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Blowup of solutions to an indirect chemotaxis system

By

Takasi Senba*

Abstract

This manuscript summarizes results by Fujie and Senba (2017, 2019). In this manuscript, we describe properties of solutions to an indirect chemotaxis system. The system is one of chemotaxis systems, and has three unknown functions. These three functions correspond to density of living thing and concentrations of two kinds of chemical substances, respectively. When the dimension of the domain is less than four, our system does not have blowup solutions. In four dimensional case, our system has blowup solutions. In this manuscript, I will describe details of these results and sketch of these proofs.

§ 1. Introduction

This manuscript is based on the joint work with Kentarou Fujie (Tohoku University).

In this manuscript, we describe some properties of solutions to an indirect chemotaxis system, which is different from Keller-Segel system. However, the indirect chemotaxis system is related to the Keller-Segel system. Then, we begin with the explanation of the Keller-Segel system.

The following system is the classical Keller-Segel system.

$$(KS)\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) & \text{in } \Omega\times(0,T), \\ \tau v_t = \Delta v - v + u & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial\nu} - \chi u\dfrac{\partial v}{\partial\nu} = 0,\ v = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \Omega. \end{cases}$$

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*Fukuoka University, Fukuoka 814-0180, Japan.
e-mail: senba@fukuoka-u.ac.jp

Here, χ and τ are positive constants, Ω is a bounded domain Ω of $\mathbf{R}^n$ $(n \geq 1)$, the boundary $\partial\Omega$ is smooth, initial conditions u_0 and v_0 are nonnegative smooth functions and T is the maximal existence time of the classical solution.

In this manuscript, we treat only classical solutions.

The system (KS) describes the aggregation of bacteria. The function u represents the density of bacteria, and the function v represents chemical concentration produced by the bacteria, and the chemical substance is an attractant. Then, the bacteria move towards higher concentration. We say this property of living things chemotaxis. In this case, the relation between living things and chemoattractant is direct. Then, we consider chemotaxis mentioned by Keller-Segel system as direct process.

The following are well known properties on solutions to Keller-Segel system:

The solutions satisfy that $u \geq 0$ in $\Omega \times (0, T)$ and that $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ $(t \in (0, T))$, where $\| \cdot \|_{L^p(\Omega)}$ is the standard L^p norm for $p \in [1, \infty]$.

If $n = 1$ or if $n = 2$ and $\|u_0\|_{L^1(\Omega)} < 8\pi/\chi$, solutions exist globally in time, and are bounded (see [4]).

If $n \geq 3$ or if $n = 2$ and $\|u_0\|_{L^1(\Omega)} > 8\pi/\chi$, there exist solutions blowing up (see [3, 5]).

Here, we say that a solution (u, v) blows up at a time T, if

$$\limsup_{t \to T} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) = \infty.$$

That is to say, two dimensional case is critical and the critical quantity is the L^1 norm of solution u. Moreover, if a solution blows up at a finite time T, T is also the maximal existence time of the classical solution.

In this manuscript, we consider classical solutions to the following system.

$$(P) \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T), \\ \tau_1 v_t = \Delta v - v + w & \text{in } \Omega \times (0, T), \\ \tau_2 w_t = \Delta w - w + u & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} = 0, \ v = w = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0, \ w(\cdot, 0) = w_0 & \text{in } \Omega. \end{cases}$$

Here, we assume the following:
χ, τ_1 and τ_2 are positive constants.
$\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ is bounded and the boundary $\partial\Omega$ is smooth.
u_0, v_0 and w_0 are nonnegative and smooth functions.

This system is one of chemotaxis system. The function u represents density of living thing, the function w represents chemical concentration and the chemical substance is produced by the living things, and the function v represents also concentration of other chemical substance, the chemical substance is produced by chemical reaction of chemical substance v, and the substance corresponding to v is attractant. Then, we regard the chemotaxis mentioned by our system (P) as indirect process. Then, we regard this system as one of indirect chemotaxis systems.

