0 \tag{3.3}$$

under the boundary conditions

$$\boldsymbol{w}|_{x_3=0,1} = \boldsymbol{0}, \quad \theta|_{x_3=0,1} = 0, \tag{3.4}$$

and

$$p,\ \boldsymbol{w} \text{ and } \theta \text{ are } \mathcal{Q}\text{-periodic in } (x_1, x_2). \tag{3.5}$$

The linearized problem for the artificial compressible system is written as

$$\varepsilon^2\partial_t p + \operatorname{div}\boldsymbol{w} = 0, \tag{3.6}$$

$$\mathrm{Pr}^{-1}\partial_t\boldsymbol{w} - \Delta\boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) + \nabla p - \sqrt{\mathrm{Ra}}\,\theta\boldsymbol{e}_3 = \boldsymbol{0}, \tag{3.7}$$

$$\partial_t\theta - \Delta\theta + \boldsymbol{v}_s\cdot\nabla\theta + \boldsymbol{w}\cdot\nabla\theta_s - \sqrt{\mathrm{Ra}}\,\boldsymbol{w}\cdot\boldsymbol{e}_3 = 0 \tag{3.8}$$

with the boundary conditions (3.4) and (3.5).

Applying the projection $\boldsymbol{P}$, we rewrite the problem (3.1)–(3.5) as

$$\mathrm{Pr}^{-1}\partial_t\boldsymbol{w} - \mathbb{P}\Delta\boldsymbol{w} + \mathrm{Pr}^{-1}\mathbb{P}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) - \sqrt{\mathrm{Ra}}\,\mathbb{P}\theta\boldsymbol{e}_3 = \boldsymbol{0}, \tag{3.9}$$

$$\partial_t\theta - \Delta\theta + \boldsymbol{v}_s\cdot\nabla\theta + \boldsymbol{w}\cdot\nabla\theta_s - \sqrt{\mathrm{Ra}}\,\boldsymbol{w}\cdot\boldsymbol{e}_3 = 0 \tag{3.10}$$

with the boundary conditions (3.4) and (3.5). We introduce the linearized operator around $\boldsymbol{U}_s = {}^\top(\boldsymbol{v}_s, \theta_s)$ associated with problem (3.9)–(3.10) under (3.4) and (3.5). We define the operator $L : L^2_{per,\sigma}\times L^2_{per} \to L^2_{per,\sigma}\times L^2_{per}$ by

$$L = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(\boldsymbol{v}_s\cdot\nabla + {}^\top(\nabla\boldsymbol{v}_s)) & -\mathrm{Pr}\sqrt{\mathrm{Ra}}\,\mathbb{P}\boldsymbol{e}_3 \\ {}^\top(\nabla\theta_s) - \sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s\cdot\nabla \end{pmatrix}$$

with domain $D(L) = [(H^2_{per}\cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per}\cap H^1_{0,per}]$.

We also introduce the linearized operator around $u_s = {}^\top(p_s, \boldsymbol{w}_s, \theta_s)$ associated with (3.6)–(3.8) under (3.4) and (3.5). We define the operator $L_\varepsilon : H^1_{per,*}\times(L^2_{per})^3\times L^2_{per} \to H^1_{per,*}\times(L^2_{per})^3\times L^2_{per}$ by

$$L_\varepsilon = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\operatorname{div} & 0 \\ \mathrm{Pr}\nabla & -\mathrm{Pr}\Delta + \boldsymbol{v}_s\cdot\nabla + {}^\top(\nabla\boldsymbol{v}_s) & -\mathrm{Pr}\sqrt{\mathrm{Ra}}\,\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & -\Delta + \boldsymbol{v}_s\cdot\nabla \end{pmatrix}$$

with domain $D(L_\varepsilon) = H^1_{per,*}\times[H^2_{per}\cap H^1_{0,per}]^3\times[H^2_{per}\cap H^1_{0,per}]$.

Since we have the translation invariance in x_1 and x_2 variables, 0 is an eigenvalue of $-L_\varepsilon$ if $\partial_{x_1}u_s \neq 0$ or $\partial_{x_2}u_s \neq 0$. If this is the case, then nonzero $\partial_{x_j}u_s$ are eigenfunctions for the eigenvalue 0. Similarly, 0 is an eigenvalue of $-L$ if $\partial_{x_1}\boldsymbol{U}_s \neq 0$ or $\partial_{x_2}\boldsymbol{U}_s \neq 0$. For definiteness we consider the case $\partial_{x_j}u_s \neq 0$ (and hence $\partial_{x_j}\boldsymbol{U}_s \neq 0$) for $j = 1, 2$.

Theorem 3.1. ([12]) *Let $\partial_{x_j}u_s \neq 0$ for $j = 1, 2$. If there exists a positive number b_0 such that $\rho(-L_{\varepsilon_n}) \supset \{\lambda\in\mathbb{C}; \operatorname{Re}\lambda \geq -b_0\}\setminus\{0\}$ for some sequence $\varepsilon_n \to 0$ as $n\to\infty$ and 0 is a semisimple eigenvalue of $-L_{\varepsilon_n}$ with $\operatorname{Ker}(-L_{\varepsilon_n}) = \operatorname{span}\{\partial_{x_1}u_s, \partial_{x_2}u_s\}$, then there exists a constant $b_1 > 0$ such that $\rho(-L) \supset \{\lambda\in\mathbb{C}; \operatorname{Re}\lambda \geq -b_1\}\setminus\{0\}$ and 0 is a semisimple eigenvalue of $-L$ with $\operatorname{Ker}(-L) = \operatorname{span}\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$.*

Theorem 3.1 implies that if u_s is obtained by Chorin's method with $0 < \varepsilon \ll 1$, then it represents an observable flow in the real world. Furthermore, we also have the following instability result.

