# A computer-assisted proof for nonlinear heat equations in the complex plane of time 

By<br>Akitoshi Takayasu*


#### Abstract

In the present article we consider a complex valued nonlinear heat equation. It is wellknown that solutions of a real valued nonlinear heat equation blow up in finite time. Our aim of this study is to find out dynamics of blow-up phenomena with computer assistance. We numerically prove that the solution has branching singularity and globally exists on the real axis except the singular point. Such a computer-assisted proof is obtained using rigorous numerics, which consists of careful blend of functional analysis, semigroup theory, numerical analysis, fixed-point theory, the Lyapunov-Perron method and interval arithmetic. This result generalizes the previous results of the complex valued nonlinear heat equation in terms of considering the different boundary condition and without the assumption of initial data being close to a constant.


## $\S$ 1. Introduction and main result

In this article, we consider a complex valued nonlinear heat equation in the complex plane of time

$$
\begin{equation*}
u_{z}=u_{x x}+u^{2}, \quad x \in(0,1), \quad \operatorname{Re}(z) \geq 0 \tag{1.1}
\end{equation*}
$$

under the periodic boundary condition in $x$ with a specific initial data $u(0, x)=50(1-$ $\cos (2 \pi x)$ ) at the origin. The subscripts $z$ and $x$ denote the complex derivative with respect to $z$ and the real derivative with respect to $x$, respectively. Here, "Re" and "Im" denote the real and imaginary part of complex values.

[^0]If we consider the real valued nonlinear heat equation, i.e., $t \geq 0$ denotes so called time variable, it is well-known that solutions of the real valued nonlinear heat equation blow up in finite time. More precisely, the $L^{\infty}$-norm of the solution tends to $\infty$ as $t$ goes to a certain $t_{B}<\infty$. Such a $t_{B}$ is called the blow-up time and there are plenty of related studies for blow-up phenomena of real valued nonlinear heat equations. One can consult previous studies by $[8,3,4,14]$ and references therein for instance.

The typical setting of complex valued nonlinear heat equation is to consider the complex $u$ valued solution $[5,11,6]$. This setting is both $t$ and $x$ are real-valued. From numerical point of view, in [18], Sulem et al. consider various evolution equations in the complex $x$ plane. Our setting in this article is the case of nonlinear heat equations whose time variable is in the complex plane.

As a pioneering work of this setting, Masuda $[9,10]$ has considered the solution of (1.1) under the Neumann boundary condition. If the initial data is close to a constant, he has proved global existence of the solution in the shaded domain of Fig. 1 (a) and has also shown that the solution is analytic in both the shaded domain and its mirror-image about the real axis (see Fig. $1(b)$ ). Furthermore, if the solution agrees in the intersection of above two domains, it is proved that the initial data is a constant. This implies that a non-constant solution can be analytically continued into the complex plane but, in such a case, is not a single-valued function in $\operatorname{Re}(z)>z_{B}$, where $z_{B} \equiv t_{B}$ denotes the blow-up point. Recently, we believe that Masuda's results give a different point of view of blow-up solutions, which come from a blend of complex analysis and modern PDE theories. On the other hand, the Masuda's study has few followers.


Figure 1. (a) Masuda [9, 10] has proved global existence of the solution in the shaded domain for $0<\theta<\pi / 2$. We denote by $z_{B}$ the blow-up point. Note that, in the domain, the real singularity is bypassed and the domain extends to infinity. (b) We plot the shaded domain (a) and its mirror-image about the real axis. Intersection of two domains is drawn by dark gray color.

Following Masuda's study, Cho-Okamoto-Shōji [2] have presented numerical observations of the solution of (1.1) under the periodic boundary condition and presented several conjectures to generalize the Masuda's results. For example, they have presented that the solution may converge to zero on a straight path $\Gamma_{\theta}:=\left\{z \in \mathbb{C}: z=t e^{\mathrm{i} \theta}, \quad t \geq\right.$ $0\}$ in the complex plane, where $\theta \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathrm{i}=\sqrt{-1}$ denotes the imaginary unit. They have also presented that the solution of (1.1) may have only one singularity on the real axis, which branches the Riemann surface of analytic function. As a conclusion of their paper, they have addressed the following two interesting conjectures:

Conjecture 1. The analytic function defined by the nonlinear heat equation (1.1) has branching singularities and only branching singularities, unless it is constant in $x$.

Conjecture 2. Nonlinear Schrödinger equation, which is the case of $\theta= \pm \pi / 2$, is globally well-posed for any real initial data, small or large.

Cho et al.'s study and their concluding conjectures are regarded as generalization of Masuda's results in the sense that they consider the periodic boundary condition instead of the Neumann boundary condition and that they consider the initial data without assumption of closeness to a constant. These results were based on numerical observations and no mathematical proof was presented. This is our main motivation for this study.

The main contribution of the present article is to give a computer-assisted proof for the complex valued nonlinear heat equation (1.1), which is a partial answer for one of Cho et al.'s conjectures; "the solution of (1.1) may have only one singularity on the real axis". Here, we firstly define the solution of (1.1) in this article.

Definition 1.1. Given $t_{s}, t_{e}$ such that $0 \leq t_{s}<t_{e} \leq \infty$, a path $\Gamma$ in the complex plane of time parameterized by $t \in\left[t_{s}, t_{e}\right]$ and $a_{0} \in \ell^{1}$ defined by a certain initial data, we say that a function $u(t, x)$ is a solution of (1.1) along the path $\Gamma$ if the Fourier coefficients of $u$, say $a(t)=\left(a_{k}(t)\right)_{k \in \mathbb{Z}}$, satisfy $a \in C\left(\left(t_{s}, t_{e}\right) ; \ell^{1}\right)$ and $a\left(t_{s}\right)=a_{0}$.

Our main theorem is given along a path ${ }^{1} \tilde{\Gamma}_{\theta}$, which is shown in Fig. 2.
Theorem 1.2 (Global existence after a branching singularity). For the complex valued nonlinear heat equation (1.1) under the periodic boundary condition with initial data $u(0, x)=50(1-\cos (2 \pi x))$, there exists a branching singularity at $z \in(0,0.0238)$. Furthermore, the solution of (1.1) along the path $\tilde{\Gamma}_{\theta}$ exits globally on the real axis for $z \geq 0.0238$ and converges to the zero function.

[^1]

Figure 2. We take a path $\tilde{\Gamma}_{\theta}:=\left\{z:[0, \infty] \rightarrow \mathbb{C}: z(t)=t e^{\mathrm{i} \theta}(0 \leq t \leq 0.0168, O \rightarrow\right.$ $\left.z_{A}\right), z=z_{A}+(t-0.0168) e^{-\mathrm{i} \theta}\left(0.0168 \leq t \leq 0.0336, z_{A} \rightarrow z_{C}\right), z=z_{C}+(t-$ $\left.0.0336)\left(0.0336 \leq t, z_{C} \rightarrow \infty\right), \theta=\pi / 4\right\}$, where $z_{A}=0.0119+0.0119 \mathrm{i}$ and $z_{C}=0.0238$.

Our computer-assisted result shows that there exists a branching singularity and after that the solution exists globally on the real axis. Furthermore, such a solution goes to zero. We remark that the analyticity of the solution with respect to the $z$ variable is not proved in this result. The problem for proving analyticity still remains. We also note that the author and his collaborator have shown the global existence of the solution of (1.1) in [19]. This result agrees with Masuda's work for the case of periodic boundary condition without assumption of closeness to a constant.

Theorem 1.3 ([19, Theorem 1.2]). For $\theta=\pi / 3, \pi / 4, \pi / 6$ and $\pi / 12$, setting a straight path $\Gamma_{\theta}: z=t e^{\mathrm{i} \theta}(t \geq 0)$ in the complex plane of time, the solution of the complex valued nonlinear heat equation (1.1) under the periodic boundary condition with initial data $u(0, x)=50(1-\cos (2 \pi x))$ exists globally along the path $\Gamma_{\theta}$ and converges to zero as $t \rightarrow \infty$.