The solutions to (P) satisfy that

$$w \geq 0, w \geq 0, w \geq 0 \quad \text{in } \Omega \times (0, T)$$

and that $\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ $(t \in (0, T))$. There exists a unique time-local classical solution (u, v, w) to (P).

Our aim is the investigation of conditions for blowup of solutions and properties of the blowup solutions. We say that a solution (u, v, w) to (P) blows up at a time $T \in (0, \infty]$, if

$$\limsup_{t \to T} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)}) = \infty.$$

The following system is a tumor invasion model.

$$(FIWY) \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta v + wz & \text{in } \Omega \times (0, \infty), \\ z_t = -\gamma wz & \text{in } \Omega \times (0, \infty), \\ w_t = \Delta w - w + u & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, v(\cdot, 0) = v_0, & \\ \quad w(\cdot, 0) = w_0, z(\cdot, 0) = z_0 & \text{in } \Omega. \end{cases}$$

Here, χ and γ are positive constants. u_0, v_0, w_0 and z_0 are nonnegative and smooth. $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ is bounded and the boundary $\partial\Omega$ is smooth.

Fujie, Ito, Winkler and Yokota [2016, DCDS] show that solutions to (FIWY) exist globally in time and are bounded if $n \leq 3$.

This system and the result motivate our research. Does the system (FIWY) have blowup solutions in the high dimensional case ? However, we think that the ODE in this system makes analysis of solutions difficult. If the constant γ is equal to 0 and if the initial condition z_0 is a positive constant in the domain, the system (FIWY) is similar to our system (P). Then, we can regard our system (P) as a simplified system of (FIWY) and we investigate solutions to (P).

§ 2. Our results

As mentioned in the previous section, our aim is the investigate of conditions for blowup of solutions to (P) and the one of properties of blowup solutions to (P). For this problem, we get the following partial results.

Theorem 2.1 ([1, 2]).
(1) If Ω is a bounded domain of $\mathbf{R}^n$ and one of the following assumptions holds:
$\cdot$ $n \leq 3$.
$\cdot$ $n = 4$ *and* $\|u_0\|_{L^1(\Omega)} < (8\pi)^2/\chi$.
Then, solutions to our system (P) exist globally in time and are uniformly bounded in time.
(2) If Ω is a bounded and convex domain of $\mathbf{R}^4$, then there exist blowup solutions to our system (P) satisfying $\|u_0\|_{L^1(\Omega)} > (8\pi)^2/\chi$.

This theorem says that our system (P) does not have any blowup solutions if $n \leq 3$. And, there exist blowup solutions in four dimensional case, and in the four dimensional case the number $(8\pi)^2/\chi$ is threshold. In this sense, the number $(8\pi)^2/\chi$ appearing in four dimensional our system corresponds to the number $8\pi/\chi$ appearing in two dimensional Keller-Segel system.

In (2) of Theorem 2.1, we can not judge whether the blowup time is finite or infinite. The following theorem guarantees the existence of finite-time blowup solutions.

Theorem 2.2.
If Ω is a bounded ball of $\mathbf{R}^4$, there exist radial solutions blowing up at a finite time T. Moreover, the solutions satisfy that

$$u(t) \to m(0)\delta_0 + f \quad \text{as } t \to T,$$

where $m(0) \geq (8\pi)^2/\chi$, $f \in L^1(\Omega)$ and δ_0 is the delta function whose support is the origin.

Moreover, for radial blowup solutions to the following parabolic-elliptic system, the weight of delta function is equal to the threshold number.

$$(PEE) \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0,T), \\ 0 = \Delta v - v + w & \text{in } \Omega \times (0,T), \\ 0 = \Delta w - w + u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} = 0, \ v = w = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases}$$

Theorem 2.3. *Let Ω is a bounded domain of $\mathbf{R}^4$ and let u_0 be nonnegative and smooth. Then, there exist solutions to (PEE) blowing up at a finite time T.*

Furthermore, if Ω is a bounded ball in $\mathbf{R}^4$ and if a radial solution to (PEE) blows up at a finite time T, the solution satisfies

$$u(\cdot,t) \to \frac{(8\pi)^2}{\chi}\delta_0 + f \qquad \text{as } t \to T,$$

Here, δ_0 is the delta function at the origin, and f is a radial and nonnegative L^1 function.