Theorem 3.2. ([12]) *Let $\partial_{x_j} u_s \neq 0$ for $j = 1, 2$. If $\sigma(-L) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$, then $\sigma(-L_\varepsilon) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\} \neq \emptyset$ for sufficiently small ε.*

Theorem 3.2 implies that unstable stationary flow cannot be obtained by Chorin's method with $0 < \varepsilon \ll 1$.

We next consider the converse question.

Theorem 3.3. ([13]) *Let $\partial_{x_j} \boldsymbol{U}_s \neq \boldsymbol{0}$ for $j = 1, 2$. Suppose that $\rho(-L) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_0\} \setminus \{0\}$ for some constant $b_0 > 0$ and 0 is a semisimple eigenvalue of $-L$ with $\operatorname{Ker}(-L) = \operatorname{span}\{\partial_{x_1}\boldsymbol{U}_s, \partial_{x_2}\boldsymbol{U}_s\}$. Then there exist constants $\varepsilon_0 > 0$, $\delta_0 > 0$ and $b_1 > 0$ such that if*

$$\inf_{\boldsymbol{w} \in H^1_{per,0}, \boldsymbol{w} \neq \boldsymbol{0}} \frac{\operatorname{Re}(\mathbb{Q}\boldsymbol{w} \cdot \nabla \boldsymbol{v}_s, \mathbb{Q}\boldsymbol{w})}{\|\nabla \mathbb{Q}\boldsymbol{w}\|_2^2} \geq -\delta_0, \tag{3.11}$$

then $\rho(-L_\varepsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -b_1\} \setminus \{0\}$ for all $0 < \varepsilon \leq \varepsilon_0$ and 0 is a semisimple eigenvalue of $-L_\varepsilon$ with $\operatorname{Ker}(-L_\varepsilon) = \operatorname{span}\{\partial_{x_1} u_s, \partial_{x_2} u_s\}$. Here δ_0 does not depend on b_0.

This gives a sufficient condition for u_s to be computed by Chorin's method with $0 < \varepsilon \ll 1$. We note that the condition (3.11) in Theorem 3.3 depends only on $\boldsymbol{v}_s$ and $\boldsymbol{w}$ but not on θ_s and θ.

Remark 1. Theorem 3.3 was proved in [13] for the incompressible Navier-Stokes equations on smooth bounded domains but not for the Oberbeck-Boussinesq equations. One can easily verify that Theorem 3.3 holds for the case of the Oberbeck-Boussinesq equations under the spatially periodic setting.

Remark 2. It is known that $-L$ is sectorial. Furthermore, $-L_\varepsilon$ is also sectorial for each $\varepsilon > 0$. (See Proposition 5.1 below.)

Remark 3. The constant ε_0 in Theorem 3.3 in general depends on b_0 and $\varepsilon_0 \to 0$ as $b_0 \to 0$, in other words, if $\sigma(-L)$ approaches to the imaginary axis, then the range of ε in Theorem 3.3 shrinks, and vanishes when $\sigma(-L)$ touches the imaginary axis. This is inconvenient to consider the stability of bifurcating stationary solutions near the bifurcation point where the spectra of the linearized operators approaches the origin. Recently, it was proved in [20] that if a stationary bifurcation from a simple eigenvalue occurs, then the range of ε does not shrink near the bifurcation point and can be taken uniformly.

The velocity fields of periodic stationary convective patterns bifurcating from the motionless state are small near the bifurcation point. Therefore, from Theorem 3.3, one can conclude the following

Corollary 3.4. *If u_s is a stable convective pattern of* (1.1)–(1.3) *bifurcating from the motionless state, then it is also stable as a solution of* (1.4)–(1.6) *for* $0 < \varepsilon \ll 1$ *when* $\mathrm{Ra} \sim \mathrm{Ra}_c$.

Remark 4. Theorems 3.1–3.3 also hold for stationary solutions of the incompressible Navier-Stokes equations (or the Oberbeck-Boussinesq system) on smooth bounded domains under nonhomogeneous boundary conditions. For example, one can apply Theorem 3.3 to the Taylor problem to conclude the stability of the Taylor vortex as a solution of the artificial compressible system, in other words, one can compute the Taylor vortex by Chorin's method. See [13] for the details.

In the remaining of this paper we will give an outline of the proof of Theorem 3.3. In section 4 we list properties of the null spaces of L and L_ε. The proof of Theorem 3.3 will be outlined in section 5. The proof of Theorems 3.1 and 3.2 are omitted here since there are no changes from [12] and no remarks to be appended there.

§ 4. The null spaces of L and L_ε

In this section we list some properties of the null spaces of L and L_ε which were given in [12, Section 4].