The proofs of these results are obtained using rigorous numerics, which is given by careful blend of functional analysis, semigroup theory, numerical analysis, fixedpoint theory, the Lyapunov-Perron method and interval arithmetic. For the details of technical procedures, we refer to the paper [19]. We directly use the method provided in this paper. We regard the main contribution of this article as a slightly new result for the complex valued nonlinear heat equation (1.1), which is shown in our main theorem (Theorem 1.2).

The organization of this article is as follows: Using the Fourier expansion of unknown function, we derive a fixed-point form in Section 2. This form corresponds to the simplified Newton operator of an operator equation, which is defined by the original complex valued nonlinear heat equation. In Section 3 we show mathematical theorems
which is used for our rigorous numerics. Combining these theorems, we put an algorithm of our rigorous integrator and proof of the global existence. The proof of our main theorem (Theorem 1.2) is shown in Section 4. First, we show there exists a point in the complex plane at which the branching singularity of solution appears. Second, after the blow-up point, we prove the global existence of the solution on the real axis. Concluding remark shows some remarks and discusses a future direction of this study.

## § 2. Fixed-point formulation for the complex valued nonlinear heat equation

In this section, we set up a fixed-point formulation to provide a rigorous enclosure of the solution of the Cauchy problem (1.1). This fixed-point operator is a foundation of our rigorous numerics procedure. To analytically continue the solution from the origin, we take a straight path $\Gamma_{\theta}:=\left\{z \in \mathbb{C}: z=t e^{\mathrm{i} \theta}, \quad t \geq 0\right\}$ for $\theta \in(-\pi / 2, \pi / 2)$. The complex valued nonlinear heat equation (1.1) is transformed into the following PDE:

$$
\begin{equation*}
u_{t}=e^{\mathrm{i} \theta}\left(u_{x x}+u^{2}\right), \quad x \in(0,1), \quad t \geq 0 \tag{2.1}
\end{equation*}
$$

under the periodic boundary condition with the given initial data $u(0, x)=50(1-$ $\cos (2 \pi x))$. We expand the unknown function by using the Fourier series

$$
\begin{equation*}
u(t, x)=\sum_{k \in \mathbb{Z}} a_{k}(t) e^{\mathrm{i} k \omega x}, \omega=2 \pi \tag{2.2}
\end{equation*}
$$

Plugging (2.2) in the initial-boundary value problem (2.1), we have the following infinitedimensional system of ODEs:

$$
\begin{equation*}
\frac{d}{d t} a_{k}(t)=e^{\mathrm{i} \theta}\left[-k^{2} \omega^{2} a_{k}(t)+(a(t) * a(t))_{k}\right] \quad(k \in \mathbb{Z}), \quad a(0)=a_{0} \tag{2.3}
\end{equation*}
$$

where "*" denotes the discrete convolution product defined by

$$
(b * c)_{k}:=\sum_{m \in \mathbb{Z}} b_{k-m} c_{m} \quad(k \in \mathbb{Z})
$$

for bi-infinite sequences $b=\left(b_{k}\right)_{k \in \mathbb{Z}}$ and $c=\left(c_{k}\right)_{k \in \mathbb{Z}}$, and $a_{0}$ is defined by

$$
\left(a_{0}\right)_{k}:=\left\{\begin{array}{cl}
-25, & k= \pm 1  \tag{2.4}\\
50, & k=0 \\
0, & \text { otherwise }
\end{array}\right.
$$

For a fixed time $h>0$ called step size, let us define $J:=(0, h)$ and the Banach space

$$
X:=C\left(J ; \ell^{1}\right), \quad\|a\|_{X}:=\sup _{t \in J}\|a(t)\|_{\ell^{1}}
$$

where $\ell^{1}:=\left\{a=\left(a_{k}\right)_{k \in \mathbb{Z}}: \sum_{k \in \mathbb{Z}}\left|a_{k}\right|<\infty, a_{k} \in \mathbb{C}\right\}$ with the norm $\|a\|_{\ell^{1}}:=\sum_{k \in \mathbb{Z}}\left|a_{k}\right|$.
Our task of rigorous numerics is to determine the Fourier coefficients $\left(a_{k}(t)\right)_{k \in \mathbb{Z}}$ of the solution of the Cauchy problem (2.3). By using Fourier spectral method, one can easily have an approximate solution ${ }^{2}\left(\bar{a}_{k}(t)\right)_{|k| \leq N}$. More precisely, we have an approximation of the Fourier coefficients with maximal wave number $N$. Let $\bar{a}(t):=$ $\left(\ldots, 0,0, \bar{a}_{-N}(t), \ldots, \bar{a}_{N}(t), 0,0, \ldots\right)$ be an approximation of $a(t)$ in $\ell^{1}$. On the basis of this approximate solution, we rigorously enclose the Fourier coefficients in the closed ball

$$
\begin{equation*}
B_{J}(\bar{a}, \varrho):=\left\{a \in X:\|a-\bar{a}\|_{X} \leq \varrho, a(0)=a_{0}\right\} \tag{2.5}
\end{equation*}
$$

Define the Laplacian operator $L$ acting on a sequence of Fourier coefficients as

$$
L b:=\left(-k^{2} \omega^{2} b_{k}\right)_{k \in \mathbb{Z}}, b=\left(b_{k}\right)_{k \in \mathbb{Z}} .
$$

The domain of the operator $L$ is defined by

$$
D(L):=\left\{a=\left(a_{k}\right)_{k \in \mathbb{Z}}: \sum_{k \in \mathbb{Z}} k^{2}\left|a_{k}\right|<\infty\right\} \subset \ell^{1} .
$$

Define an operator acting on $a \in C^{1}(J ; D(L))$ as

$$
(F(a))(t):=\frac{d}{d t} a(t)-e^{\mathrm{i} \theta}(L a(t)+a(t) * a(t)) .
$$

Hence, considering the operator equation $F(a)=0$, we define the following simplified Newton operator:

$$
\begin{equation*}
(T(a))(t):=\mathscr{A}_{a(0)}\left[e^{\mathrm{i} \theta}(a(t) * a(t)-2 \bar{a}(t) * a(t))\right], \quad T: X \rightarrow X . \tag{2.6}
\end{equation*}
$$

It is expected that the simplified Newton operator has a fixed point $\tilde{a} \in B_{J}(\bar{a}, \varrho)$ such that $\tilde{a}=T(\tilde{a})$. Here, $\mathscr{A}_{a(0)}$ is called the solution map operator in [19], which is defined by the solution representation of the following inhomogeneous linearized Cauchy problem in $\ell^{1}$ :

$$
\begin{equation*}
\frac{d}{d t} b_{k}(t)+e^{\mathrm{i} \theta}\left[k^{2} \omega^{2} b_{k}(t)-2(\bar{a}(t) * b(t))_{k}\right]=g_{k}(t) \quad(k \in \mathbb{Z}) \tag{2.7}
\end{equation*}
$$

with any $\ell^{1}$ initial sequence $b_{k}(s)=\phi_{k}(0 \leq s \leq t)$, where $\left(g_{k}(t)\right)_{k \in \mathbb{Z}} \in \ell^{1}$ is an arbitrary forcing term. More precisely, we define the solution map operator as

$$
\begin{equation*}
\mathscr{A}_{\phi} g:=U(t, s) \phi+\int_{s}^{t} U(t, r) g(r) d r, \tag{2.8}
\end{equation*}
$$

[^2]where $\{U(t, s)\}_{0 \leq s \leq t \leq h}$ is the evolution operator of (2.7) without the forcing term. This is nothing but the solution representation via the variation of constants formula for (2.7). Furthermore, $\mathscr{A}_{\phi} 0=U(t, s) \phi$ holds.