§ 3. Key properties of solutions to (P)

Here and henceforth, we assume $\tau = \tau_1 = \tau_2 = 1$.

Because, the positivity of these constant is important. However, the quantity is essentially independent of properties of solutions. In order to show our results, we use the following properties.

We describe key properties of solutions to (P), and we use these properties for the proofs of our results.

Conservation law. For solutions (u, v, w) to (P), the following equation holds.

$$\frac{d}{dt}\mathcal{F}(u(t), v(t)) + \mathcal{D}(u(t), v(t)) = 0.$$

Here,

$$\begin{aligned}
\mathcal{F}(u,v) &= \int_\Omega (u\log u - \chi uv)dx + \frac{\chi}{2}\int_\Omega |v_t|^2 dx + \frac{\chi}{2}\int_\Omega |(-\Delta+1)v|^2 dx,\\
\mathcal{D}(u,v) &= 2\chi\int_\Omega \left(|\nabla v_t|^2 + |v_t|^2\right) dx + \int_\Omega u|\nabla(\log u - \chi v)|^2 dx.
\end{aligned}$$

The function $\mathcal{F}$ is referred to as Lyapunov function. Since the function $\mathcal{D}$ is nonnegative, then the Lyapunov function $\mathcal{F}$ decreases with respect to time t.

Adams type inequality (Four dimensional case). Let Ω be a bounded domain of $\mathbf{R}^4$. There exists some constant $C > 0$ such that for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\log\left(\int_\Omega e^{v(x)}\,dx\right) \le \frac{1}{2\cdot(8\pi)^2}\|(-\Delta+1)v\|^2_{L^2(\Omega)} + C.$$

The threshold number appearing in (1) of Theorem 2.1 comes from the Adams type inequality.

Stationary solutions to (P). Stationary solutions to (P) satisfy the following system.

$$(SP)\begin{cases} 0 = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ 0 = \Delta v - v + w & \text{in } \Omega, \\ 0 = \Delta w - v + u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} - \chi u \dfrac{\partial v}{\partial \nu} = 0, \quad v = w = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

The first equation can be rewritten as

$$0 = \nabla \cdot u(\nabla \log u - \chi v) = \nabla \cdot u \nabla \log \frac{u}{e^{\chi v}}.$$

This and the boundary condition lead us to the relation $ue^{-\chi v} = C$ with some positive constant C and the constant C can be rewritten as

$$C = \frac{\|u\|_{L^1(\Omega)}}{\int_\Omega e^{\chi v} dx}.$$

From this and the second and third equations of (SP), we obtain that stationary solutions (u, v, v) satisfy the following;

$$(SP2)\begin{cases} (-\Delta + 1)^2 v = \lambda \dfrac{e^{\chi v}}{\int_\Omega e^{\chi v}} & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Here, $\lambda = \|u\|_{L^1(\Omega)} > 0$, $w = (-\Delta + 1)v$ and

$$u = \lambda \frac{e^{\chi v}}{\int_\Omega e^{\chi v}}.$$

Stationary solutions satisfy the following property.

Lemma 3.1. *Let $\{v_k\}$ be a sequence of solutions to (SP2) with a positive constant λ. If $\lim_{k\to\infty} \|v_n\|_{L^\infty(\Omega)} = \infty$, then there exist an positive integer J and a set of points $\{Q_j\}_{j=1}^J \subset \Omega$ such that*

$$\lambda = \frac{(8\pi)^2}{\chi} J$$

and

$$u_k = \lambda \frac{e^{\chi v_k}}{\int_\Omega e^{\chi v_k}} \to \sum_{j=1}^{J} \frac{(8\pi)^2}{\chi} \delta_{Q_j} \text{ in } \mathcal{M}(\overline{\Omega}) \text{ as } k \to \infty.$$

The threshold number appearing in (2) of Theorem 2.1 comes from Lemma 3.1.