We introduce the adjoint operator $L^* : L^2_{per,\sigma} \times L^2_{per} \to L^2_{per,\sigma} \times L^2_{per}$ of L:

$$L^* = \begin{pmatrix} -\mathrm{Pr}\mathbb{P}\Delta + \mathbb{P}(-\boldsymbol{v}_s \cdot \nabla + (\nabla \boldsymbol{v}_s)) & \mathrm{Pr}\mathbb{P}((\nabla\theta_s) - \sqrt{\mathrm{Ra}}\,\boldsymbol{e}_3) \\ -\sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & -\Delta - \boldsymbol{v}_s \cdot \nabla \end{pmatrix}$$

with domain $D(L^*) = [(H^2_{per} \cap H^1_{0,per})^3 \cap L^2_{per,\sigma}] \times [H^2_{per} \cap H^1_{0,per}]$.

From now on we suppose that 0 is a semisimple eigenvalue of $-L$ and the corresponding eigenspace $\mathrm{Ker}\,(-L)$ is spanned by $\partial_{x_1}\boldsymbol{U}_s$ and $\partial_{x_2}\boldsymbol{U}_s$.

As is well known, $-L$ is a sectorial operator with compact resolvent and 0 is a semisimple eigenvalue of $-L$. We thus have the following results on the spectrum of $-L$.

Proposition 4.1. *Set $\boldsymbol{U}_j^{(0)} = \partial_{x_j}\boldsymbol{U}_s$ for $j = 1, 2$. Then the following assertions hold.*

(i) *There exist $\boldsymbol{U}_j^* = {}^\top(\boldsymbol{w}_j^*, \theta_j^*) \in D(L^*)$ such that $L^*\boldsymbol{U}_j^* = \boldsymbol{0}$ and $\langle \boldsymbol{U}_j^{(0)}, \boldsymbol{U}_k^* \rangle = \delta_{jk}$ for $j, k = 1, 2$. Furthermore,*

$$L^2_{per,\sigma} \times L^2_{per} = X_0 \oplus X_1,$$

where $X_0 = \mathrm{Ker}\,(-L)$ *and*

$$X_1 = R(-L) = \{\boldsymbol{U} \in L^2_{per,\sigma} \times L^2_{per}; \langle \boldsymbol{U}, \boldsymbol{U}_j^* \rangle = 0, j = 1, 2\}.$$

The eigenprojection $\boldsymbol{\Pi}_0$ *for the eigenvalue* 0 *of* $-L$ *is given by*

$$\boldsymbol{\Pi}_0 \boldsymbol{U} = \langle \boldsymbol{U}, \boldsymbol{U}_1^* \rangle \boldsymbol{U}_1^{(0)} + \langle \boldsymbol{U}, \boldsymbol{U}_2^* \rangle \boldsymbol{U}_2^{(0)}.$$

(ii) *Set* $\boldsymbol{\Pi}_0^c = I - \boldsymbol{\Pi}_0$. *There exist constants* $a_0 > 0$ *and* $c_0 \in \mathbb{R}$ *such that*

$$\Sigma \setminus \{0\} \subset \rho(-L),$$

where

$$\Sigma := \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \geq -a_0 |\mathrm{Im}\,\lambda|^2 + c_0\},$$

and the estimates

$$\|(\lambda + L)^{-1} \boldsymbol{F}\|_2 \leq C \left\{ \frac{1}{|\lambda|} \|\boldsymbol{\Pi}_0 \boldsymbol{F}\|_2 + \frac{1}{|\lambda| + 1} \|\boldsymbol{\Pi}_0^c \boldsymbol{F}\|_2 \right\},$$

$$\|\partial_x^2 (\lambda + L)^{-1} \boldsymbol{F}\|_2 \leq C \left\{ \frac{1}{|\lambda|} \|\boldsymbol{\Pi}_0 \boldsymbol{F}\|_2 + \|\boldsymbol{\Pi}_0^c \boldsymbol{F}\|_2 \right\}$$

hold uniformly for $\lambda \in \Sigma \setminus \{0\}$. *Furthermore, if* $\boldsymbol{\Pi}_0 \boldsymbol{F} = 0$, *then* $\boldsymbol{\Pi}_0 (\lambda + L)^{-1} \boldsymbol{F} = 0$ *and the above estimates hold for any* $\lambda \in \Sigma$.

We set

$$Y = H^1_{per,*} \times (L^2_{per})^3 \times L^2_{per}.$$

We define an operator $\mathscr{L}_{\varepsilon,\lambda} : Y \to Y$ by

$$D(\mathscr{L}_{\varepsilon,\lambda}) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}],$$

$$\mathscr{L}_{\varepsilon,\lambda} = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ \mathrm{Pr}\nabla & \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s \cdot \nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{Pr}\sqrt{\mathrm{Ra}}\,\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla \theta_s) - \sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & \lambda - \Delta + \boldsymbol{v}_s \cdot \nabla \end{pmatrix},$$

and its adjoint $\mathscr{L}^*_{\varepsilon,\lambda} : Y \to Y$ by

$$D(\mathscr{L}^*_{\varepsilon,\lambda}) = H^1_{per,*} \times [H^2_{per} \cap H^1_{0,per}]^3 \times [H^2_{per} \cap H^1_{0,per}],$$

$$\mathscr{L}^*_{\varepsilon,\lambda} = \begin{pmatrix} 0 & -\frac{1}{\varepsilon^2}\mathrm{div} & 0 \\ -\mathrm{Pr}\nabla & \overline{\lambda} - \mathrm{Pr}\Delta - \boldsymbol{v}_s \cdot \nabla + (\nabla \boldsymbol{v}_s) & \mathrm{Pr}(\nabla \theta_s) - \mathrm{Pr}\sqrt{\mathrm{Ra}}\,\boldsymbol{e}_3 \\ 0 & -\sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & \overline{\lambda} - \Delta - \boldsymbol{v}_s \cdot \nabla \end{pmatrix}.$$