To use the solution map operator for our rigorous numerics, we require the explicit value of a positive constant $\boldsymbol{W}_{\boldsymbol{h}}>0$ satisfying

$$
\begin{equation*}
\sup _{0 \leq s \leq t \leq h}\|U(t, s) \phi\|_{\ell^{1}} \leq \boldsymbol{W}_{\boldsymbol{h}}\|\phi\|_{\ell^{1}}, \quad \forall \phi \in \ell^{1} \tag{2.9}
\end{equation*}
$$

which is based on generation theory (cf. [12]) of the evolution operator on Banach space $\ell^{1}$. We show the explicit value of $\boldsymbol{W}_{\boldsymbol{h}}$ in the next section.

Remark. Suppose $\bar{a} \in C^{1}(J ; D(L))$, we take $\mathscr{A}_{\phi}^{\dagger}$ as the Fréchet derivative of $F: C^{1}(J ; D(L)) \rightarrow X$ at $\bar{a}$, say $\mathscr{A}_{\phi}^{\dagger}=D F[\bar{a}]$. We obtain that the solution map operator $\mathscr{A}_{\phi}$ satisfies $\mathscr{A}_{\phi} \mathscr{A}_{\phi}^{\dagger} a=a$ if $a \in C^{1}(J ; D(L))$ with the initial data $a(s)=\phi(0 \leq s \leq t)$. Furthermore, $\mathscr{A}_{\phi_{1}} g_{1}+\mathscr{A}_{\phi_{2}} g_{2}=\mathscr{A}_{\phi_{1}+\phi_{2}}\left(g_{1}+g_{2}\right)$ holds for $\phi_{1}, \phi_{2} \in \ell^{1}, g_{1}, g_{2} \in X$. Then, if $a \in C^{1}(J ; D(L))$ with $a(0)=\phi$, the simplified Newton operator satisfies

$$
\begin{aligned}
T(a) & =\mathscr{A}_{\phi}\left[e^{\mathrm{i} \theta}(a * a-2 \bar{a} * a)\right] \\
& =\mathscr{A}_{\phi}\left[\frac{d}{d t} a-e^{\mathrm{i} \theta}(L a+2 \bar{a} * a)-\frac{d}{d t} a+e^{\mathrm{i} \theta}(L a+a * a)\right] \\
& =\mathscr{A}_{\phi}\left(\mathscr{A}_{\phi}^{\dagger} a-F(a)\right)=a-\mathscr{A}_{0} F(a) .
\end{aligned}
$$

We note that this form $a-\mathscr{A}_{0} F(a)$ is defined only on $C^{1}(J ; D(L))$ but the simplified Newton operator (2.6) can be defined on $X$. Moreover, from the bootstrap property of the solution map operator $\mathscr{A}_{\phi}: X \rightarrow C^{1}(J ; D(L)) \subset X$ [19, Remark 3.4], the fixed point $\tilde{a}$ of $T$ satisfies $\tilde{a} \in C^{1}(J ; D(L))$ if such a fixed point is obtained.

## § 3. Tools of rigorous numerics

This section displays technical theorems for our rigorous numerics. We note that each hypothesis of these theorems can be rigorously checked by numerical computations based on interval arithmetic (cf., e.g., [16]). For the proofs of these theorems, let us refer to [19].

## §3.1. A uniform bound of the evolution operator

As previously mentioned, given a step size $h>0$, our task to use the solution map operator for rigorous numerics is to show the existence of the evolution operator $U(t, s)$ by computing a constant $\boldsymbol{W}_{\boldsymbol{h}}$ satisfying (2.9). To achieve this task we separate the
equations (2.7) by considering yet another homogeneous Cauchy problem with respect to the sequence $c(t)=\left(c_{k}(t)\right)_{k \in \mathbb{Z}}$

$$
\begin{array}{ll}
\frac{d}{d t} c_{k}(t)+e^{\mathrm{i} \theta}\left[k^{2} \omega^{2} c_{k}(t)-2\left(\bar{a}(t) * c^{(m)}(t)\right)_{k}\right]=0 & (|k| \leq m) \\
\frac{d}{d t} c_{k}(t)+e^{\mathrm{i} \theta}\left[k^{2} \omega^{2} c_{k}(t)-2\left(\bar{a}(t) * c^{(\infty)}(t)\right)_{k}\right]=0 & (|k|>m) \tag{3.2}
\end{array}
$$

where $c^{(m)}(t)$ and $c^{(\infty)}(t)$ are defined by $c^{(m)}(t):=\left(\ldots, 0, c_{-m}(t), \ldots, c_{m}(t), 0, \ldots\right)$ and $c^{(\infty)}(t):=\left(\ldots, c_{-m-1}(t), 0, \ldots, 0, c_{m+1}(t), \ldots\right)$, respectively. This new decoupled formulation, while not being equivalent to (2.7), will be used to control the evolution operator associated to (2.7). Denote by $C^{(m)}(t, s)$ and $C^{(\infty)}(t, s)$ the evolution operators of the $(2 m+1)$-dimensional equation (3.1) and the infinite dimensional equation (3.2), respectively. We extend the action of the operator $C^{(m)}(t, s)\left(\right.$ resp. $\left.C^{(\infty)}(t, s)\right)$ on $\ell^{1}$ by introducing the operator $\bar{U}^{(m)}(t, s)$ (resp. $\left.\bar{U}^{(\infty)}(t, s)\right)$ as follows. Given $\phi \in \ell^{1}$, define $\bar{U}^{(m)}(t, s): \ell^{1} \rightarrow \ell^{1}$ and $\bar{U}^{(\infty)}(t, s): \ell^{1} \rightarrow \ell^{1}$ by

$$
\begin{align*}
\left(\bar{U}^{(m)}(t, s) \phi\right)_{k} & = \begin{cases}\left(C^{(m)}(t, s)\left(\phi_{k}\right)_{|k| \leq m}\right)_{k}, & |k| \leq m \\
0, & |k|>m\end{cases} \\
\left(\bar{U}^{(\infty)}(t, s) \phi\right)_{k} & = \begin{cases}0, & |k| \leq m \\
\left(C^{(\infty)}(t, s)\left(\phi_{k}\right)_{|k|>m}\right)_{k}, & |k|>m\end{cases} \tag{3.3}
\end{align*}
$$

The proof of existence of the evolution operator $U(t, s)$ of the original linearized problem (2.7) is presented in the following theorem by showing the explicit bound $\boldsymbol{W}_{\boldsymbol{h}}$ satisfying (2.9).

Theorem 3.1 ([19, Theorem 3.2]). Let $s, t \in J=(0, h)$ satisfying $0 \leq s \leq t \leq$ $h$. Assume that there exists a constant $W_{m}>0$ such that

$$
\begin{equation*}
\sup _{0 \leq s \leq t \leq h}\left\|\bar{U}^{(m)}(t, s)\right\|_{B\left(\ell^{1}\right)} \leq W_{m} \tag{3.4}
\end{equation*}
$$

Assume that $C^{(\infty)}(t, s)$ exists and that $\bar{U}^{(\infty)}(t, s)$ defined in (3.3) satisfies

$$
\left\|\bar{U}^{(\infty)}(t, s)\right\|_{B\left(\ell^{1}\right)} \leq W^{(\infty)}(t, s):=e^{-\mu_{m+1}(t-s)+2 \int_{s}^{t}\|\bar{a}(\tau)\|_{\ell^{1}} d \tau}
$$

where $\mu_{m+1}:=(m+1)^{2} \omega^{2} \cos \theta$. Define the constants $W_{\infty} \geq 0, \bar{W}_{\infty} \geq 0, W_{\infty}^{\text {sup }}>0$ as

$$
\begin{aligned}
W_{\infty} & :=\frac{e^{\left(2\|\bar{a}\|_{X}-\mu_{m+1}\right) h}-1}{2\|\bar{a}\|_{X}-\mu_{m+1}} \\
\bar{W}_{\infty} & :=\frac{W_{\infty}-h}{2\|\bar{a}\|_{X}-\mu_{m+1}} \\
W_{\infty}^{\sup } & := \begin{cases}1, & \mu_{m+1} \geq 2\|\bar{a}\|_{X} \\
e^{\left(\mu_{m+1}-2\|\bar{a}\|_{X}\right) h}, & \mu_{m+1}<2\|\bar{a}\|_{X}\end{cases}
\end{aligned}
$$

respectively. Define $\bar{a}^{(\boldsymbol{s})}(t) \in \ell^{1}$ component-wisely by

$$
\bar{a}_{k}^{(\boldsymbol{s})}(t)= \begin{cases}0, & |k| \leq N \text { and } k=0 \\ \bar{a}_{k}(t), & |k| \leq N \text { and } k \neq 0 \\ 0, & |k|>N\end{cases}
$$

