RIMS Kôkyûroku Bessatsu
B82 (2020), 29 - 46

Global solutions to the Boltzmann equation without angular cutoff and the Landau equation with Coulomb potential

By

Renjun DUAN*, Shuangqian LIU**, Shota SAKAMOTO*** and Robert M. STRAIN†

Abstract

This report succinctly summarizes results proved in the authors' recent work [7] where the unique existence of solutions to the Boltzmann equation without angular cut-off and the Landau equation with Coulomb potential are studied in a perturbation framework. A major feature is the use of the Wiener space $A(\Omega)$, which can be expected to play a similar role to L^∞. Compared to the L^2-based solution spaces that were employed for prior known results, this function space enables us to establish a new global existence theory. One further feature is that, not only an initial value problem, but also an initial boundary value problem whose boundary conditions can be regarded as physical boundaries in some simple situation, are considered for both equations. In addition to unique existence, large-time behavior of the solutions and propagation of spatial regularity are also proved. In the end of report, key ideas of the proof will be explained in a concise way.

Received September 17, 2019. Revised Febuary 26, 2020.
2010 Mathematics Subject Classification(s): 35Q20, 82D10, 82C40, 35A01
Key Words: *Kinetic theory, Landau equation, Boltzmann equation without angular cut-off, Global solutions.*
*Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, P.R. China.
e-mail: `rjduan@math.cuhk.edu.hk`
**School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P.R. China
and Department of Mathematics, Jinan University, Guangzhou 510632, P.R. China.
e-mail: `tsqliu@jnu.edu.cn`
***Mathematical Institute, Tohoku University, 6-3 Aoba, Aramaki, Aoba-ku, Sendai, 980-0856, Japan.
e-mail: `shota.sakamoto.e1@tohoku.ac.jp`
†Department of Mathematics, University of Pennsylvania, David Rittenhouse Lab, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA.
e-mail: `strain@math.upenn.edu`

§ 1. Introduction

We study the following kinetic equation which includes the collisional operator:

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = Q(F,F)(t,x,v). \tag{1.1}$$

Here the unknown $F = F(t,x,v)$ is a non-negative density function for particles with position $x = (x_1, x_2, x_3)$ in a given domain $\Omega \subset \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t > 0$. Partial differential equations of this form are sometimes called Kinetic equations, they describe the dynamics of rarefied gases in various settings. In particular the Boltzmann and Landau equations, which are regarded as fundamental Kinetic equations, will be investigated in this report.

When the bilinear collision operator $Q(F,G)$ takes the form

$$Q(F,G)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \psi(v-u) \left[F(u)\nabla_v G(v) - G(v)\nabla_u F(u) \right] du \right\}, \tag{1.2}$$

then (1) is called the Landau equation, where the Landau collision kernel ψ in (1) is a non-negative symmetric 3×3 matrix-valued function defined for $z = (z_1, z_2, z_3) \in \mathbb{R}^3 \backslash \{0\}$ as

$$\psi^{jm}(z) = \left\{ \delta_{jm} - \frac{z_j z_m}{|z|^2} \right\} |z|^{\gamma+2}, \quad j, m = 1, 2, 3. \tag{1.3}$$

Here δ_{jm} is the Kronecker delta and $-3 \leq \gamma \leq 1$ is a parameter determined by the interaction potential between particles. The case $\gamma = -3$ corresponds to the classical Coulomb potential (see [13, 19]).

If $Q(F,G)$ instead takes the form

$$Q(F,G)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \sigma) \left[F(u')G(v') - F(u)G(v) \right] d\sigma du, \tag{1.4}$$

then the equation is called the Boltzmann equation, where the velocity pairs (v,u) and (v',u') satisfy

$$\begin{cases} v' = \dfrac{v+u}{2} + \dfrac{|v-u|}{2}\sigma, \\ u' = \dfrac{v+u}{2} - \dfrac{|v-u|}{2}\sigma, \end{cases} \quad \sigma \in \mathbb{S}^2.$$

The Boltzmann collision kernel $B(v-u,\sigma)$ is a non-negative function, depending on the relative velocity $|v-u|$ and the deviation angle θ between σ and $(v-u)/|v-u|$, determined via

$$\cos\theta = \left\langle \sigma, \frac{v-u}{|v-u|} \right\rangle.$$

In this report we assume that B is given by

$$B(v-u,\sigma) = C_B|v-u|^\gamma b(\cos\theta),$$

for a constant $C_B > 0$, where $|v-u|^\gamma$ is called the kinetic part with $\gamma > -3$, and $b(\cos\theta)$ is called the angular part satisfying that there are $C_b > 0$, $0 < s < 1$ such that

$$b(\cos\theta)\sin\theta \sim C_b\theta^{-1-2s} \text{ as } \theta \downarrow 0.$$

We further assume, by convention as in [20], that $b(\cos\theta)$ is supported on $[0, \pi/2]$. This can be assumed without loss of genrality since one can always replace $b(\cos\theta)$ by

$$[b(\cos\theta) + b(\cos(\pi-\theta))]\mathbf{1}_{[0,\pi/2]}(\theta).$$

This follows from a standard change of variables.

One physical example is given when the collision kernel is derived from a spherical intermolecular repulsive potential of the inverse power law form $\phi(r) = r^{-(\ell-1)}$ with $2 < \ell < \infty$ corresponding to which B satisfies the assumptions above with $\gamma = (\ell-5)/(\ell-1)$ and $s = 1/(\ell-1)$, cf. [4]. Also, for the connections between the Boltzmann equation with long range interactions and the Landau equation with Coulomb potential, we refer the reader to [20].

Throughout this report, in the Boltzmann case we further require that

$$\gamma > \max\left\{-3, -\frac{3}{2} - 2s\right\} \tag{1.5}$$

due to the mild regularity setting of the results in this article. Notice this is satisfied for the inverse power law model for any $2 < \ell < \infty$.

In this report we focus on two kinds of specific bounded domains $\Omega \subset \mathbb{R}^3$. We consider either a torus, or a finite channel with prescribed boundary conditions.

- For the torus domain case, we set

$$\Omega = \mathbb{T}^3 := [0, 2\pi]^3.$$

  Correspondingly, $F(t,x,v)$ is assumed to be spatially periodic in $x \in \mathbb{T}^3$.