Note that

$$\mathscr{L}_{\varepsilon,0} = L_\varepsilon.$$

We also set

$$\boldsymbol{A}_\lambda = \begin{pmatrix} \mathrm{Pr}\nabla & \lambda - \mathrm{Pr}\Delta + \boldsymbol{v}_s\cdot\nabla + {}^\top(\nabla \boldsymbol{v}_s) & -\mathrm{Pr}\sqrt{\mathrm{Ra}}\,\boldsymbol{e}_3 \\ 0 & {}^\top(\nabla\theta_s) - \sqrt{\mathrm{Ra}}\,{}^\top\boldsymbol{e}_3 & \lambda - \Delta + \boldsymbol{v}_s\cdot\nabla \end{pmatrix}.$$

We have the following characterization of the null space of L_ε which were proved in [12].

Proposition 4.2. ([12]) *Let $\varepsilon > 0$. The following assertions hold.*

(i) *Let $u_j^{(0)} = \partial_{x_j} u_s$ $(j = 1, 2)$. Then*

$$\mathrm{Ker}\,(\mathscr{L}_{\varepsilon,0}) = \mathrm{Ker}\,(L_\varepsilon) = \mathrm{span}\,\{u_1^{(0)}, u_2^{(0)}\}.$$

(ii) *For each $j = 1, 2$, there exists a unique $p_j^* \in H^1_{per,*}$ such that $\mathscr{L}^*_{\varepsilon,0} u_j^* = 0$, $u_j^* = {}^\top(p_j^*, \boldsymbol{w}_j^*, \theta_j^*)$, where ${}^\top(\boldsymbol{w}_j^*, \theta_j^*)$ $(j = 1, 2)$ are the functions given in Proposition 4.1 (i). Furthermore,*

$$R(\mathscr{L}_{\varepsilon,0}) = R(L_\varepsilon) = \{u \in Y; \langle\langle u, u_j^*\rangle\rangle_\varepsilon = 0, j = 1, 2\}.$$

(iii) *There exists a positive constant ε_1 such that if $0 < \varepsilon \le \varepsilon_1$, then*

$$Y = \mathrm{Ker}\,(L_\varepsilon) \oplus R(L_\varepsilon).$$

Therefore, 0 is a semisimple eigenvalue of L_ε. Furthermore, if $F = {}^\top(f, \boldsymbol{g}, h) \in R(L_\varepsilon)$, then there exists a unique solution $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon) \cap R(L_\varepsilon)$ of $L_\varepsilon u = F$ and u satisfies

$$\|u\|_{H^1\times H^2\times H^2} \le C\{\varepsilon^2 \|f\|_{H^1} + \|\boldsymbol{F}\|_2\},$$

where $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$.

Remark 5. To prove Propositions 4.1 and 4.2, we do not use any particular form of the eigenfunctions $\partial_{x_1} u_s$ and $\partial_{x_2} u_s$ but only use the fact that 0 is a semisimple eigenvalue of $-L$. Therefore, Propositions 4.1 and 4.2 hold if 0 is a semisimple eigenvalue of $-L$ no matter what the eigenspace is; and if 0 is a semisimple eigenvalue of $-L$, one can restate Propositions 4.1 and 4.2 in terms of eigenfunctions forming the basis of the eigenspace for the eigenvalue 0.

§ 5. Proof of Theorem 3.3

In this section we give an outline of a proof of Theorem 3.3 following the arguments in [12, 13].

Let us consider the resolvent problem for $-L_\varepsilon$:

$$\lambda u + L_\varepsilon u = F, \tag{5.1}$$

where $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$ and $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. Problem (5.1) is written as

$$\varepsilon^2 \lambda p + \operatorname{div} \boldsymbol{w} = \varepsilon^2 f, \tag{5.2}$$

$$\mathrm{Pr}^{-1} \lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \mathrm{Pr}^{-1}(\boldsymbol{v}_s \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_s) + \nabla p - \sqrt{\mathrm{Ra}}\, \theta \boldsymbol{e}_3 = \mathrm{Pr}^{-1} \boldsymbol{g}, \tag{5.3}$$

$$\lambda \theta - \Delta \theta + \boldsymbol{v}_s \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla \theta_s - \sqrt{\mathrm{Ra}}\, \boldsymbol{w} \cdot \boldsymbol{e}_3 = h, \tag{5.4}$$

and $u = {}^\top(p, \boldsymbol{w}, \theta)$ satisfies the boundary conditions (3.4) and (3.5).