If

$$
\kappa:=1-4 W_{m} \bar{W}_{\infty}\left\|\bar{a}^{(\boldsymbol{s})}\right\|_{X}^{2}>0,
$$

then the evolution operator $U(t, s)$ exists and the following estimate holds

$$
\sup _{0 \leq s \leq t \leq h}\|U(t, s) \phi\|_{\ell^{1}} \leq \boldsymbol{W}_{\boldsymbol{h}}\|\phi\|_{\ell^{1}}, \quad \forall \phi \in \ell^{1}
$$

where $\boldsymbol{W}_{\boldsymbol{h}}>0$ is defined by the following 1-norm of $2 \times 2$ matrix:

$$
\boldsymbol{W}_{\boldsymbol{h}}:=\left\|\left[\begin{array}{cc}
W_{m} \kappa^{-1} & 2 W_{m} W_{\infty}\left\|\bar{a}^{(\boldsymbol{s})}\right\|_{X} \kappa^{-1}  \tag{3.5}\\
2 W_{m} W_{\infty}\left\|\bar{a}^{\boldsymbol{s})}\right\|_{X} \kappa^{-1} W_{\infty}^{\sup }+4 W_{m} W_{\infty}^{2}\left\|\bar{a}^{(\boldsymbol{s})}\right\|_{X}^{2} \kappa^{-1}
\end{array}\right]\right\|_{1}
$$

## §3.2. Rigorous enclosure of solution in short time

The following theorem guarantees the existence of the solution of (2.3), which is equivalent to the original complex valued nonlinear heat equation (1.1) on the straight line $\Gamma_{\theta}$, in the neighborhood of numerically computed solution $B_{J}(\bar{a}, \varrho)$ defined in (2.5). The proof of this theorem is based on the Banach fixed point theorem.

Theorem 3.2 ([19, Theorem 4.1]). For a given initial sequence $a(0)$ and its approximation $\bar{a}(0)$, assume that there exists $\varepsilon \geq 0$ such that $\|a(0)-\bar{a}(0)\|_{\ell^{1}} \leq \varepsilon$. Assume also that $\bar{a} \in C^{1}(J ; D(L))$ and any $a \in B_{J}(\bar{a}, \varrho)$ satisfies

$$
\sup _{t \in J} \sum_{k \in \mathbb{Z}}\left|(T(a)(t)-\bar{a}(t))_{k}\right| \leq f_{\varepsilon}(\varrho),
$$

where $f_{\varepsilon}(\varrho)$ is defined by

$$
f_{\varepsilon}(\varrho):=\boldsymbol{W}_{\boldsymbol{h}}\left[\varepsilon+h\left(2 \varrho^{2}+\delta\right)\right] .
$$

Here, $\boldsymbol{W}_{\boldsymbol{h}}>0$ and $\delta>0$ satisfy $\sup _{0 \leq s \leq t \leq h}\|U(t, s)\|_{B\left(\ell^{1}\right)} \leq \boldsymbol{W}_{\boldsymbol{h}}$ and $\|F(\bar{a})\|_{X} \leq \delta$, respectively. If $f_{\varepsilon}(\varrho) \leq \varrho$ holds, then the Fourier coefficients $\tilde{a}$ of the solution of (2.1) are rigorously included in $B_{J}(\bar{a}, \varrho)$ and are unique in $B_{J}(\bar{a}, \varrho)$.

To check the hypothesis of Theorem 3.2, let us put a sketch of the proof. For the complete proof and details how we obtain the bounds $\varepsilon$ and $\delta$, we refer to [19, Section 4].

Firstly, for a sequence $a \in B_{J}(\bar{a}, \varrho)$, we have using (2.6)

$$
\begin{align*}
T(a)-\bar{a} & =T(a)-T(\bar{a})+T(\bar{a})-\bar{a}  \tag{3.6}\\
& =\mathscr{A}_{a(0)-\bar{a}(0)}\left[e^{\mathrm{i} \theta}(a * a-\bar{a} * \bar{a}-2 \bar{a} *(a-\bar{a}))\right]-\mathscr{A}_{0} F(\bar{a}) \\
& =\mathscr{A}_{z(0)}\left[2 e^{\mathrm{i} \theta} \int_{0}^{1} \eta d \eta(z * z)\right]-\mathscr{A}_{0} F(\bar{a}),
\end{align*}
$$

where we denote $z:=a-\bar{a}$. Thus, (3.6) is represented by

$$
T(a)-\bar{a}=\mathscr{A}_{z(0)}\left(2 e^{\mathrm{i} \theta} \int_{0}^{1} \eta d \eta(z * z)_{k}-F_{k}(\bar{a})\right)_{k \in \mathbb{Z}}
$$

where $\mathscr{A}_{z(0)}$ is the solution map operator defined in (2.8) and

$$
F_{k}(\bar{a})= \begin{cases}\frac{d}{d t} \bar{a}_{k}-e^{\mathrm{i} \theta}\left((L \bar{a})_{k}+(\bar{a} * \bar{a})_{k}\right), & |k| \leq N \\ -e^{\mathrm{i} \theta}(\bar{a} * \bar{a})_{k}, & |k|>N .\end{cases}
$$

Taking $\ell^{1}$ norm of $T(a)-\bar{a}$, we have from (2.8)

$$
\begin{align*}
\|T(a)-\bar{a}\|_{\ell^{1}} & =\sum_{k \in \mathbb{Z}}\left|(T(a)-\bar{a})_{k}\right|  \tag{3.7}\\
& =\left\|U(t, 0) z(0)+\int_{0}^{t} U(t, s) g(s) d s\right\|_{\ell^{1}} \\
& \leq\|U(t, 0) z(0)\|_{\ell^{1}}+\int_{0}^{t}\|U(t, s) g(s)\|_{\ell^{1}} d s,
\end{align*}
$$

where $g(s):=2 e^{\mathrm{i} \theta} \int_{0}^{1} \eta d \eta(z(s) * z(s))-(F(\bar{a}))(s)$. Taking $\ell^{1}$-norm of $g$, we have

$$
\|g(s)\|_{\ell^{1}} \leq \sum_{k \in \mathbb{Z}}\left|2 e^{\mathrm{i} \theta}(z(s) * z(s))_{k}\right|+\|(F(\bar{a}))(s)\|_{\ell^{1}} \leq 2\|z(s)\|_{\ell^{1}}^{2}+\delta,
$$

where $\delta$ satisfies $\sup _{s \in J}\|(F(\bar{a}))(s)\|_{\ell^{1}} \leq \delta$. Since $a \in B_{J}(\bar{a}, \varrho),\|z\|_{X} \leq \varrho$ holds. Consequently, (3.7) is bounded by using the uniform bound $\boldsymbol{W}_{\boldsymbol{h}}$ (2.9) as

$$
\begin{aligned}
\sup _{t \in J} \sum_{k \in \mathbb{Z}}\left|((T(a))(t)-\bar{a}(t))_{k}\right| & \leq \sup _{t \in J}\|U(t, 0) z(0)\|_{\ell^{1}}+\sup _{t \in J} \int_{0}^{t}\|U(t, s) g(s)\|_{\ell^{1}} d s \\
& \leq \boldsymbol{W}_{\boldsymbol{h}}\left[\varepsilon+h\left(2 \varrho^{2}+\delta\right)\right]=f_{\varepsilon}(\varrho),
\end{aligned}
$$

where $\varepsilon$ is the upper bound of the initial error such that $\|z(0)\|_{\ell^{1}} \leq \varepsilon$. From the assumption $f_{\varepsilon}(\varrho) \leq \varrho, T(a) \in B_{J}(\bar{a}, \varrho)$ holds for any $a \in B_{J}(\bar{a}, \varrho)$.