- For the finite channel case, we set

$$\Omega = I \times \mathbb{T}^2 = \left\{x = (x_1, \bar{x}), x_1 \in I := (-1,1), \bar{x} := (x_2, x_3) \in \mathbb{T}^2 = [0, 2\pi]^2\right\}.$$

  Correspondingly, $F(t,x,v)$ is assumed to be spatially periodic for $\bar{x} \in \mathbb{T}^2$ and satisfy either of the following two boundary conditions at $x_1 = \pm 1$:

– *Inflow* boundary condition:

$$F(t,-1,\bar{x},v)|_{v_1>0} = G_-(t,\bar{x},v), \quad F(t,1,\bar{x},v)|_{v_1<0} = G_+(t,\bar{x},v), \tag{1.6}$$

or

– *Specular reflection* boundary condition:

$$\begin{aligned} F(t,-1,\bar{x},v_1,\bar{v})|_{v_1>0} &= F(t,-1,\bar{x},-v_1,\bar{v}), \\ F(t,1,\bar{x},v_1,\bar{v})|_{v_1<0} &= F(t,1,\bar{x},-v_1,\bar{v}), \end{aligned} \tag{1.7}$$

where $\bar{v} = (v_2, v_3)$.

Let us consider (1) in a perturbation framework. It is well-known that the following global Maxwellian equilibrium state

$$\mu = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

is a spatially homogeneous steady solution to (1) for both the Landau case (1) and the Boltzmann case (1). Then the form of the solution that we will look for is

$$F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v),$$

and the new unknown $f = f(t,x,v)$ satisfies

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f,f) \tag{1.8}$$

with initial data

$$f(0,x,v) = f_0(x,v) := \mu^{-1/2}[F_0(x,v) - \mu]. \tag{1.9}$$

Here the linear and nonlinear parts of the collision operator Q are given by

$$Lf = -\mu^{-\frac{1}{2}} \left\{ Q(\mu, \mu^{\frac{1}{2}} f) + Q(\mu^{\frac{1}{2}} f, \mu) \right\},$$

and

$$\Gamma(f,g) = \mu^{-\frac{1}{2}} Q(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g), \tag{1.10}$$

respectively.

In the case of the finite channel, the boundary conditions (1) and (1) along x_1 are also recast as

$$f(t,-1,\bar{x},v)|_{v_1>0} = g_-(t,\bar{x},v), \quad f(t,1,\bar{x},v)|_{v_1<0} = g_+(t,\bar{x},v) \tag{1.11}$$

with $g_{\pm} = \mu^{-1/2}[G_{\pm} - \mu]$ for the inflow boundary, and

$$\begin{aligned} f(t,-1,\bar{x},v_1,\bar{v})|_{v_1>0} &= f(t,-1,\bar{x},-v_1,\bar{v}), \\ f(t,1,\bar{x},v_1,\bar{v})|_{v_1<0} &= f(t,1,\bar{x},-v_1,\bar{v}), \end{aligned} \tag{1.12}$$

for the specular reflection boundary.

We further assume that the total mass, momentum and energy of the system are conserved over time. In other words, we assume

$$\int_{\Omega}\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mu^{\frac{1}{2}} f(t,x,v)\, dvdx = 0 \in \mathbb{R}^5 \tag{1.13}$$

for each $t \geq 0$. This corresponds to the form of the pertubation (1) and the expectation that the solution will converge to the Maxwellian μ as $t \to \infty$ in a sense to be specified below. Here we use the domains:

- $\Omega = \mathbb{T}^3$, or
- $\Omega = (-1,1) \times \mathbb{T}^2$ with the specular reflection boundary condition (1).

In the latter case, we further assume that the initial data F_0 is even with respect to (x_1, v_1) so that the following holds

$$F_0(x_1,\bar{x},v_1,\bar{v}) = F_0(-x_1,\bar{x},-v_1,\bar{v}). \tag{1.14}$$

It can be formally shown that, if (1) is assumed, the same evenness holds for a solution F at time $t > 0$. This condition deduces the conservation law of the first component of momentum for the specular reflection boundary condition case.

For convenience of terminology in relation to (1.1) below, in the rest of this report we will call them *soft* potentials if either $\gamma + 2 < 0$ in the Landau case or $\gamma + 2s < 0$ in the Boltzmann case, and we will call them the *hard* potentials otherwise.

There has been a large number of research works studying the existence of a global solutions to (1) for small data. For the Landau equation including the case with Coulomb interaction potential, Guo [13] proved that there exists a global classical solution $f \in L^\infty_t(0,\infty; H^8_{x,v})$, where H^8-Sobolev regularity was required to control the nonlinearity of the equation by the Sobolev embedding theorem.

For the Boltzmann equation with the long range interaction, Gressman-Strain [12] and AMUXY [1, 2] independently proved global existence when $\Omega = \mathbb{T}^3$ and $\Omega = \mathbb{R}^3$, respectively. In [12] one solution space used is $L^\infty_t(0,\infty; L^2_v H^2_x)$ for the hard potential case and the soft potential case under the condition (1), and another solution space is $L^\infty_t(0,\infty; H^4_{x,v})$ for the more singular soft potential case for γ below (1). Regularity conditions for the solution spaces are determined by the use of the Sobolev embedding for this equation.

Later, the embedding theorems for Besov spaces are employed to improve these results in view of functional spaces. Duan-Liu-Xu [8] was first to use a so-called Chemin-Lerner type space for the problem. Denoting the Besov space $B^s_{2,1}(\mathbb{R}^3)$ by B^s_x for $s \in \mathbb{R}$, they established a unique global solution in $\tilde{L}^\infty_T B^{3/2}_x L^2_v$ for the cutoff hard potential case (i.e. $b(\cos\theta)$ is integrable over $\mathbb{S}^2$ and $0 \le \gamma \le 1$). Here, $\tilde{L}^\infty_T B^{3/2}_x L^2_v$ is a set of tempered distributions whose

$$\|f\|_{\tilde{L}^\infty_T B^{3/2}_x L^2_v} := \sum_{j=-1}^{\infty} 2^{js} \sup_{0\le t\le T} \|\Delta_j f(t,\cdot,\cdot)\|_{L^2_{x,v}}$$

is finite. Here $\{\Delta_j\}_{j=-1}^\infty$ is the inhomogeneous Littlewood-Paley decomposition. The advantage of this space for the problem is twofold: First, it is simpler to control the nonlinear part in this space rather than in the usual Bochner space $L^\infty_T B^{3/2}_x L^2_v$ because the summation over j comes after all the L^p norms with respect to t, x and v, which defines a stronger topology. Second, the embeddings $B^{3/2} \hookrightarrow L^\infty$ and $B^{3/2} \hookrightarrow H^{3/2}$ hold, while $H^{3/2} \not\hookrightarrow L^\infty$ in dimension three. Morimoto-Sakamoto [17] extended their result by solving the problem in the same function space for the non-cutoff hard and soft potential cases restricted by (1), and Duan-Sakamoto [9] established the cutoff soft potential case (i.e. $b(\cos\theta)$ integrable and $-3 < \gamma \le 0$). Notice that the embedding to L^∞ plays a key role for all of these results.