Proposition 5.1. *There exist constants $a_1 > 0$ and $b_1 > 0$ as $\varepsilon \to 0$ such that $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -a_1 \varepsilon^2 |\operatorname{Im} \lambda|^2 + b_1\} \subset \rho(-L_\varepsilon)$ for all $0 < \varepsilon \leq 1$.*

This proposition can be proved by the Matsumura-Nishida energy method ([15]). See [12] for the details.

We next show that the spectrum of $-L_\varepsilon$ in a disc with radius $O(\varepsilon^{-1})$ can be viewed as a perturbation of the one of $-L$. Under the assumption of Theorem 3.3, we see that the constant c_0 in Proposition 4.1 (ii) can be taken in such a way that $c_0 < 0$ by changing $a_0 > 0$ suitably. In what follows we fix these $a_0 > 0$ and $c_0 < 0$. Since 0 is a semisimple eigenvalue of $-L$, we have the following estimates for $\mathscr{L}_{\varepsilon,\lambda}^{-1}$. We set

$$Y_{1,\varepsilon} = R(L_\varepsilon) = \{u \in Y; \langle\langle u, u_j^* \rangle\rangle_\varepsilon = 0, j = 1, 2\}.$$

Proposition 5.2. *Let $\varepsilon > 0$. If $\lambda \in \Sigma \setminus \{0\}$, then $\mathscr{L}_{\varepsilon,\lambda}$ has a bounded inverse $\mathscr{L}_{\varepsilon,\lambda}^{-1}$ and ${}^\top(p, \boldsymbol{v}, \theta) = \mathscr{L}_{\varepsilon,\lambda}^{-1} F$ for $F = {}^\top(f, \boldsymbol{g}, h) \in Y$ satisfies*

$$\|\boldsymbol{U}\|_2 \leq \frac{C}{|\lambda|} \sum_{j=1}^{2} \left| \langle\langle F, u_j^* \rangle\rangle_\varepsilon \right| + C \left\{ \varepsilon^2 \|f\|_{H^1} + \frac{1}{|\lambda| + 1} \|\boldsymbol{F}\|_2 \right\},$$

$$\|\partial_x^2 \boldsymbol{U}\|_2 + \|\partial_x p\|_2 \leq \frac{C}{|\lambda|} \sum_{j=1}^{2} \left| \langle\langle F, u_j^* \rangle\rangle_\varepsilon \right| + C \left\{ \varepsilon^2 (|\lambda| + 1) \|f\|_{H^1} + \|\boldsymbol{F}\|_2 \right\},$$

where $\boldsymbol{U} = {}^\top(\boldsymbol{w}, \theta)$ and $\boldsymbol{F} = {}^\top(\boldsymbol{g}, h)$. Furthermore, if $F \in Y_{1,\varepsilon}$, then ${}^\top(p, \boldsymbol{v}, \theta) = \mathscr{L}_{\varepsilon,\lambda}^{-1} F$ exists uniquely in $Y_{1,\varepsilon}$ and the above estimates hold with $\lambda = 0$ and $\langle\langle F, u_j^* \rangle\rangle_\varepsilon = 0$ $(j = 1, 2)$.

See [12, Section 6] for a proof of Proposition 5.2.

Proposition 5.3. *There exist positive numbers ε_1 and a_2 such that*

$$\Sigma \cap \{\lambda \in \mathbb{C}; 0 < |\lambda| \leq a_2\varepsilon^{-1}\} \subset \rho(-L_\varepsilon)$$

and the following estimates

$$\|(\lambda + L_\varepsilon)^{-1}F\|_{H^1\times L^2\times L^2} \leq C\frac{1}{|\lambda|}\left\{\varepsilon^2\|f\|_{H^1} + \|\boldsymbol{g}\|_2 + \|h\|_2\right\},$$

$$\|(\lambda + L_\varepsilon)^{-1}F\|_{H^1\times H^2\times H^2} \leq C\left(\frac{1}{|\lambda|} + 1\right)\left\{\varepsilon^2\|f\|_{H^1} + \|\boldsymbol{g}\|_2 + \|h\|_2\right\}$$

hold for all $\lambda \in \Sigma \cap \{\lambda \in \mathbb{C}; 0 < |\lambda| \leq a_2\varepsilon^{-1}\}$ and $0 < \varepsilon \leq \varepsilon_1$. Furthermore,

$$\Sigma \cap \{\lambda \in \mathbb{C}; |\lambda| \leq a_2\varepsilon^{-1}\} \subset \rho(-L_\varepsilon|_{Y_{1,\varepsilon}}).$$

Remark 6. In general, the constant ε_1 depends on b_0 and $\varepsilon_1 \to 0$ as $b_0 \to 0$, in other words, if $\sigma(-L)$ approaches to the imaginary axis, then the range of ε in Proposition 5.2 shrinks, and vanishes when $\sigma(-L)$ touches the imaginary axis.

Proof of Proposition 5.3. Proposition 5.3 can be proved similarly to the proof of [12, Proposition 6.3], but we here give a proof since a small correction is needed there.