Secondly, for sequences $a_{1}, a_{2} \in B_{J}(\bar{a}, \varrho)$, we define the distance in $B_{J}(\bar{a}, \varrho)$ as $\mathbf{d}\left(a_{1}, a_{2}\right):=\left\|a_{1}-a_{2}\right\|_{X}$. The analogous discussion above yields $\mathbf{d}\left(T\left(a_{1}\right), T\left(a_{2}\right)\right) \leq$
$\left(2 \boldsymbol{W}_{\boldsymbol{h}} h \varrho\right) \mathbf{d}\left(a_{1}, a_{2}\right)$. Taking $\kappa=2 \boldsymbol{W}_{\boldsymbol{h}} h \varrho$, it follows $\kappa<f_{\varepsilon}(\varrho) / \varrho \leq 1$ from the assumption of theorem. It is proved that the simplified Newton operator $T$ becomes the contraction mapping on $B_{J}(\bar{a}, \varrho)$.

## $\S$ 3.3. Proof of global existence based on the Lyapunov-Perron method

After guaranteeing the local existence of the solution, we try to prove global existence in time of the solution by checking a hypothesis of the following theorem. That corresponds to a calculation of a part of center-stable manifold. Starting from the PDE (2.1), we consider the system of differential equations in $\ell^{1}$ given by (2.3). To show global existence of the solution, we use the Lyapunov-Perron method to compute a foliation of portion of the center-stable manifold of the equilibrium at $a \equiv 0$. A good reference for this method in ODEs is [1], and for PDEs see [17]. In [20] this method is applied to give computer assisted proofs of the stable manifold theorem in the Swift-Hohenberg PDE.

Let us define subspaces

$$
X_{c}:=\left\{a \in \ell^{1} \mid a_{k}=0 \forall k \neq 0\right\}, \quad X_{s}:=\left\{a \in \ell^{1} \mid a_{0}=0\right\} .
$$

We rewrite (2.3) into the following system:

$$
\begin{align*}
& \dot{\mathrm{x}}_{c}=\mathcal{N}_{c}\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right)  \tag{3.8}\\
& \dot{\mathrm{x}}_{s}=\mathfrak{L x}_{s}+\mathcal{N}_{s}\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right),
\end{align*}
$$

where for $a=\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right)$ and we define

$$
\begin{aligned}
(\mathfrak{L} a)_{k} & :=-e^{i \theta} k^{2} \omega^{2} a_{k}, \\
\mathcal{N}_{c}\left(a_{c}, a_{s}\right) & :=e^{i \theta} \sum_{k=0}^{\infty} a_{k} a_{-k}, \\
\left(\mathcal{N}_{s}\left(a_{c}, a_{s}\right)\right)_{k} & :=e^{i \theta} \sum_{\substack{k_{1}+k_{2}=k \\
k_{1}, k_{2} \in \mathbb{Z}}} a_{k_{1}} a_{k_{2}} .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left\|e^{\mathfrak{L} t}\right\|_{B\left(\ell^{1}\right)} & \leq e^{-\omega^{2} \cos \theta t} \\
\left|\mathcal{N}_{c}\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right)\right| & \leq\left|\mathrm{x}_{c}\right|^{2}+\left\|\mathrm{x}_{s}\right\|_{\ell^{1}}^{2} \\
\left\|\mathcal{N}_{s}\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right)\right\|_{\ell^{1}} & \leq 2\left|\mathrm{x}_{c}\right|\left\|\mathrm{x}_{s}\right\|_{\ell^{1}}+\left\|\mathrm{x}_{s}\right\|_{\ell^{1}}^{2}
\end{aligned}
$$

holds. Furthermore, let us define $\mu=\omega^{2} \cos \theta$.
For the equilibrium at zero, the center manifold is precisely $X_{c}$. Restricted to this subspace $X_{c}$, we can solve (3.8) by

$$
\begin{equation*}
\dot{\mathrm{x}}_{c}=e^{i \theta} \mathrm{x}_{c}^{2}, \tag{3.9}
\end{equation*}
$$

using the fact that the differential equation is separable. For an initial condition $\phi \in \mathbb{C}$, the solution of (3.9) is given by

$$
\Phi(t, \phi):=\frac{\phi}{1-\phi t e^{i \theta}} .
$$

For $r_{c}, r_{s} \in \mathbb{R}_{+}$, let us define the following sets

$$
\begin{aligned}
& B_{c}\left(r_{c}\right):=\left\{\mathrm{x}_{c} \in X_{c}:\left|\mathrm{x}_{c}\right| \leq r_{c}, \operatorname{Re}\left(e^{i \theta} \mathrm{x}_{c}\right) \leq 0\right\} \\
& B_{s}\left(r_{s}\right):=\left\{\mathrm{x}_{s} \in X_{s}:\left\|\mathrm{x}_{s}\right\|_{\ell^{1}} \leq r_{s}\right\} .
\end{aligned}
$$

Note that if $\phi \in B_{c}\left(r_{c}\right)$ then $\Phi(t) \in B_{c}\left(r_{c}\right)$ for all $t \geq 0$, and additionally $|\Phi(t, \phi)| \leq r_{c}$ holds. For a fixed $\rho \in \mathbb{R}_{+}$, we define the following set of functions:
$\mathcal{B}=\left\{\alpha \in \operatorname{Lip}\left(B_{c}\left(r_{c}\right) \times B_{s}\left(r_{s}\right), X_{c}\right): \alpha\left(\mathrm{x}_{c}, 0\right)=\mathrm{x}_{c},\left|\alpha\left(\mathrm{x}_{c}, \mathrm{x}_{1}\right)-\alpha\left(\mathrm{x}_{c}, \mathrm{x}_{2}\right)\right| \leq \rho\left|\mathrm{x}_{1}-\mathrm{x}_{2}\right|\right\}$.
Continuing with the Lyapunov-Perron method, for a fixed $\alpha \in \mathcal{B}, \phi \in B_{c}\left(r_{c}\right), \xi \in$ $B_{s}\left(r_{s}\right)$, we define $x(t, \phi, \xi, \alpha)$ as a solution of the following differential equation:

$$
\dot{\mathrm{x}}_{s}=\mathfrak{L} \mathrm{x}_{s}+\mathcal{N}_{s}\left(\alpha\left(\Phi(t, \phi), \mathrm{x}_{s}\right), \mathrm{x}_{s}\right)
$$

with initial conditions $(\phi, \xi)$.
Now we define the Lyapunov-Perron Operator for $\alpha \in \mathcal{B}$ as follows:

$$
\Psi[\alpha](\phi, \xi)=-\int_{0}^{\infty} \mathcal{N}_{c}(\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha)), x(t, \phi, \xi, \alpha)) d t .
$$

It is shown [19, Proposition 6.10] that $\Psi: \mathcal{B} \rightarrow \mathcal{B}$ is a well defined operator. If $\Psi[\alpha]=\alpha$ holds, then the trajectory $\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha))$ satisfies (3.8) for all $(\phi, \xi) \in B_{c} \times B_{s}$. This is because the Lyapunov-Perron operator is defined as the variation of constants formula. Hence, it implies that $(\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha)), x(t, \phi, \xi, \alpha))$ satisfies our original equation (2.3).

Since $\Phi(t, \phi)$ limits to zero, such a fixed point $\alpha=\Psi[\alpha]$ gives us a foliation of the center stable manifold over $B_{c}\left(r_{c}\right)$. Therefore, we obtain an explicit neighborhood within which all points limit to the zero equilibrium. To prove the existence of a fixed point to our Lyapunov-Perron operator, and obtain explicit bounds, we prove the following theorem.