On the other hand, the study of the non-cutoff Boltzmann equation in $L^\infty_{x,v}$ have not been fully explored yet. In this case a difficulty arises from the observation that the L^∞ norm of a solution can no longer be controlled via embedding theorems. Instead, we focus on a property that L^∞ is a Banach algebra, i.e., if f and g belong to L^∞, then the L^∞ norm of the product fg is bounded by the product of L^∞ norms of f and g. From this point of view, recently some works are carried out to utilize the Wiener space $A(\Omega)$, denoted by L^1_k or $L^1_{\bar{k}}$ in this report, as a solution space for the x variable. We refer to [5, 6, 11, 14, 15, 18] as examples of an application of the Wiener space to solvability of various partial differential equations. Here a function f on $\Omega = \mathbb{T}^3$ is in L^1_k if

$$\|f\|_{L^1_k} := \int_{\mathbb{Z}^3} |\hat{f}(k)|\, d\Sigma(k) := \sum_{k\in\mathbb{Z}^3} |\hat{f}(k)| < \infty.$$

It is easy to see that L^1_k is indeed a Banach algebra because

$$\|fg\|_{L^1_k} = \sum_k |\hat{f} * \hat{g}(k)| \le \sum_k \sum_\ell |\hat{f}(\ell)\hat{g}(k-\ell)| = \|f\|_{L^1_k} \|g\|_{L^1_k} < \infty,$$

provided f and g are in L^1_k. We are motivated by these works to apply the theory of the Wiener space for our initial and initial-boundary value problems.

In order to study the well-posedness of the problems, we introduce a function space X_T with $0 < T \leq \infty$, which is a key point in this work. For the problem in a torus $\mathbb{T}^3$, we define

$$X_T := L^1_k L^\infty_T L^2_v \tag{1.15}$$

with norm

$$\|f\|_{X_T} := \int_{\mathbb{Z}^3} \sup_{0\leq t\leq T} \|\hat{f}(t,k,\cdot)\|_{L^2_v} \, d\Sigma(k) < \infty, \tag{1.16}$$

where

$$\hat{f}(t,k,v) = \mathcal{F}_x f(t,k,v) = \int_{\mathbb{T}^3} e^{-ik\cdot x} f(t,x,v)\, dx, \quad k \in \mathbb{Z}^3$$

denotes the Fourier transform of $f(t,x,v)$ with respect to $x \in \mathbb{T}^3$.

For the problem in a finite channel, we define

$$X_T := L^1_{\bar{k}} L^\infty_T L^2_{x_1,v} \tag{1.17}$$

with norm

$$\|f\|_{X_T} := \int_{\mathbb{Z}^2_{\bar{k}}} \sup_{0\leq t\leq T} \|\hat{f}(t,\bar{k},\cdot)\|_{L^2_{x_1,v}} \, d\Sigma(\bar{k}) < \infty. \tag{1.18}$$

Here we take the Fourier transform with respect to $\bar{x} = (x_2, x_3)$, that is,

$$\hat{f}(t,x_1,\bar{k},v) = \mathcal{F}_{\bar{x}} f(t,x_1,\bar{k},v) = \int_{\mathbb{T}^2} e^{-i\bar{k}\cdot\bar{x}} f(t,x_1,\bar{x},v)\, d\bar{x}$$

for $\bar{k} = (\bar{k}_2, \bar{k}_3) \in \mathbb{Z}^2$. We remark that we have used the measure integral in (1) and (1) to denote the summation over $k \in \mathbb{Z}^3$ and $\bar{k} \in \mathbb{Z}$, respectively. This notation will simplify the notion of solution spaces as in (1) and (1). Furthermore, our proof for the case of $\mathbb{T}^3$ can be applied to that of the whole space $\mathbb{R}^3$ with appropriate modification. We intend to use the notation that can be used for both case with replacement of $\mathbb{Z}^3_k$ and $d\Sigma(k)$ by $\mathbb{R}^3_\xi$ and $d\xi$, respectively.

We also introduce the velocity weighted norm

$$\begin{aligned} \|f\|_{X^w_T} := &\int_{\mathbb{Z}^3} \sup_{0\leq t\leq T} \|w\hat{f}(t,k,\cdot)\|_{L^2_v} \, d\Sigma(k) \\ \text{or } &\int_{\mathbb{Z}^2} \sup_{0\leq t\leq T} \|w\hat{f}(t,\bar{k},\cdot)\|_{L^2_{x_1,v}} \, d\Sigma(\bar{k}), \end{aligned} \tag{1.19}$$

for the problems in a torus and finite channel, respectively. Here $w = w_{q,\vartheta}(v)$ is a velocity weight function defined as

$$w_{q,\vartheta}(v) = e^{\frac{q\langle v\rangle^\vartheta}{4}}, \quad \langle v\rangle = \sqrt{1+|v|^2}, \tag{1.20}$$

with two parameters q and ϑ. We write $w_{q,\vartheta}(v) = w(v)$ to simplify the notation whenever there would be no confusion. In the situation when $q = 0$, we have $w_{q,\vartheta}(v) \equiv 1$ and hence the function space X_T^w with velocity weight is reduced to X_T without weight. Throughout this report we require that (q, ϑ) satisfies the following hypothesis in terms of γ and s:

$$
\textbf{(H)} \begin{cases} \text{Landau case: if } -2 \leq \gamma \leq 1 \text{ then } q = 0; \\ \qquad\qquad\qquad \text{if } -3 \leq \gamma < -2 \text{ then } q > 0 \text{ and } 0 < \vartheta \leq 2 \text{ with} \\ \qquad\qquad\qquad \text{the restriction that } 0 < q < 1 \text{ if } \vartheta = 2. \\ \\ \text{Boltzmann case: if } \gamma + 2s \geq 0 \text{ then } q = 0; \\ \qquad\qquad\qquad\quad \text{if } -3 < \gamma < -2s \text{ then } q > 0 \text{ and } \vartheta = 1. \end{cases}
$$

To obtain the rate of convergence, under the hypothesis **(H)**, associated with the velocity weight function $w = w_{q,\vartheta}(v)$ and γ, we define a parameter κ in the Landau case as

$$
\kappa = \begin{cases} 1 & \text{for } q = 0,\ -2 \leq \gamma \leq 1, \\ \dfrac{\vartheta}{\vartheta + |\gamma + 2|} & \text{for } q > 0,\ -3 \leq \gamma < -2, \end{cases} \tag{1.22}
$$

and in the non-cutoff Boltzmann case we define

$$
\kappa = \begin{cases} 1 & \text{for } q = 0,\ \gamma + 2s \geq 0, \\ \dfrac{\vartheta}{\vartheta + |\gamma + 2s|} & \text{for } q > 0,\ -3 < \gamma < -2s,\ \vartheta = 1. \end{cases} \tag{1.23}
$$

Note that $\kappa \in (0, 1]$.