We write the resolvent problem

$$(\lambda + L_\varepsilon)u = F$$

on Y as

$$\mathscr{L}_{\varepsilon,\lambda}u + \lambda Ju = F, \tag{5.5}$$

where $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. If $\lambda \in \Sigma\setminus\{0\}$, then it follows from Proposition 5.2 that (5.5) is written as

$$\mathscr{L}_{\varepsilon,\lambda}(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)u = F,$$

and, furthermore, we have

$$\|\mathscr{L}_{\varepsilon,\lambda}^{-1}JF\|_{H^1\times H^2\times H^2} \leq \varepsilon^2 C_1\left(|\lambda| + \frac{1}{|\lambda|}\right)\|f\|_{H^1}$$

for all $F = {}^\top(f, \boldsymbol{g}, h) \in Y$. It then follows that there exists $\varepsilon_1 > 0$ such that if $\lambda \in \Sigma\setminus\{0\}$ and $|\lambda| \leq 1/(4\sqrt{C_1}\varepsilon)$, then $\mathscr{L}_{\varepsilon,\lambda}^{-1}JF \in D(\mathscr{L}_{\varepsilon,\lambda}) = D(L_\varepsilon)$ and $\|\lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}JF\|_{H^1\times H^2\times H^2} \leq \frac{1}{2}\|F\|_{H^1\times L^2\times L^2}$ for $0 < \varepsilon \leq \varepsilon_1$. Therefore, $(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)$ is boundedly invertible both on Y and $D(L_\varepsilon)$ with estimates

$$\|(I + \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)^{-1}F\|_{H^1\times L^2\times L^2} \leq 2\|F\|_{H^1\times L^2\times L^2}$$

for $F \in Y$ and

$$\|(I+\lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J)^{-1}F\|_{H^1\times H^2\times H^2} \le 2\|F\|_{H^1\times H^2\times H^2}$$

for $F \in D(L_\varepsilon)$. With this, together with Proposition 5.2, we find that $\lambda + L_\varepsilon = \mathscr{L}_{\varepsilon,\lambda} + \lambda J$ has a bounded inverse $(\lambda+L_\varepsilon)^{-1} = (\mathscr{L}_{\varepsilon,\lambda}+\varepsilon^2\lambda J)^{-1}$ on Y which satisfies

$$(\lambda+L_\varepsilon)^{-1} = \mathscr{L}_{\varepsilon,\lambda}^{-1} - \lambda\mathscr{L}_{\varepsilon,\lambda}^{-1}J\sum_{N=0}^{\infty}(-\lambda)^N(\mathscr{L}_{\varepsilon,\lambda}^{-1}J)^N\mathscr{L}_{\varepsilon,\lambda}^{-1}$$

and the desired estimates. This shows the first assertion.

As for the second assertion, we see from Proposition 4.2 that $0 \in \rho(-L_\varepsilon|_{Y_{1,\varepsilon}})$ and $\|(-L_\varepsilon|_{Y_{1,\varepsilon}})^{-1}F\|_{H^1\times H^2\times H^2} \le C\|F\|_{H^1\times L^2\times L^2}$ uniformly for $0<\varepsilon\le 1$. The desired result is now obtained by a standard perturbation argument. This completes the proof. □

Theorem 3.3 follows from Propositions 4.2, 5.1 and 5.3 if $\sqrt{b_1/a_1} < a_2$ for $0<\varepsilon\ll 1$. If $\sqrt{b_1/a_1} \ge a_2$, we still need to show that some range of λ near the imaginary axis with $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ belongs to $\rho(-L_\varepsilon)$.

In the remaining we consider the case $\sqrt{b_1/a_1} \ge a_2$. We first show that the θ-component is of order $O(\varepsilon)$ if $\mathrm{Im}\,\lambda = O(\varepsilon^{-1})$. Recall that the Poincaré inequality

$$\|\nabla\theta\|_2 \ge \beta\|\theta\|_2$$

holds for $\theta \in H^1_{0,per}$ with some positive constant β.

Proposition 5.4. *Let* ${}^\top(p,\boldsymbol{w},\theta) \in D(L_\varepsilon)$ *satisfy* (5.2)–(5.4). *If* $\mathrm{Re}\,\lambda \ge -\frac{\beta^2}{2}$, *then the following estimate holds:*

$$\|\theta\|_2 \le \frac{1}{|\mathrm{Im}\,\lambda|}\left(1+\frac{2\|\boldsymbol{v}_s\|_\infty}{\beta}\right)\left\{(\|\nabla\theta_s\|_\infty+\sqrt{\mathrm{Ra}})\|\boldsymbol{w}\|_2+\|h\|_2\right\}.$$

This estimate can be obtained by a standard energy method. See [12] for a proof. The idea is that $-\Delta$ with zero-Dirichlet boundary condition is self-adjoint, and hence, $\|(\lambda-\Delta)^{-1}\| = O(|\mathrm{Im}\,\lambda|^{-1})$ as $|\mathrm{Im}\,\lambda|\to\infty$.

Combining Proposition 5.4 and the argument in [13, Section 3], one can prove the following estimate of the velocity component $\boldsymbol{w}$ of a solution $u = {}^\top(p,\boldsymbol{w},\theta)$ of (5.2)–(5.4) for λ near the imaginary axis with $\mathrm{Im}\,\lambda = O(\varepsilon^{-1})$. .