Theorem 3.3 ([19, Corollary 6.3]). Fix $r_{c}, r_{s}, \rho>0$ and write $B_{c}=B_{c}\left(r_{c}\right)$ and $B_{s}=B_{s}\left(r_{s}\right)$. Define the set $U \subseteq B_{c} \times B_{s}$ as

$$
U:=\left\{\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right) \in B_{c} \times B_{s}: \rho\left\|\mathrm{x}_{s}\right\|_{\ell^{1}}<\operatorname{dist}\left(\mathrm{x}_{c}, \partial B_{c}\right)\right\},
$$

where $\partial B_{c}$ denotes the boundary of $B_{c}$. Suppose the following constants

$$
\begin{aligned}
\delta_{1} & :=2 r_{c}+(1+2 \rho) r_{s}, \\
\delta_{2} & :=2 r_{c}+2(1+\rho) r_{s}, \\
\delta_{3} & :=2\left(\rho\left(r_{c}+\rho r_{s}\right)+r_{s}\right), \\
\delta_{4} & :=2\left(r_{c}+2 \rho r_{s}+r_{s}\right), \\
\lambda & :=\frac{2\left(\rho\left(r_{c}+\rho r_{s}\right)+r_{s}\right) 2 r_{s}}{\left(\mu-\delta_{1}\right)\left(\mu-\delta_{4}\right)}+\frac{2\left(r_{c}+\rho r_{s}\right)}{\mu-\delta_{1}}
\end{aligned}
$$

satisfy

$$
\begin{equation*}
\delta_{1}, \delta_{2}, \delta_{4}<\mu, \quad \frac{\delta_{3}}{\mu-\delta_{2}}<\rho, \quad \lambda<1 . \tag{3.10}
\end{equation*}
$$

If the initial data $a_{0} \in \ell^{1}$ satisfy

$$
\left(\left(a_{0}\right)_{k=0},\left(a_{0}\right)_{k \neq 0}\right) \in U,
$$

then the solution $a(t)$ of (2.3) with the initial data $a_{0}$ is globally defined and converges to zero.

Remark. For practical implementation, in order to construct a trapping region $U$ in Theorem 3.3 which might contain $a_{0}$, we fix mildly inflated radii constants

$$
r_{s}:=\left\|\left(a_{0}\right)_{k \neq 0}\right\|_{\ell^{1}}, \quad r_{c}:=\left|\left(a_{0}\right)_{k=0}\right|+0.02 r_{s} .
$$

Then we numerically check the condition of $U$ by checking the inequality

$$
\rho r_{s}<\operatorname{dist}\left(\left(a_{0}\right)_{k=0}, \partial B_{c}\right)<\min \left\{r_{c}-\left|\left(a_{0}\right)_{k=0}\right|,-\operatorname{Re}\left(e^{i \theta} \cdot\left(a_{0}\right)_{k=0}\right)\right\} .
$$

If the hypothesis of Theorem 3.3 holds, then the explicit neighborhood $U$ is contained in the $\alpha$-skew image of $B_{c} \times B_{s}$, i.e., $U \subseteq\left\{\left(\alpha\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right), \mathrm{x}_{s}\right):\left(\mathrm{x}_{c}, \mathrm{x}_{s}\right) \in B_{c} \times B_{s}\right\}$. Furthermore, if the initial data $a_{0} \in U$, then $a_{0}$ is in the center stable manifold of the zero equilibrium.

## § 3.4. Algorithm of rigorous integration and global existence

Summing up this section, we put an algorithm for proving local inclusion of the solution of (2.1) and showing global existence on the straight path. Let $0=t_{0}<t_{1}<\ldots$ be grid points of the time variable. We call $J_{i}=\left(t_{i-1}, t_{i}\right)$ the $i$ th time step and let $t_{i}=i h(i=1,2, \ldots)$ with the stepsize $h$. Algorithm 1 shows our procedure of rigorous integration and proof of global existence on the straight path, which is combination of theorems introduced in this section.

```
Algorithm 1 Rigorous integrator and proof of global existence.
Input: \(N\) (maximal wave number of Fourier in space), \(n\) (number of Chebyshev series
    in time), \(h\) (step size), \(\theta\) (angle of the path), \(m\) (truncation size of the linearized
    Cauchy problem shown in Section 3.1), \(a_{0}\) (initial sequence defined in (2.4))
    while \(i=1,2, \ldots\) do
        Set the time step \(J \equiv J_{i}=\left(t_{i-1}, t_{i}\right)\) with \(t_{i}=i h\).
        Get \(\bar{a}\) using Chebyshev-Fourier spectral methods for (2.3) with \(|k| \leq N\).
        Obtain \(\delta \geq 0\) satisfying \(\|F(\bar{a})\|_{X} \leq \delta\) and \(\varepsilon \geq 0\) such that \(\left\|a\left(t_{i-1}\right)-\bar{a}\left(t_{i-1}\right)\right\|_{\ell^{1}} \leq\)
        \(\varepsilon\).
        Obtain a uniform bound of finite dimensional evolution operator \(W_{m}>0\) satis-
        fying (3.4) using the radii-polynomial approach [7].
        if the hypothesis of Theorem 3.1 holds then
            Theorem 3.1 guarantees the existence of the solution map operator \(\mathscr{A}_{\phi}\) defined
            in (2.8).
            Compute \(\boldsymbol{W}_{\boldsymbol{h}}\) by (3.5).
        else
            return fail in getting uniform bound of the evolution operator.
        end if
        if the hypothesis of Theorem 3.2 holds then
            Theorem 3.2 proves the existence of the solution of \((2.3)\) in \(B_{J}(\bar{a}, \varrho)\) defined in
            (2.5).
        else
            return fail in proving existence of the solution locally in time.
        end if
        if the hypothesis of Theorem 3.3 holds then
            return succeed in proving the global existence of the solution of (2.1) via
            Theorem 3.3.
        end if
        Update \(a_{0}=a\left(t_{i}\right)\).
    end while
```

In practical implementation, to finish Algorithm 1, we check the hypothesis of Theorem 3.3 after numerical verification of the local inclusion in time using Theorem 3.2. If we obtain the hypothesis is true, we prove the global existence of the solution of (2.1) via Theorem 3.3. Otherwise, we continue our rigorous integration to the next time step using Theorem 3.1 and Theorem 3.2. For this purpose the initial sequence in the next time step is replaced by a sequence at the endpoint of the current time step (e.g., $a_{0}=a(h)$ for the first time step). Replacing $J=(h, 2 h)$, we apply numerical verification for the Cauchy problem on the next time step and repeat this process recursively.

## §4. Proof of Theorem 1.2

In this section, we show our computer-assisted proof of Theorem 1.2. Our proof is based on rigorous numerics. All computations were carried out on Windows 10, $\operatorname{Intel}(\mathrm{R})$ Core(TM) i7-6700K CPU @ 4.00GHz, and MATLAB 2019a with INTLAB - INTerval LABoratory [15] version 11 and Chebfun - numerical computing with functions [13] version 5.7.0.

The proof consists of two steps. First, we prove the branching singularity, which is proved in [19, Theorem 1.1]. For the first step, we follow the same proof in [19]. Second, after proving the branching singularity, we prove the global existence on the real axis beyond the blow-up point via Theorem 3.3.

## §4.1. Proof of existence of branching singularity

Let us follow the proof of branching singularity given in [19, Section 7.1]. Because it is obvious that the solution of (1.1) has a symmetry $u\left(t e^{-\mathrm{i} \theta}, x\right)=\overline{u\left(t e^{\mathrm{i} \theta}, x\right)}$ for a real $t$, it is sufficient to prove that the imaginary part of $u(z, x)$ becomes a non-zero function at a certain point $z \in \mathbb{R}$ satisfying $z_{B}<z$, where $z_{B}$ denotes the blow-up point of (1.1). We took a path $\tilde{\Gamma}_{\theta}$ for analytical continuation, which bypasses the blow-up point $z_{B}$ as shown in Fig. 2. Here, Cho et al. [2] has shown $z_{B} \approx 0.0119$ under the periodic boundary condition with the initial data $u(0, x)=50(1-\cos (2 \pi x))$.