To state the main results in the next section, we will now introduce some more notation. Recall that to characterize the energy functional for the problem, we have introduced the function space

$$
X_T = L^1_k L^\infty_T L^2_v \text{ or } L^1_{\bar{k}} L^\infty_T L^2_{x_1,v},
$$

respectively, as well as the velocity weighted space X_T^w as in (1), where the velocity weight $w = w_{q,\vartheta}(v)$ is defined in (1) under the assumption **(H)** as in (1.1). In what follows, we further define the corresponding energy dissipation rate functionals. From now on we will also use f, g and h as generic smooth real valued functions in our estimates, when f is not being used as the solution to an equation such as (1). Then since we are taking the Fourier transform we will also use the standard complex conjugate as $\bar{f}$. Now, for the Landau equation, we recall the Landau kernel in (1). Then we define

$$
\sigma^{jm} = \sigma^{jm}(v) = \int_{\mathbb{R}^3} \psi^{jm}(v - u)\mu(u)du.
$$

It is convenient to define the following velocity weighted D-norm:

$$|w_{q,\vartheta} f|_D^2 = \sum_{j,m=1}^{3} \int_{\mathbb{R}^3} w_{q,\vartheta}^2 \left\{ \sigma^{jm} \partial_{v_j} f \partial_{v_m} \bar{f} + \frac{1}{4} \sigma^{jm} v_j v_m f \bar{f} \right\} dv.$$

In the case of the finite channel, we also define

$$\|w_{q,\vartheta} f\|_D^2 = \sum_{j,m=1}^{3} \int_I \int_{\mathbb{R}^3} w_{q,\vartheta}^2 \left\{ \sigma^{jm} \partial_{v_j} f \partial_{v_m} \bar{f} + \frac{1}{4} \sigma^{jm} v_j v_m f \bar{f} \right\} dv dx_1,$$

by including an extra integration in $x_1 \in I = (-1, 1)$. For the case of the non-cutoff Boltzmann equation, we define accordingly

$$\begin{aligned}
&|w_{q,\vartheta} f|_D^2 = \\
&\quad \int_{\mathbb{R}^3_v} \int_{\mathbb{R}^3_u} \int_{\mathbb{S}^2} B(v-u,\sigma) w_{q,\vartheta}^2(v) \mu(u) (f(v') - f(v)) \overline{(f(v') - f(v))} \, d\sigma du dv \\
&\qquad + \int_{\mathbb{R}^3_v} \int_{\mathbb{R}^3_u} \int_{\mathbb{S}^2} B(v-u,\sigma) w_{q,\vartheta}^2(v) f(u) \overline{f(u)} \left(\mu^{\frac{1}{2}}(v') - \mu^{\frac{1}{2}}(v) \right)^2 d\sigma du dv,
\end{aligned}$$

and in the presence of the spatial variable in the finite channel we have

$$\begin{aligned}
&\|w_{q,\vartheta} g\|_D^2 = \\
&\quad \int_I \int_{\mathbb{R}^3_v} \int_{\mathbb{R}^3_u} \int_{\mathbb{S}^2} B(v-u,\sigma) w_{q,\vartheta}^2(v) \mu(u) (f(v') - f(v)) \overline{(f(v') - f(v))} \, d\sigma du dv dx_1 \\
&\qquad + \int_I \int_{\mathbb{R}^3_v} \int_{\mathbb{R}^3_u} \int_{\mathbb{S}^2} B(v-u,\sigma) w_{q,\vartheta}^2(v) f(u) \overline{f(u)} \left(\mu^{\frac{1}{2}}(v') - \mu^{\frac{1}{2}}(v) \right)^2 d\sigma du dv dx_1.
\end{aligned}$$

Then, corresponding to the energy functional X_T^w, we define the weighted dissipation rate functionals:

$$\|w_{q,\vartheta} f\|_{L^1_k L^2_T L^2_{v,D}} = \int_{\mathbb{Z}^3_k} \left(\int_0^T |w_{q,\vartheta} \mathcal{F}_x f(t,k)|_D^2 \, dt \right)^{1/2} d\Sigma(k)$$

and

$$\|w_{q,\vartheta} f\|_{L^1_{\bar{k}} L^2_T L^2_{x_1,v,D}} = \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \|w_{q,\vartheta} \mathcal{F}_{\bar{x}} f(t,\bar{k})\|_D^2 \, dt \right)^{1/2} d\Sigma(\bar{k}),$$

for the torus and finite channel domains, respectively. In the case of a finite channel, we need to include an extra first-order derivative in x, and thus we define the total energy functional and energy dissipation rate functional respectively as

$$\mathcal{E}_{T,w}(f) = \sum_{|\alpha| \le 1} \|w_{q,\vartheta} \partial^\alpha f\|_{L^1_{\bar{k}} L^\infty_T L^2_{x_1,v}}, \tag{1.24}$$

and

$$\mathcal{D}_{T,w}(f) = \sum_{|\alpha|\leq 1} \|\partial^\alpha[a,b,c]\|_{L^1_{\bar{k}}L^2_T L^2_{x_1}} + \sum_{|\alpha|\leq 1} \|w_{q,\vartheta}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_{L^1_{\bar{k}}L^2_T L^2_{x_1,v,D}}. \tag{1.25}$$

Here $\partial^\alpha = \partial^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ a standard multi-index, $\mathbf{P}$ is the projection from L^2_v into the null space of the linear operator L, and the macroscopic part (a, b, c) of f is defined via $\mathbf{P}$ by

$$\mathbf{P}f = \Big\{a + b\cdot v + \frac{1}{2}(|v|^2 - 3)c\Big\}\mu^{\frac{1}{2}}.$$

Further the norm $\|\cdot\|_{L^1_{\bar{k}}L^2_T L^2_{x_1}}$ in only the t and x variables for a function $g = g(t,x)$ is understood in the same integration order as the norm $\|\cdot\|_{L^1_{\bar{k}}L^2_T L^2_{x_1,v,D}}$:

$$\|g\|_{L^1_{\bar{k}}L^2_T L^2_{x_1}} = \int_{\mathbb{Z}^2_{\bar{k}}} \left(\int_0^T \int_{-1}^1 |\mathcal{F}_{\bar{x}} g(t,\bar{k})|^2 dx_1 dt\right)^{1/2} d\Sigma(\bar{k}).$$