Proposition 5.5. *Let* $\lambda = \mu + i\frac{\eta}{\varepsilon}$ *with* $\mu,\eta\in\mathbb{R}$. *Suppose that* $u = {}^\top(p,\boldsymbol{w},\theta)$ *is a solution of* (5.2)–(5.4). *For given positive numbers* μ_1 *and* η_* *there exist positive constants* δ_1 *and* $C' = C'(\|\boldsymbol{v}_s\|_{C^1},\|\theta_s\|_{C^1},\beta,\Omega)$ *such that if*

$$\inf\left\{\frac{\mathrm{Re}\,(\nabla\varphi\cdot\nabla\boldsymbol{v}_s,\nabla\varphi)}{\|\Delta\varphi\|_2^2};\varphi\in H^2_*(\Omega),\ \varphi\ne 0,\ \frac{\partial\varphi}{\partial\boldsymbol{n}}|_{\partial\Omega}=0\right\} \ge -\delta_1$$

and

$$-c_1\beta^2 \le \mu \le \mu_1,\ \eta_* \le \eta \le C'\varepsilon^{-1},$$

then

$$(\eta^3+\beta^2\eta)\|\boldsymbol{w}\|_2^2 + \eta\|\nabla \boldsymbol{w}\|_2^2 \le C_\eta\left\{\varepsilon^2\|f\|_{H^1}^2 + \|\boldsymbol{g}\|_2^2 + \|h\|_2^2\right\}$$

for all $0<\varepsilon \le C'\min\{1,\eta_*,\sqrt{\frac{\eta_*}{\mu_*}},\frac{\eta_*}{\mu_*},\eta_*\mu_*^{-\frac{2}{3}},\sqrt{\frac{1}{\mu_*}}\}$ *with* $\mu_*=\max\{c_1\beta^2,\,\mu_1\}$. *Here* c_1 *is a positive constant depending on* Pr *and* Ra.

The proof of Theorem 3.3 is now complete if we take η_* in Proposition 5.5 in such a way that $\eta_*=\frac{a_2}{2}$.

We outline a proof of Proposition 5.5. The details of the following argument can be found in [13, Section 3].

We see from (5.2) that

$$p = -\frac{1}{\varepsilon^2\lambda}\mathrm{div}\,\boldsymbol{w} + \frac{1}{\lambda}f. \tag{5.6}$$

Substituting (5.6) into (5.3), we have

$$\frac{\varepsilon^2\lambda^2}{\mathrm{Pr}}\boldsymbol{w} - \varepsilon^2\lambda\Delta\boldsymbol{w} - \nabla\mathrm{div}\,\boldsymbol{w} + \frac{\varepsilon^2\lambda}{\mathrm{Pr}}(\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s) - \varepsilon^2\lambda\sqrt{\mathrm{Ra}}\,\theta\boldsymbol{e}_3 = \varepsilon^2\boldsymbol{G}_\lambda, \tag{5.7}$$

where $\boldsymbol{G}_\lambda = \frac{\lambda}{\mathrm{Pr}}\boldsymbol{g} - \nabla f$.

In a similar manner to the proof of [12, Proposition 6.5], one can show the following estimate by applying an energy method to (5.7) and using Proposition 5.4.

Proposition 5.6. *Let* μ_1 *and* η_* *be given positive numbers. Suppose that* $u = {}^\top(p,\boldsymbol{w},\theta)\in D(L_\varepsilon)$ *is a solution of* (5.2)–(5.4) *with* $\lambda=\mu+i\frac{\eta}{\varepsilon}$, $\mu,\eta\in\mathbb{R}$. *There exists a positive constant* $C'=C'(\|\boldsymbol{v}_s\|_{C^1},\|\theta_s\|_{C^1}\beta,\Omega)$ *such that if*

$$\varepsilon \le C'\min\{1,\eta_*,\frac{\eta_*}{\mu_*},\frac{1}{\sqrt{\mu_1}}\},\ -c_1\beta^2\le\mu\le\mu_1,\ \eta_*\le\eta\le\frac{1}{4\varepsilon}$$

with $\mu_*=\max\{c_1\beta^2,\,\mu_1\}$, *then*

$$\begin{aligned}&(\eta^3+2\beta^2\eta)\|\boldsymbol{w}\|_2^2+\eta\|\nabla\boldsymbol{w}\|_2^2\\&\le -64\eta\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w}) + C_\eta\left\{\varepsilon^2\|f\|_{H^1}^2+\|\boldsymbol{g}\|_2^2+\|h\|_2^2\right\}.\end{aligned}$$

We need to replace $\boldsymbol{w}$ in $\mathrm{Re}\,(\boldsymbol{w}\cdot\nabla\boldsymbol{v}_s,\boldsymbol{w})$ on the right-hand side of the estimate in Proposition 5.6 by $\mathbb{Q}\boldsymbol{w}$. To this end, we use the following estimate of the incompressible part of $\boldsymbol{w}$.