We divided each segment $\left(O \rightarrow z_{A}\right.$ and $z_{A} \rightarrow z_{C}$ in Fig. 2) into 16 steps and, by using our rigorous integrator, analytically continued to the $z_{C}$ point in Fig. 2. From $O$ to $z_{A}$, we set $\theta=\pi / 4$ in (2.1). After that, we changed the value of $\theta$ as $\theta=-\pi / 4$ from $z_{A}$ to $z_{C}$. We set the approximate solution defining the maximal wave number $N=15$. Then, for each time stepping, we got a numerical solution $\bar{u}$ by using Chebfun [13] described by

$$
\bar{u}(t, x)=\sum_{|k| \leq N}\left(\bar{a}_{0, k}+2 \sum_{\ell=1}^{n-1} \bar{a}_{\ell, k} T_{\ell}(t)\right) e^{\mathrm{i} k \omega x}, \omega=2 \pi
$$

with $n=13$, where $T_{\ell}(t)$ denotes the $\ell$ th order Chebyshev polynomial of first kind. We also set $m=2$ in Theorem 3.1, which decides the truncation size of linearized Cauchy problem. The profiles of numerically computed $\operatorname{Re}(\bar{u})$ and $\operatorname{Im}(\bar{u})$ were plotted in Fig. 3 .


Figure 3. Profiles of numerically computed solutions on each segment ((a) $O \rightarrow z_{A}$ and (b) $z_{A} \rightarrow z_{C}$ ) were plotted. At the point $z_{C}$, the imaginary part of $\bar{u}\left(z_{C}, x\right)$ became obviously non-zero function.

Our task of the proof here is to show that the imaginary part of $u\left(z_{C}, x\right)$ is a non-zero function. To prove this, we use the $\ell^{1}$ norm of the Fourier coefficients, say

$$
\|v\|:=\|c\|_{\ell^{1}}=\sum_{k \in \mathbb{Z}}\left|c_{k}\right| \quad \text { for } \quad v(x)=\sum_{k \in \mathbb{Z}} c_{k} e^{\mathrm{i} k \omega x} .
$$

Let the solution of (1.1) at $z_{C}$ point and its numerically computed solution be denoted by

$$
u\left(z_{C}, x\right)=\sum_{k \in \mathbb{Z}} a_{k}^{z_{C}} e^{\mathrm{i} k \omega x} \quad \text { and } \quad \bar{u}\left(z_{C}, x\right)=\sum_{|k| \leq N} \bar{a}_{k}^{z_{C}} e^{\mathrm{i} k \omega x},
$$

respectively. Denote two bi-infinite complex valued sequences by $a^{z_{C}}=\left(a_{k}^{z_{C}}\right)_{k \in \mathbb{Z}} \in$ $\ell^{1}$ and $\bar{a}^{z_{C}}=\left(\ldots, 0, \bar{a}_{-N}^{z_{C}}, \ldots, \bar{a}_{N}^{z_{C}}, 0 \ldots\right) \in \ell^{1}$, whose existence was proved by our numerical verification via Theorem 3.2. The norm of imaginary part of the solution at $z_{C}$ point follows

$$
\begin{aligned}
\left\|\operatorname{Im}\left(u\left(z_{C}, \cdot\right)\right)\right\| & =\left\|\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)+\operatorname{Im}\left(u\left(z_{C}, \cdot\right)\right)-\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)\right\| \\
& \geq\left\|\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)\right\|-\left\|\operatorname{Im}\left(u\left(z_{C}, \cdot\right)-\bar{u}\left(z_{C}, \cdot\right)\right)\right\| \\
& \geq\left\|\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)\right\|-\left\|a^{z_{C}}-\bar{a}^{z_{C}}\right\|_{\ell^{1}} \\
& \geq\left\|\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)\right\|-\varepsilon^{z_{C}},
\end{aligned}
$$

where $\varepsilon^{z_{C}}$ is the rigorous error of the solution at the $z_{C}$ point, which is given by our rigorous integrator. Hence our rigorous integrator gave the value $\varepsilon^{z_{C}}=1.10236 \times 10^{-4}$. Furthermore, the imaginary part of $\bar{u}\left(z_{C}, x\right)$ is presented by

$$
\begin{equation*}
\operatorname{Im}\left(\bar{u}\left(z_{C}, x\right)\right)=\frac{1}{2} \sum_{|k| \leq N}\left[\operatorname{Im}\left(\bar{a}_{k}^{z_{C}}\right)+\operatorname{Im}\left(\bar{a}_{-k}^{z_{C}}\right)-\mathrm{i}\left(\operatorname{Re}\left(\bar{a}_{k}^{z_{C}}\right)-\operatorname{Re}\left(\bar{a}_{-k}^{z_{C}}\right)\right)\right] e^{\mathrm{i} k \omega x} . \tag{4.1}
\end{equation*}
$$

We rigorously computed the coefficients of (4.1) using interval arithmetic and obtained the following inclusion:

$$
\left\|\operatorname{Im}\left(\bar{u}\left(z_{C}, \cdot\right)\right)\right\|-\varepsilon^{z_{C}} \in[126.9994,126.9995]
$$

where $[\cdot, \cdot]$ denotes the real interval. This implies $\left\|\operatorname{Im}\left(u\left(z_{C}, \cdot\right)\right)\right\|>0$.
Consequently, we prove that the imaginary part of $u\left(z_{C}, x\right)$ is the non-zero function. Then, there exists at least one branching singularity on the real line at $z \in\left(0, z_{C}\right)$ with $z_{C}=0.0238$. The results of analytical continuation is shown in Fig. 4.


Figure 4. For each time stepping $J_{i}(i=1, \ldots, 32)$, we plotted results of analytical continuation. (a) The rigorous error bound $\varepsilon_{i}$ such that $\left\|a\left(t_{i}\right)-\bar{a}\left(t_{i}\right)\right\|_{\ell^{1}} \leq \varepsilon_{i}$ for $i=0, \ldots, 32$. (b) The radius of the neighborhood $B_{J_{i}}\left(\bar{a}, \varrho_{i}\right)$ in which the exact solution of (2.3) is included for $i=1, \ldots, 32$.

## $\S$ 4.2. Proof of global existence beyond the blow-up point

Next, we show the global existence after the branching point via Theorem 3.3. From the $z_{C}$ point, we took the value of $\theta$ as $\theta=0$ and continued our rigorous integrator on the real axis. As shown in Fig. 5, profiles of the solution after the blow-up point was no longer the real valued solution even if the path is set on the real axis. Nevertheless, our rigorous integrator succeeded in analytically continuing on the real axis $z_{C} \rightarrow \infty$ in Fig. 2.

To prove the global existence, we check the hypothesis of Theorem 3.3. More precisely, at $t=t_{i}$ (after $i$ th time stepping), we rigorously computed

$$
r_{s}=\left\|\bar{a}^{(\boldsymbol{s})}\left(t_{i}\right)\right\|_{\ell^{1}}+\varepsilon_{i}, \quad r_{c}=\left|\bar{a}_{0}\left(t_{i}\right)\right|+\varepsilon_{i}+0.02 r_{s},
$$

where $\bar{a}^{(\boldsymbol{s})}(t)=\left(\ldots, 0, \bar{a}_{-N}(t), \ldots, \bar{a}_{-1}(t), 0, \bar{a}_{1}(t), \ldots, \bar{a}_{N}(t), 0, \ldots\right) \in \ell^{1}$. Then, for $\mu=\omega^{2} \cos \theta$, we tried to find the $\rho$ in (3.10). If such $\rho$ is not obtained, then the foliation of center stable manifold is failed to validate by Theorem 3.3. In such a case, we consider rigorous integration on the next time step, i.e. $J_{i+1}$.