For any given $t \geq 0$, we define the following norms in x and v:

$$\|w_{q,\vartheta} f(t)\|_{L^1_k L^2_v} = \int_{\mathbb{Z}^3_k} \|w_{q,\vartheta}\mathcal{F}_x f(t,k)\|_{L^2_v}\, d\Sigma(k),$$

and

$$\|w_{q,\vartheta} f(t)\|_{L^1_{\bar{k}} L^2_{x_1,v}} = \int_{\mathbb{Z}^2_{\bar{k}}} \|w_{q,\vartheta}\mathcal{F}_{\bar{x}} f(t,\bar{k})\|_{L^2_{x_1,v}}\, d\Sigma(\bar{k}).$$

The corresponding high-order norms are defined by

$$\|w_{q,\vartheta} f(t)\|_{L^1_{k,m} L^2_v} = \int_{\mathbb{Z}^3_k} \langle k\rangle^m \|w_{q,\vartheta}\mathcal{F}_x f(t,k)\|_{L^2_v}\, d\Sigma(k),$$

and

$$\|w_{q,\vartheta} f(t)\|_{L^1_{\bar{k},m} L^2_{x_1,v}} = \int_{\mathbb{Z}^2_{\bar{k}}} \langle \bar{k}\rangle^m \|w_{q,\vartheta}\mathcal{F}_{\bar{x}} f(t,\bar{k})\|_{L^2_{x_1,v}}\, d\Sigma(\bar{k}),$$

where m is an integer.

For the inflow boundary value problem in the finite channel case, we also define the following norms to capture the boundary effect of the given functions $g_\pm$:

$$\begin{aligned} E_{\bar{k}}(\widehat{g_\pm}) = {} & \sum_\pm \int_0^T \int_{\pm v_1<0} |v_1|^{-1}|\widehat{\partial_t g_\pm}|^2 dv dt + \sum_\pm \int_0^T \int_{\pm v_1<0} |v_1|^{-1}|\bar{k}\cdot\bar{v}|^2 |\widehat{g_\pm}|^2 dv dt \\ & + \sum_\pm \int_0^T \int_{\pm v_1<0} |v_1|^{-1}|L\widehat{g_\pm}|^2 dv dt + \sum_\pm \int_0^T \int_{\pm v_1<0} |v_1|^{-1}|\Gamma(\widehat{g_\pm},\widehat{g_\pm})|^2 dv dt \\ & + \sum_\pm \int_0^T \int_{\pm v_1<0} |v_1|(1+|\bar{k}|^2)|\widehat{g_\pm}|^2 dv dt, \end{aligned}$$

and

$$E(\widehat{g_\pm}) = \int_{\mathbb{Z}^2_{\bar{k}}} \sqrt{E_{\bar{k}}(\widehat{g_\pm})}\, d\Sigma(\bar{k}). \tag{1.26}$$

Finally, throughout this report, C denotes a generic positive (generally large) uniform constant that may take different values in different places. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leqslant CB$.

§ 2. Main Results

To state the main results, we are first concerned with the problem for the Landau equation or the non-cutoff Boltzmann equation in the *torus*. In the following statements, we point out that an omitted constant C in the notation $\lesssim$ is always uniformly independent of $t \geq 0$.

Theorem 2.1 (Existence and large-time behavior in the torus). *Let $\Omega = \mathbb{T}^3$, Q be given in the form* (1) *or* (1)*, and $w_{q,\vartheta}$ be chosen under the assumption* **(H)**. *There is $\epsilon_0 > 0$ such that if $F_0(x,v) = \mu + \mu^{\frac{1}{2}} f_0(x,v) \geq 0$ and*

$$\|w_{q,\vartheta} f_0\|_{L^1_k L^2_v} \leq \epsilon_0,$$

then the initial value problem (1)-(1) *possesses a unique global mild solution f in*

$$\left\{ f \in L^1_k L^\infty_T L^2_v \cap L^1_k L^2_T L^2_{v,D} \;\middle|\; f(t,x,v) \text{ satisfies the conservation laws } (1) \right\}.$$

This solution satisfies $F(t,x,v) = \mu + \mu^{\frac{1}{2}} f(t,x,v) \geq 0$ and

$$\|w_{q,\vartheta} f\|_{L^1_k L^\infty_T L^2_v} + \|w_{q,\vartheta} f\|_{L^1_k L^2_T L^2_{v,D}} \lesssim \|w_{q,\vartheta} f_0\|_{L^1_k L^2_v}$$

for any $T > 0$.

Moreover, let $\kappa \in (0,1]$ be defined in (1) *or* (1) *for the Landau case or the non-cutoff Boltzmann case, respectively, then there is a $\lambda > 0$ such that the solution also enjoys the time decay estimate*

$$\|f(t)\|_{L^1_k L^2_v} \lesssim e^{-\lambda t^\kappa} \|w_{q,\vartheta} f_0\|_{L^1_k L^2_v}, \tag{2.1}$$

for any $t \geq 0$.

Theorem 2.2 (Propagation of spatial regularity). *Under the same hypotheses as in Theorem 2.1, for any integer $m \geq 0$, there exists an $\epsilon_1 > 0$ such that if*

$$\|w_{q,\vartheta} f_0\|_{L^1_{k,m} L^2_v} \leq \epsilon_1,$$

then the solution $f(t,x,v)$ *to* (1)-(1) *established in Theorem 2.1 satisfies*

$$\|w_{q,\vartheta} f\|_{L^1_{k,m} L^\infty_T L^2_v} + \|w_{q,\vartheta} f\|_{L^1_{k,m} L^2_T L^2_{v,D}} \lesssim \|w_{q,\vartheta} f_0\|_{L^1_{k,m} L^2_v} \tag{2.2}$$

for any $T > 0$.