Proof. Let $\{(u_k, v_k, w_k)\}$ be a sequence of stationary solutions satisfying

$$\lim_{k\to\infty} \|v_k\|_{L^\infty(\Omega)} = \infty.$$

Using the moving plane method, we can find positive constants d_* and C such that

$$u_k(x), v_k(x), w_k(x) \le C\lambda \quad \text{for } k \ge 1 \text{ and } x \in \Omega \text{ with } d(x, \partial\Omega) < d_*.$$

Here,

$$d(x, \partial\Omega) = \min_{y\in\partial\Omega} |x - y|.$$

Let $\mathcal{G}$ be the Green function of $(\Delta - 1)^2$ in Ω with $\cdot = \Delta\cdot = 0$ on $\partial\Omega$, and let $\mathcal{G}_e = (8\pi^2)^{-1}\log(R/|x-y|)$, where $R = \sup\{|x - y| : x, y \in \Omega\}$. Then, $0 < \mathcal{G} \le \mathcal{G}_e$ in $\Omega \times \Omega$.

Let Q be a blowup point of $\{v_k\}$. Then, $Q \in \Omega$. We choose $\delta > 0$ such that $B(2\delta) = \{x \in \Omega; |x - Q| < 2\delta\} \subset \Omega$. Let $p > 1$.

$$\begin{aligned}
&\int_{B(\delta)} |(\Delta - 1)^2 v_k|^p dx = \int_{B(\delta)} |f_k|^p dx \\
&\le \frac{\lambda^p}{|\Omega|^p} \int_{B(\delta)} \exp\left(\int_\Omega \mathcal{G}(x,y) p\chi f_k \, dy\right) dx \\
&\le C_\delta \int_{B(\delta)} \exp\left(\int_{B(\delta)} \|f_k\|_{L^1(B(2\delta))} \frac{p\chi}{8\pi^2} dx \left(\log \frac{R}{|x-y|}\right) \frac{f_k}{\|f_k\|_{L^1(B(\delta))}} dy\right) dx \\
&\le C_\delta \int_{B(\delta)} \left(\int_{B(2\delta)} \left(\frac{R}{|x-y|}\right)^{A_k} \frac{f_k}{\|f_k\|_{L^1(B(2\delta))}} dy\right) dx.
\end{aligned}$$

Here,

$$f_k = \lambda \frac{e^{\chi v_k}}{\int_\Omega e^{\chi v_k}}$$

and

$$A_k = \frac{p\chi}{8\pi^2} \|f_k\|_{L^1(B(2\delta))}.$$

Then, $\liminf_{k\to\infty} A_k \ge 4$, since $W^{4,p}(B(\delta)) \subset L^\infty(B(\delta))$. This means that

$$\liminf_{k\to\infty} \int_{B(2\delta)} \lambda \frac{e^{\chi v_k}}{\int_\Omega e^{\chi v_k}} \ge \frac{4 \cdot (8\pi^2)}{\chi}$$

for any sufficiently small $\delta > 0$.

Moreover, we see that

$$\liminf_{k\to\infty} e^{\chi v_k} \ge \frac{O(1)}{|x - Q|^4} \text{ near } Q,$$

since

$$\liminf_{k\to\infty} \chi v_k \ge \chi \cdot \frac{4 \cdot (8\pi^2)}{\chi} \cdot \frac{1}{8\pi^2} \log \frac{R}{|x - Q|} + O(1) \text{ near } Q.$$

Then,

$$\liminf_{k\to\infty}\int_\Omega e^{\chi v_k}=\infty$$

and there exists a subsequence $\{v_{k'}\}\subset\{v_k\}$ satisfying

$$\lim_{k'\to\infty}\lambda\frac{e^{\chi v_{k'}}}{\int_\Omega e^{\chi v_{k'}}}=\sum_{j=1}^{J}m_j\delta_{Q_j}$$

with some positive integer J and $m_j\geq\frac{4\cdot(8\pi^2)}{\chi}$.Then, $\lambda=\sum_{j=1}^{J}m_j$.