Fig. 6 shows verified results of our rigorous integration on $\tilde{\Gamma}_{\theta}$. After 121 steps of the rigorous integration (at $t=0.1273$ ), the hypothesis of Theorem 3.3 held for $r_{c}=9.8129$,


Figure 5. The ( $x, t, u$ )-plot of the numerically computed solution on the path $\tilde{\Gamma}_{\theta}$. After the $z_{C}$ point, the solution gradually changed to be a constant and converged to zero.
$r_{s}=0.0181, \rho=0.0137$ and $\lambda=0.9895$. The execute time was 375 sec . Finally, we succeeded in proving the global existence of the solution of (1.1) along the path $\tilde{\Gamma}_{\theta}$. This completes the proof of Theorem 1.2.



Figure 6. Results of rigorous integration on $\tilde{\Gamma}_{\theta}$ : The value of $X$ norm of $\bar{a}$ was plotted as piecewise constant on $J_{i}$ (left). The rigorous error $\varepsilon_{i}$ was plotted at each $t_{i}$ (right). After 121 steps, the hypothesis of Theorem 3.3 held at $t=0.1273$. The step size is equidistantly taken as $h=1.0518 \times 10^{-3}$. We remark that we set $N=15$ (maximum wave number of Fourier), $n=13$ (number of Chebyshev basis) and $m=2$.

## Concluding remarks

In the present article we show a computer-assisted proof of the complex valued nonlinear heat equation, which gives a mathematical proof of Cho et al.'s numerical observation. It is shown that the solution of complex valued nonlinear heat equation under the periodic boundary condition has at least one branching singularity. After the singular point, there exists a certain $z_{C} \in \mathbb{R}$ such that the solution has no singularity from $z_{C}$ to infinity on the real axis. The proof is based on rigorous numerics which numerically gives constructive proofs and provides explicit error bounds nearby a numerically computed solution. The key idea is to show existence and local uniqueness of a fixed point for the simplified Newton operator corresponding to the Cauchy
problem, which requires explicit bounds of the evolution operator for the linearized Cauchy problem. We introduce a method of rigorous integration of the complex valued nonlinear heat equation and apply the provided rigorous integrator for proving our computer-assisted proof on a path in the complex plane of time. Finally, using the Lyapunov-Perron method to calculate part of a center-stable manifold, we introduce a method of proving global existence of the solution which converges to the zero function. Applying this method, our computer-assisted proof is completed, which is presented in details in Section 4.

We conclude this paper by putting some remarks and future directions of studying blow-up phenomena of nonlinear heat equations. Firstly, it is interesting that the branching singularity appears at the blow-up point but the branched Riemann surface after the blow-up point is close to zero. This indicates that the branched surface seems to be glued at infinity. Secondly, changing the nonlinearity or/and dimension of the space, one may characterize a different behavior of blow-up solution so called peaking solution from a view point of complex analysis. Thirdly, as Cho et al.'s conjecture 2, the global existence of the solution in the case of $\theta=\pi / 2$ (nonlinear Schödinger equation under the periodic boundary condition) is challenging problem both numerically and mathematically. Indeed, our rigorous integrator works for a few time steps for proving local inclusion of the solution in the neighborhood of a numerical solution. On the other hand, Theorem 3.3 cannot work in the case of calculating only center manifolds. We will try to prove global existence of the nonlinear Schödinger equation using rigorous numerics in future.

## Acknowledgement

The author express his sincere gratitude to Jean-Philippe Lessard (McGill University), Jonathan Jaquette (Brandeis University) and Hisashi Okamoto (Gakushuin University) for making progress of our project in [19]. This article is deeply depending on this paper. The author was partially supported by JSPS Grant-in-Aid for EarlyCareer Scientists (18K13453) and Grant-in-Aid for Scientific Research (B) (16H03950).

## References

[1] Chicone, C., Ordinary Differential Equations with Applications, Springer, 2006.
[2] Cho, C.-H., Okamoto, H. and Shōji, M., A blow-up problem for a nonlinear heat equation in the complex plane of time, Japan J. Indust. Appl. Math., 33 (2016), 145-166.
[3] Deng, K. and Levine, H. A., The role of critical exponents in blow-up theorems: The sequel, J. Math. Anal. Appl., 243 (2000), $85-126$.
[4] Fila, M. and Matano, H., Blow-up in nonlinear heat equations from the dynamical systems point of view, Handbook of dynamical systems, vol. 2, North-Holland, Amsterdam, 2002, pp. 723-758.
[5] Guo, J.-S., Ninomiya, H., Shimojo, M. and Yanagida, E., Convergence and blow-up of solutions for a complex-valued heat equation with a quadratic nonlinearity, Trans. Amer. Math. Soc., 365 (2013), 2447-2467.
[6] Harada, J., Blowup profile for a complex valued semilinear heat equation, J. Funct. Anal., 270 (2016), 4213 - 4255.
[7] Lessard, J.-P. and Reinhardt, C., Rigorous Numerics for Nonlinear Differential Equations Using Chebyshev Series, SIAM J. Numer. Anal., 52 (2014), 1-22.
[8] Levine, H. A., The role of critical exponents in blowup theorems, SIAM Rev., $\mathbf{3 2}$ (1990), 262-288.
[9] Masuda, K., Blow-up of solutions of some nonlinear diffusion equations, Nonlinear Partial Differential Equations in Applied Science; Proceedings of The U.S.-Japan Seminar, Tokyo, 1982, North-Holland Mathematics Studies, North-Holland, 81 (1983), 119 - 131.
[10] Masuda, K., Analytic solutions of some nonlinear diffusion equations, Math. Z., 187 (1984), 61-73.
[11] Nouaili, N. and Zaag, H., Profile for a simultaneously blowing up solution to a complex valued semilinear heat equation, Comm. Partial Differential Equations, 40 (2015), 11971217.
[12] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 1983.
[13] Platte, R. B. and Trefethen, L. N., Chebfun: a new kind of numerical computing, Progress in industrial mathematics at ECMI 2008, vol. 15, Springer, Heidelberg, 2010, pp. 69-87.
[14] Quittner, P. and Souplet, P., Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhäuser, 2007.
[15] Rump, S. M., INTLAB - INTerval LABoratory, Developments in Reliable Computing, Kluwer Academic Publishers, Dordrecht, 1999, pp. 77-104.
[16] Rump, S. M., Verification methods: Rigorous results using floating-point arithmetic, Acta Numer., 19 (2010), 287-449.
[17] Sell, G. R. and You, Y., Dynamics of evolutionary equations, vol. 143, Springer Science \& Business Media, 2002.
[18] Sulem, C., Sulem, P.-L. and Frisch, H., Tracing complex singularities with spectral methods, J. Comput. Phys., 50 (1983), $138-161$.
[19] Takayasu, A., Lessard, J.-P., Jaquette, J. and Okamoto, H., Rigorous numerics for nonlinear heat equations in the complex plane of time, arXiv:1910.12472, 2019.
[20] van den Berg, J. B., Jaquette, J. and Mireles James, J. D., Validated numerical approximation of stable manifolds for parabolic PDEs, In Preparation, 2019.


[^0]:    Received September, 30, 2019. Revised January 18, 2020.
    2010 Mathematics Subject Classification(s): 35A20, 35B40, 35B44, 35K55, 65G40, 65M15, 65M70
    Key Words: blow-up solutions for nonlinear heat equations, branching singularity, global existence, evolution operator, rigorous numerics.
    *Faculty of Engineering, Information and Systems, University of Tsukuba, Ibaraki 305-8573, Japan. e-mail: takitoshi@risk.tsukuba.ac.jp

[^1]:    ${ }^{1}$ We may take $\theta \approx 0$. However, in this case, the solution is near the singularity and it is difficult to prove our main result. We require a sufficiently large angle $\theta$.

[^2]:    ${ }^{2}$ Throughout this article, we note that $\bar{a}$ represents an approximation of $a$, not the complex conjugate.