Proposition 5.7. *Let μ_0, μ_1 and η_* be given positive numbers. Suppose that $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$ is a solution of* (5.2)–(5.4) *with $\lambda = \mu + i\frac{\eta}{\varepsilon}$, $-\mu_0 \le \mu \le \mu_1$, $\eta \ge \eta_*$. If $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ is the Helmholtz decomposition of $\boldsymbol{w}$, then*

$$\|\boldsymbol{v}\|_2^2 \le C\left\{\frac{\varepsilon^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\|\nabla\varphi\|_{H^1}^2 + \frac{\varepsilon^2}{\eta^2}\|\nabla\varphi\|_{H^2}^2 + \frac{\varepsilon^2}{\eta^2}\|\boldsymbol{g}\|_2^2 + \frac{\varepsilon^2}{\eta^2}\|\nabla\boldsymbol{w}\|_2^2 + \frac{\varepsilon^2}{\eta^2}\|\boldsymbol{w}\|_2^2\right\},$$

$$\|\boldsymbol{v}\|_{H^2}^2 \le C\left\{\frac{\eta^{\frac{3}{2}}}{\varepsilon^{\frac{3}{2}}}\|\nabla\varphi\|_{H^1}^2 + \|\nabla\varphi\|_{H^2}^2 + \|\boldsymbol{g}\|_2^2 + \|\nabla\boldsymbol{w}\|_2^2 + \|\boldsymbol{w}\|_2^2\right\}.$$

Proposition 5.7 is proved by the following estimate for the Stokes system with nonhomogeneous boundary data.

Lemma 5.8. *Suppose that ${}^\top(p, \boldsymbol{v}) \in H_*^1(\Omega) \times H^2(\Omega)$ is a solution of*

$$\begin{cases} \operatorname{div}\boldsymbol{v} = 0, \\ \lambda\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla p = \boldsymbol{g}, \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{\psi}, \end{cases} \tag{5.8}$$

with $\lambda \in \{\lambda \in \mathbb{C};\ |\arg\lambda| \le \pi - \omega\}$ for some $0 < \omega < \frac{\pi}{2}$, $\boldsymbol{g} \in L^2(\Omega)$ and $\boldsymbol{\psi} \in H^{\frac{3}{2}}(\partial\Omega)$ satisfying $\boldsymbol{\psi}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. Then there exists a positive constant $C = C(\omega, \Omega)$ such that

$$|\lambda|\|\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{H^2} + \|p\|_{H^1} \le C\{\|\boldsymbol{g}\|_2 + |\lambda|^{\frac{3}{4}}\|\boldsymbol{\psi}\|_{L^2(\partial\Omega)} + \|\boldsymbol{\psi}\|_{H^{\frac{3}{2}}(\partial\Omega)}\}.$$

See [13, Section 4] for a proof of Lemma 5.8.

If $u = {}^\top(p, \boldsymbol{w}, \theta) \in D(L_\varepsilon)$ is a solution of (5.2)–(5.4) and $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ is the Helmholtz decomposition of $\boldsymbol{w}$, then

$$\begin{cases} \operatorname{div}\boldsymbol{v} = 0, \\ \lambda\boldsymbol{v} - \mathrm{Pr}\Delta\boldsymbol{v} + \nabla q = \boldsymbol{g} - (\boldsymbol{v}_s\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_s - \mathrm{Pr}\sqrt{\mathrm{Ra}}\,\theta\boldsymbol{e}_3), \\ \boldsymbol{v}|_{\partial\Omega} = -\nabla\varphi|_{\partial\Omega}. \end{cases}$$

Here

$$q = \lambda\varphi - \mathrm{Pr}\Delta\varphi + p.$$

Applying Lemma 5.8, we obtain the estimates in Proposition 5.7.

We further need to estimate the irrotational flow part $\nabla\varphi$. Noting that $\operatorname{div}\boldsymbol{w} = \Delta\varphi$, one can see from (5.7) that φ satisfies the following estimates.

Proposition 5.9. *If $\boldsymbol{w} = \boldsymbol{v} + \nabla\varphi$ is the Helmholtz decomposition of $\boldsymbol{w}$ as in Proposition 5.7, then there exists a positive constant $C' = C'(\|\boldsymbol{v}_s\|_{C^1}, \|\theta_s\|_{C^1})$ such that for $0 < \varepsilon \le C' \min\{1, \frac{\eta_*}{\mu_*}, \eta_*\}$ with $\mu_* = \max\{\mu_0, \mu_1\}$, the following estimates hold true:*

$$\|\Delta\varphi\|_2^2 \le C\left\{\eta^2\|\boldsymbol{w}\|_2^2 + \varepsilon\eta\|\nabla\boldsymbol{w}\|_2^2\right\} + C_\eta\varepsilon^2\left\{\varepsilon^2\|f\|_{H^1}^2 + \|\boldsymbol{g}\|_2^2 + \|h\|_2^2\right\},$$

$$\frac{1}{\eta^2}\|\nabla\Delta\varphi\|_2^2 \le C\left\{\eta^2\|\boldsymbol{w}\|_2^2 + \varepsilon\eta\|\nabla\boldsymbol{w}\|_2^2 + \varepsilon^2\|\Delta\boldsymbol{v}\|_2^2\right\} + C_\eta\varepsilon^2\left\{\varepsilon^2\|f\|_{H^1}^2 + \|\boldsymbol{g}\|_2^2 + \|h\|_2^2\right\}.$$

Combining Propositions 5.6, 5.7 and 5.9, one can obtain the estimate in Proposition 5.5. See [13, Section 3] for the details.

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