Moreover, the Pohozaev's identity

$$\int_{B(\delta)}(\Delta-1)^2v_k(x)\,(x\cdot\nabla v_k(x))\,dx=\int_{B(\delta)}\lambda\frac{e^{\chi v_k(x)}}{\int_\Omega e^{\chi v_k}}\,(x\cdot\nabla v_k(x))\,dx$$

leads us to $m_j=\frac{(8\pi)^2}{\chi}$. Then, $\lambda\in\frac{(8\pi)^2}{\chi}\mathcal{N}$. The proof is complete. □

§ 4. Sketch of Proof of Theorem 2.1

In this section, we describe sketch of proof of Theorem 2.1. First, we describe the proof of (1) of Theorem 2.1. Since the statement in the case of $n\leq 3$ can be shown by the standard energy method, then we describe only the proof in the case where $n=4$.

Sketch of Proof of (1) of Theorem 2.1 in the case where $n=4$. By the Jensen inequality, for $\alpha>0$ we have that

$$\begin{aligned}\int_\Omega u(\alpha v-\log u)&=\int_\Omega u\log\frac{e^{\alpha v}}{u}dx\\&=\|u_0\|_{L^1(\Omega)}\int_\Omega\left(\log\frac{e^{\alpha v}}{u}\right)\cdot\left(\frac{u}{\|u_0\|_{L^1(\Omega)}}\right)dx\\&\leq\|u_0\|_{L^1(\Omega)}\log\left(\int_\Omega\frac{e^{\alpha v}}{u}\cdot\frac{u}{\|u_0\|_{L^1(\Omega)}}\,dx\right)\\&=\|u_0\|_{L^1(\Omega)}\log\left(\int_\Omega e^{\alpha v}dx\right)-\left(\|u_0\|_{L^1(\Omega)}\right)\log\left(\|u_0\|_{L^1(\Omega)}\right)\\&\leq\|u_0\|_{L^1(\Omega)}\log\left(\int_\Omega e^{\alpha v}dx\right)+\frac{1}{e}.\end{aligned}$$

Let ε be a positive constant. Put $\alpha=\chi+2\varepsilon$. Since we assume that $\|u_0\|_{L^1(\Omega)}<(8\pi)^2/\chi$, for any sufficiently small $\varepsilon>0$ Adams type inequality lead us to

$$\begin{aligned}(\chi+2\varepsilon)\int_\Omega u(t)v(t)dx&\leq\int_\Omega u(t)\log u(t)dx+\|u_0\|_{L^1(\Omega)}\log\left(\int_\Omega e^{\alpha v(t)}dx\right)+\frac{1}{e}\\&\leq\int_\Omega u(t)\log u(t)dx+\left\{\|u_0\|_{L^1(\Omega)}\frac{\alpha^2}{2\cdot(8\pi)^2}\right\}\|(-\Delta+1)v\|^2_{L^2(\Omega)}+C\\&\leq\int_\Omega u(t)\log u(t)dx+\frac{\chi}{2}\|(-\Delta+1)v\|^2_{L^2(\Omega)}+C.\end{aligned}$$

Therefore, the following inequality holds.

$$\chi\int_\Omega u(t)v(t)dx \le \frac{\chi}{\chi+\varepsilon}\int_\Omega u(t)\log u(t)dx \\ + \frac{\chi}{\chi+\varepsilon}\frac{\chi}{2}\|(-\Delta+1)v\|_{L^2(\Omega)}^2 - \frac{\varepsilon\chi}{\chi+\varepsilon}\int_\Omega u(t)v(t)dx + C.$$

Combining this with Lyapunov function, we imply that

$$\frac{\varepsilon}{\chi+\varepsilon}\int_\Omega (u\log u + \chi uv)dx + \frac{\chi}{2}\int_\Omega (v_t)^2 dx + \frac{\varepsilon\chi}{2(\chi+\varepsilon)}\int_\Omega |(-\Delta+1)v|^2 dx - C \\ \le \mathcal{F}(u,v) = \int_\Omega (u\log u - \chi uv)dx + \frac{\chi}{2}\int_\Omega (v_t)^2 dx + \frac{\chi}{2}\int_\Omega |(-\Delta+1)v|^2 dx \\ \le \mathcal{F}(u_0,v_0).$$

This means that each term in the Lyapunov function $\mathcal{F}(u(t),v(t))$ is bounded. Using the boundedness and the standard energy argument, we get the boundedness of the solution (u,v,w). Then, the proof is complete.