Next, we are concerned with the problem for the Landau equation or the non-cutoff Boltzmann equation in *finite channel.*

Theorem 2.3 (Inflow boundary condition). *Let* $\Omega = I \times \mathbb{T}^2$ *and* $w_{q,\vartheta}$ *be chosen under the assumption* **(H)**. *There is* $\epsilon_0 > 0$ *such that if*

$$\begin{aligned} &F_0(x_1,\bar{x},v) = \mu + \mu^{\frac12} f_0(x_1,\bar{x},v) \geq 0,\\ &F(t,\pm 1,\bar{x},v) = \mu + \mu^{\frac12} g_\pm(t,\bar{x},v) \geq 0\\ &\qquad \text{for } v_1 > 0 \text{ at } x_1 = -1 \text{ and } v_1 < 0 \text{ at } x_1 = 1, \end{aligned}$$

and

$$\sum_{|\alpha|\leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}} + E(w_{q,\vartheta}\widehat{g_\pm}) \leq \epsilon_0, \tag{2.3}$$

then there exists a unique mild solution

$$f \in L^1_{\bar{k}} L^\infty_T L^2_{x_1,v} \cap L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D}$$

to the inflow boundary problem (1), (1) *and* (1) *for the Landau equation or the non-cutoff Boltzmann equation.*

The solution satisfies $F(t,x_1,\bar{x},v) = \mu + \mu^{\frac12} f(t,x_1,\bar{x},v) \geq 0$ *and*

$$\mathcal{E}_{T,w}(f) + \mathcal{D}_{T,w}(f) \lesssim \left\{ \sum_{|\alpha|\leq 1} \|w_{q,\vartheta}\partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}} + E(w_{q,\vartheta}\widehat{g_\pm}) \right\}$$

for any $T > 0$, *where* $\mathcal{E}_{T,w}(f)$, $\mathcal{D}_{T,w}(f)$, *and* $E(w_{q,\vartheta}\widehat{g_\pm})$ *are defined in* (1), (1), *and* (1), *respectively.*

Moreover, let $\kappa \in (0,1]$ *be defined in* (1) *or* (1) *for the Landau case or the non-cutoff Boltzmann case, respectively, then there is* $\lambda > 0$ *such that if*

$$E(w_{q,\vartheta}\widehat{g_\pm}) + \sup_{s>0} E(e^{\lambda s^\kappa}\widehat{g_\pm}) \leq \epsilon_0$$

for $\epsilon_0 > 0$ *further small enough, then it holds that*

$$\begin{aligned} \sum_{|\alpha|\leq 1} \|\partial^\alpha f(t)\|_{L^1_{\bar{k}} L^2_{x_1,v}} &\lesssim e^{-\lambda t^\kappa} \sum_{|\alpha|\leq 1} \|w_{q,\vartheta}\partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}} \\ &\quad + e^{-\lambda t^\kappa} \left\{ E(w_{q,\vartheta}\widehat{g_\pm}) + \sup_{s>0} E(e^{\lambda s^\kappa}\widehat{g_\pm}) \right\} \end{aligned}$$

for any $t \geq 0$.

Theorem 2.4 (Specular reflection boundary condition). *Let $\Omega = I \times \mathbb{T}^2$ and $w_{q,\vartheta}$ be chosen under the assumption* **(H)**. *There is $\epsilon_0 > 0$ such that if $F_0(x_1, \bar{x}, v) = \mu + \mu^{\frac{1}{2}} f_0(x_1, \bar{x}, v) \geq 0$ and*

$$\sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}} \leq \epsilon_0, \tag{2.4}$$

then there exists a unique mild solution $f(t, x_1, \bar{x}, v)$ in

$$\{f \in L^1_{\bar{k}} L^\infty_T L^2_{x_1,v} \cap L^1_{\bar{k}} L^2_T L^2_{x_1} L^2_{v,D} \mid f(t, x_1, \bar{x}, v) \text{ satisfies } (1) \text{ and } (1)\}$$

to the specular reflection boundary problem (1), (1) *and* (1) *for the Landau equation or the non-cutoff Boltzmann equation. The solution satisfies $F(t, x_1, \bar{x}, v) = \mu + \mu^{\frac{1}{2}} f(t, x_1, \bar{x}, v) \geq 0$ and*

$$\mathcal{E}_{T,w}(f) + \mathcal{D}_{T,w}(f) \lesssim \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}}$$

for any $T > 0$, where $\mathcal{E}_{T,w}(f)$ and $\mathcal{D}_{T,w}(f)$ are defined in (1) *and* (1), *respectively.*

Moreover, let $\kappa \in (0, 1]$ be defined in (1) *or* (1) *for the Landau case or the non-cutoff Boltzmann case, respectively, then there is $\lambda > 0$ such that*

$$\sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_{L^1_{\bar{k}} L^2_{x_1,v}} \lesssim e^{-\lambda t^\kappa} \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k}} L^2_{x_1,v}}$$

for any $t \geq 0$.

Theorem 2.5 (Propagation of spatial regularity in $\bar{x}$). *Let all of the conditions in Theorem 2.3 and Theorem 2.4 be satisfied and let f be the solution obtained in the theorems. Then for any integer $m \geq 0$, there is $\epsilon_1 > 0$ such that if*

$$\sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k},m} L^2_{x_1,v}} + E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g_\pm}) \leq \epsilon_1$$

and

$$\sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k},m} L^2_{x_1,v}} \leq \epsilon_1$$

hold in the place of (2.3) *and* (2.4), *respectively, then we obtain that*

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k},m} L^\infty_T L^2_{x_1,v}} + \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k},m} L^2_T L^2_{x_1} L^2_{v,D}} \\ \lesssim \left\{ \sum_{|\alpha| \leq 1} \|w_{q,\vartheta} \partial^\alpha f_0\|_{L^1_{\bar{k},m} L^2_{x_1,v}} + E(w_{q,\vartheta} \langle \bar{k} \rangle^m \widehat{g_\pm}) \right\} \end{aligned}$$

for the inflow boundary condition, and

$$\sum_{|\alpha|\le 1} \|w_{q,\vartheta}\partial^\alpha f_0\|_{L^1_{\bar{k},m}L^\infty_T L^2_{x_1,v}} + \sum_{|\alpha|\le 1} \|w_{q,\vartheta}\partial^\alpha f_0\|_{L^1_{\bar{k},m}L^2_T L^2_{x_1}L^2_{v,D}} \lesssim \sum_{|\alpha|\le 1} \|w_{q,\vartheta}\partial^\alpha f_0\|_{L^1_{\bar{k},m}L^2_{x_1,v}}$$

for the specular reflection boundary condition, respectively.

Here we give two remarks on the theorem: First, as far as we know, these are the first results which shows the existence of solutions to initial boundary problems of the Landau and non-cutoff Boltzmann equation with physical boundaries. Second, for the torus domain case, the solution space we have used is larger than those employed in the preceding works [12, 13]. Our proof for the torus case $\Omega = \mathbb{T}^3$ can be modified to the case of the whole domain $\Omega = \mathbb{R}^3$, and the solution space is also larger than those used in previous work. In place of L^1_k, we define the space L^1_ξ as a set of tempered distributions on $\mathbb{R}^3$ whose Fourier transform is integrable. Then one can show that $\tilde{L}^\infty_T B^{3/2}_x L^2_v \subset L^1_\xi L^\infty_T L^2_v$ for any $T > 0$. Indeed the Wiener algebra L^1_ξ is known to contain continuous functions with arbitrarily low orders of regularity.

§ 3. Proof Outline

In this report we do not give the whole proof of the theorems, instead we propose an outline of it. For this purpose, in this section we mainly consider the Boltzmann equation for the hard potential case in a torus. In this simplest case we do not have to use the weight functions due to (1.1), (1) and (1) and to control the terms coming from the boundary conditions so that we can focus on looking into crucial points of the proof. Here we just remark that such more complicated terms can be also controlled by the methods we will explain below, and the case of the Landau equation is treated in the same way as that of the Boltzmann equation because the collision terms Q for each case share important properties we will essentially rely on (see [13]).

Proof strategy is based on the energy method (see also [16] for one of the earliest use of this method to the Boltzmann equation). Therefore we shall show uniform a priori estimates of solutions in X_T given by (1) and the unique existence of local solutions, then we combine them to extend the local solutions to global ones by a standard continuation argument. For the uniform estimates, a key idea is to apply the Fourier transform with respect to x to the nonlinear term Γ in (1). Then we can define the bilinear operator $\hat{\Gamma}$ as

$$\hat{\Gamma}(\hat{f},\hat{g})(k,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\sigma)\mu^{1/2}(u)\Big([\hat{f}(u')*\hat{g}(v')](k) - [\hat{f}(u)*\hat{g}(v)](k)\Big)\,d\sigma du.$$

Using this expression, we can apply the usual energy method to

$$\partial_t \hat{f}(t,k,v) + iv\cdot k\hat{f}(t,k,v) + L\hat{f}(t,k,v) = \hat{\Gamma}(\hat{f},\hat{f})(t,k,v). \tag{3.1}$$

Indeed, since L acts only on the velocity variable, for the Boltzmann case we can apply the estimates proven in [2] and [12] to have

$$\delta_0 |\{\mathbf{I}-\mathbf{P}\}\hat{f}|_D^2 \le \mathrm{Re}(L\hat{f},\hat{f})_{L^2_v} \tag{3.2}$$

for some $\delta_0 > 0$, where $\mathrm{Re}(\cdot)$ stands for taking the real part of a complex number. Similarly, for the nonlinear term, one can prove the estimate

$$\left| \Big(\hat{\Gamma}(\hat{f},\hat{g})(k),\hat{h}(k)\Big)_{L^2_v} \right| \le C\int_{\mathbb{Z}^3} \|\hat{f}(k-l)\|_{L^2_v} |\hat{g}(l)|_D |\hat{h}(k)|_D \, d\Sigma(l).$$

To prove this we crucially use the trilinear estimates from [3] and [12].

Due to the weak dissipation effect of the linear term L in (3), we also need a priori estimates of the macroscopic part $Pf \sim (a,b,c)$. In order to obtain them, we derive the fluid-like system, which is a system of partial differential equations of (a,b,c) and microscopic parts represented by higher-order moment functions. This system comes from the conservation laws in (1). The usual energy method may be applied, however, we followed the dual argument employed in [10] would give a simpler and unified proof of all cases. The necessary a priori estimates are obtained in this way.

For the proof of local existence, we construct a solution via the Hahn-Banach extension theorem. In this respect we are highly motivated by [1] and [17] to employ this method, especially for the initial boundary value problem in a finite channel. We will find a solution to the linear inhomogeneous problem

$$\begin{cases} \partial_t g + v\cdot\nabla_x g + \mathscr{L}_1 g - \Gamma(h,g) = -\mathscr{L}_2 h, \\ g(0,x,v) = g_0(x,v), \end{cases} \tag{3.3}$$

where

$$\mathscr{L}_1 g = -\mu^{1/2} Q(\mu, \mu^{1/2} g) \quad \text{and} \quad \mathscr{L}_2 h = -\mu^{1/2} Q(\mu^{1/2} h, \mu).$$

We can show that, if an initial datum g_0 is sufficiently regular and a given function h is sufficiently regular and small, then there exists $T_* > 0$ such that the Cauchy problem admits a unique global local solution in

$$L^1_k L^\infty_{T_*} L^2_v \cap L^1_k L^2_{T_*} L^2_{v,D} \tag{3.4}$$

for the torus case. A key idea of this proof is to first establish a weak solution in $L^\infty_{T_*} L^2_{x,v}$ and then show that the solution additionally satisfies the desired regularity condition. This idea comes from the observation that

$$X_{T_*} = L^1_k L^\infty_{T_*} L^2_v \subset L^\infty((0,T_*)\times\mathbb{T}^3; L^2_v) \subset L^\infty_{T_*} L^2_{x,v}$$

and that $L^2_{x,v}$ is a Hilbert space, which is more suitable for the weak formulation of the equation in the L^2 framework. Let χ_ε be a mollifier over $\mathbb{T}^3$ parametrized with $\varepsilon > 0$. The approximated initial value problem of (3) is

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_1 g - \Gamma(h_\varepsilon, g) = -\mathscr{L}_2 h_\varepsilon, \\ g(0,x,v) = g_{0,\varepsilon}(x,v), \end{cases} \tag{3.5}$$

where $g_{0,\varepsilon} = g_0 *_x \chi_\varepsilon$ and $h_\varepsilon = h *_x \chi_\varepsilon$. Then one can show that (3) has a weak solution $g_\varepsilon \in L^\infty(0,T_0; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. To see this, define

$$\mathcal{G} := -\partial_t + (v \cdot \nabla_x + \mathscr{L}_1 - \Gamma(h_\varepsilon, \cdot))^*,$$

where the asterisk stands for taking the adjoint operator with respect to the inner product $L^2_{x,v}$. Then the solvability of (3) is reduced to show that $\mathcal{G}$ is an injection over

$$\mathbb{W}_1 = \big\{ g : \; g \in H^1(0,T_0; \mathcal{S}(\mathbb{T}^3 \times \mathbb{R}^3_v)) \text{ such that } g(T_0,x,v) \equiv 0 \big\}.$$

Here $\mathcal{S}$ stands for the usual Schwartz class. This is proven by showing that

$$\|g_\varepsilon(t)\|^2 + \lambda \int_t^{T_0} \|g_\varepsilon\|_D^2 d\tau \lesssim \int_t^{T_0} \|\mathcal{G} g_\varepsilon\| \, d\tau, \tag{3.6}$$

which follows from the application of the energy method to (3).