Next, we describe the sketch of proof of (2) of Theorem 2.1. The result essentially comes from the following two lemmas.

Lemma 4.1. *For $\lambda > 0$, put*

$$F_{\inf}(\lambda) = \inf\left\{\mathcal{F}(u,v) : (u,v,w) \text{ is a stationary solution with } \|u\|_{L^1(\Omega)} = \lambda\right\}.$$

If $\lambda \notin \frac{(8\pi)^2}{\chi}\mathbf{N}$, then $F_{\inf}(\lambda)$ is bounded.

Lemma 4.2. *If $\lambda > \frac{(8\pi)^2}{\chi}$ and $\lambda \notin \frac{(8\pi)^2}{\chi}\mathbf{N}$, there exists a pair of nonnegative and smooth functions (u_0,v_0,w_0) satisfying $\|u_0\|_{L^1(\Omega)} = \lambda$ and $\mathcal{F}(u_0,v_0) < F_{\inf}(\lambda)$.*

Lemma 4.1 comes from Lemma 3.1. Then, we describe the proof of Lemma 4.2.

Proof of Lemma 4.2. Put $A = 2^7\cdot 3$. Let $\mu > 0$,

$$\overline{u}_\mu(x) = \frac{A}{\chi}\frac{\mu^4}{(1+\mu^2|x|^2)^4},$$

$$\overline{v}_\mu(x) = \frac{4}{\chi}\log\frac{\mu A^{1/4}}{1+\mu^2|x|^2}$$

and let

$$\overline{w}_\mu(x) = \frac{\mu^2}{\chi}\frac{16(2+\mu^2|x|^2)}{(1+\mu^2|x|^2)^2}.$$

These functions satisfy that

$$\begin{aligned}
\Delta^2 \overline{v}_\mu &= \overline{u}_\mu \quad \text{in } \mathbf{R}^4, \\
-\Delta \overline{v}_\mu &= \overline{w}_\mu \text{ in } \mathbf{R}^4, \\
\int_{\mathbf{R}^4} \overline{u}_\mu &= \frac{(8\pi)^2}{\chi}, \\
\lim_{\mu \to \infty} \overline{u}_\mu &= \frac{(8\pi)^2}{\chi} \delta_0
\end{aligned}$$

and that

$$\lim_{\mu \to \infty} \mathcal{F}(a\overline{u}_\mu, a\overline{v}_\mu) = -\infty \text{ for } a > 1.$$

Here, we regard v_t as $\frac{1}{\tau_1}(\Delta \overline{v}_\mu - \overline{v}_\mu + \overline{w}_\mu)$. By using these functions, we can construct a desired initial functions (u_0, v_0, w_0). Then, the proof is complete. □

(2) of Theorem 2.1 comes from Lemmas 4.1 and 4.2.

Proof of (2) of Theorem 2.1. Let (u_0, v_0, w_0) be the triple in Lemma 4.2 and let (u, v, w) be the corresponding solutions to (P).

We assume that the solution exists globally in time and is uniformly bounded. Then, there exists a sequence $\{t_k\} \subset (0, \infty)$ such that $(u(t_k), v(t_k), w(t_k))$ converges to a stationary solution $(u_\infty, v_\infty, w_\infty)$. Since the Lyapunov function decreases with respect to t, then we see that $\mathcal{F}(u_\infty, v_\infty) < F_{\inf}(\lambda)$. It contradicts the definition of $F_{\inf}(\lambda)$. Then, $T < \infty$ or (u, v, w) is unbounded. This means that the solution blows up. Then, the proof is complete. □

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