Moreover, we can show that the following functional is bijective

$$\mathcal{M}(w_\varepsilon) = (u(0), g_{0,\varepsilon}) - \int_0^{T_*} (\mathcal{L}_2 h_\varepsilon, u) dt$$

over

$$\mathbb{W}_2 = \big\{ w : w = \mathscr{G} u, \; u \in \mathbb{W}_1 \big\} \subset L^1(0,T_0; L^2(\mathbb{T}^3 \times \mathbb{R}^3)),$$

where $h_\varepsilon \in \mathbb{W}_1$ is uniquely determined by $w_\varepsilon \in \mathbb{W}_2$ as $\mathscr{G} : \mathbb{W}_1 \to \mathbb{W}_2$. Using (3), one can show that $\mathscr{M} : \mathbb{W}_2 \to \mathbb{C}$ can be extended to be a bounded linear functional on $L^1(0,T_0; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. Therefore by the Hahn-Banach theorem there exists $g_\varepsilon \in L^\infty(0,T_0; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$ such that

$$\mathcal{M}(w_\varepsilon) = \int_0^{T_0} (g_\varepsilon(t), w_\varepsilon(t)) dt, \quad \forall \, w_\varepsilon \in L^1(0,T_0; L^2(\mathbb{T}^3 \times \mathbb{R}^3)),$$

which is a weak solution to (3). By the same scheme we can further prove that g_ε has sufficient regularity. Next, we show that the sequence of approximate solutions $\{g_\varepsilon\}$ strongly converges to a function in the solution space (3), which turns out to be a local solution to (3). Finally, by the iterative scheme

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + \mathscr{L}_1 f^{n+1} - \Gamma(f^n, f^{n+1}) = -\mathscr{L}_2 f^n, \\ f^{n+1}(0,x,v) = f_0(x,v) \end{cases}$$

for $n \in \mathbb{N}$, we show that (1)-(1) has a local solution.

Combining the local existence of a solution and the a priori estimates obtained above, we can show the unique global existence of a solution in X_T.

Propagation of regularity is similarly proven by means of the same energy method that was used to prove the unique global existence. We can then obtain (2.2). Indeed, multiplication of the weight function $\langle k \rangle^m$ to the equation is harmless to the aforementioned argument.

In order to obtain the large time behavior, we set $\hat{h} := e^{\lambda t} \hat{f}$, where $\hat{f}$ is a solution to (3). Then $\hat{h}$ solves

$$\partial_t \hat{h} + ik \cdot v \hat{h} + L\hat{h} = e^{-\lambda t} \hat{\Gamma}(\hat{h}, \hat{h}) + \lambda \hat{h}.$$

Hence by the energy method we obtain

$$\|h\|_{L^1_k L^\infty_T L^2_v} + \|h\|_{L^1_k L^2_T L^2_{v,D}} \lesssim \|h\|_{L^1_k L^\infty_T L^2_v},$$

which leads to the desired decay rate (2.1).

Acknowledgments

The first author was supported by the General Research Fund (Project No. 14302716) from RGC of Hong Kong and a Direct Grant from CUHK. The second author was supported by grants from the National Natural Science Foundation of China (contracts: 11471142, 11571063 and 11731008). The third author was supported by JSPS Kakenhi Grant (No. 18J00285). The last author was partially supported by the NSF grants DMS-1500916 and DMS-1764177 of the USA. Finally, the authors would thank the anonymous referee for the valuable and helpful comments on this manuscript.

References

[1] Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang, *Global existence and full regularity of the Boltzmann equation without angular cutoff*, Comm. Math. Phys. **304** (2011), no. 2, 513–581.

[2] —, *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential*, J. Funct. Anal. **262** (2012), no. 3, 915–1010.

[3] —, *Local existence with mild regularity for the Boltzmann equation*, Kinet. Relat. Models **6** (2013), no. 4, 1011–1041.

[4] Carlo Cercignani, Reinhard Illner, and Mario Pulvirenti, *The mathematical theory of dilute gases*, Applied Mathematical Sciences, vol. 106, Springer-Verlag, New York, 1994.

[5] Peter Constantin, Diego Córdoba, Francisco Gancedo, Luis Rodríguez-Piazza, and Robert M. Strain, *On the Muskat problem: global in time results in 2D and 3D*, Amer. J. Math. **138** (2016), no. 6, 1455–1494.

[6] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain, *On the global existence for the Muskat problem*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 1, 201–227,

[7] Renjun Duan, Shuangqian Liu, Shota Sakamoto, and Robert M. Strain, *Global mild solutions of the Landau and non-cutoff Boltzmann equation*, Comm. Pure Appl. Math., to appear, 84 pages, arXiv:1904.12086.

[8] Renjun Duan, Shuangqian Liu, and Jiang Xu, *Global well-posedness in spatially critical Besov space for the Boltzmann equation*, Arch. Ration. Mech. Anal. **220** (2016), no. 2, 711–745.

[9] Renjun Duan and Shota Sakamoto, *Solution to the Boltzmann equation in velocity-weighted Chemin-Lerner type spaces*, Kinet. Relat. Models **11** (2018), no. 6, 1301–1331.

[10] Raffaele Esposito, Yan Guo, Chanwoo Kim, and Rossana Marra, *Non-isothermal boundary in the Boltzmann theory and Fourier law*, Comm. Math. Phys. **323** (2013), no. 1, 177–239.

[11] Francisco Gancedo, Eduardo García-Juárez, Neel Patel, and Robert M. Strain, *On the Muskat problem with viscosity jump: Global in time results*, Adv. Math. **345** (2019), 552–597.

[12] Philip T. Gressman and Robert M. Strain, *Global classical solutions of the Boltzmann equation without angular cut-off*, J. Amer. Math. Soc. **24** (2011), no. 3, 771–847.

[13] Yan Guo, *The Landau equation in a periodic box*, Comm. Math. Phys. **231** (2002), no. 3, 391–434.

[14] Zhen Lei and Fanghua Lin, *Global mild solutions of Navier-Stokes equations*, Comm. Pure Appl. Math. **64** (2011), no. 9, 1297–1304.

[15] Jian-Guo Liu and Robert M. Strain, *Global stability for solutions to the exponential PDE describing epitaxial growth*, Interfaces Free Bound. **21** (2019), no. 1, 61–86.

[16] Tai-Ping Liu, Tong Yang, and Shih-Hsien Yu, *Energy method for Boltzmann equation*, Phys. D **188** (2004), no. 3-4, 178–192.

[17] Yoshinori Morimoto and Shota Sakamoto, *Global solutions in the critical Besov space for the non-cutoff Boltzmann equation*, J. Differential Equations **261** (2016), no. 7, 4073–4134.

[18] Neel Patel and Robert M. Strain, *Large time decay estimates for the Muskat equation*, Comm. Partial Differential Equations **42** (2017), no. 6, 977–999.

[19] Cédric Villani, *On the Cauchy problem for Landau equation: sequential stability, global existence*, Adv. Differential Equations **1** (1996), no. 5, 793–816.

[20] —, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 2002, pp. 71–305.