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On the derivation of the mean field equation of the Gibbs distribution function for equilibrium vortices in an external field

By

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Abstract

Motivated by several experimental facts, we are interested in the linear response of equilibrium vortices. In order to study the phenomenon, here we investigate the mean field limit of equilibrium vortices perturbed by an external field and derive the mean field equation of the Gibbs distribution function. Similar limits for classical point particles with bounded interactions were studied by Messer-Spohn [14] and later the results were extended to the system of vortices, which interact via the singular logarithmic potential, by Caglioti et al [2] and Kiessling [10] . In this paper, we start with the review of these results in some detail and extend their arguments to the case for vortices perturbed by an external field.

§ 1. Introduction

In this paper, we are interested in the (canonical) Gibbs distribution function for equilibrium of a large number of vortices confined in a bounded container $\Lambda \subset \mathbf{R}^2$ in an external field. To simplify the presentation, we assume that Λ is simply connected and with a smooth boundary throughout this paper unless we mention otherwise.

The study of Gibbs distribution function for physical systems seems to be a central topic of statistical physics, see [5] for example. Concerning the vortices in two dimensional incompressible non-viscous fluid, the importance of the study of the distribution function seems to be realized when Onsager pointed out the possibility of negative temperature states of equilibrium vortices, which are introduced to explain the reason why

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the large scale structures, such as the great red spot on the surface of Jupiter, often maintain in two dimensional fluid [15].

This kind of *self-organization* phenomenon attracted many researchers and they try to derive so-called *the mean field equation* that describe the distribution function for infinitely many vortices, see [8, 9, 17, 18] for example. Recently, it was found that Onsager himself also derived the equation, see [3].

These results, however, are obtained by rather heuristically. Later mathematically rigorous derivation of the mean field equation is developed by Caglioti, Lions, Marchioro, Pulvirenti[2] and Kiessling [10], see also [13, 12]. They used and improved the argument established by Messer-Spohn [14] for system of classical point particles with bounded interactions, which does not cover the logarithmic interaction of vortices.

The purpose of this paper is to derive mathematically rigorously the mean field equation of equilibrium vortices perturbed by an external field. To this purpose, we review the theory of vortices, the Messer-Spohn argument, and it's improvement by Caglioti et al. and Kiessling in detail for the readers convenience. Then we derive the mean field equation (5.3) for the system described by the Hamiltonian with an external field, which might not be completely new but seems not to be mentioned previously to the author's knowledge. Our motivation for this study is to study *the linear response* of equilibrium vortices, which is our on going project with several physicists, see [16] for our progress so far.

§ 2. Dynamics of vortices

We assumed that $\Lambda \subset \mathbf{R}^2$ is a simply connected bounded domain with smooth boundary $\partial\Lambda$. Then the motion of non-viscous incompressible fluid in Λ is described by the equation of the vorticity field $\omega = \operatorname{curl} \boldsymbol{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$:

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \omega = 0. \tag{2.1}$$

Here $\boldsymbol{v} = (v_1(x,t), v_2(x,t))$ is the velocity field determined by the vorticity field ω from the incompressible condition and the usual slip boundary condition:

$$\operatorname{div} \boldsymbol{v} := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Lambda.$$

In fact, using the solution ψ of the Poisson equation

$$-\Delta\psi = \omega \quad \text{in } \Lambda, \qquad \psi = 0 \quad \text{on } \partial\Lambda,$$

we are able to recover the velocity field $\boldsymbol{v}$ from the vorticity field:

$$\boldsymbol{v} = \nabla^{\perp}\psi = \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right)$$

since Λ is simply connected. The function ψ is called *the stream function* of the velocity field $\boldsymbol{v}$ and is uniquely determined by ω under appropriate assumptions.

The N-points *vortices* are the set $\{(x_j(t), \Omega_j)\}_{j=1,\cdots,N} \subset \Lambda \times \mathbf{R}$ that composes the vorticity field $\omega(x,t) = \sum_{j=1}^N \Omega_j \delta_{x_j(t)}$ *satisfying* the vorticity equation (2.1), where δ_p is the Dirac measure supported at $p \in \Lambda$. We call Ω_j the intensity of j-th vortex. From Kelvin's circulation law, the intensity Ω_j and the form $\sum_{j=1}^N \Omega_j \delta_{x_j}$ are considered to be preserved during the time evolution. However, it still seems to exist a problem how to recognize the vortices as a solution of vorticity equation (2.1) since the singularity of vorticity field like $\sum_{j=1}^N \Omega_j \delta_{x_j}$ is too strong to assume $\omega(x,t) = \sum_{j=1}^N \Omega_j \delta_{x_j(t)}$ to be a solution of (2.1) even in a weak sense.

Nevertheless, vortices are considered to obey the following system of ordinary differential equations, see [1, 6, 13] for example:

$$\Omega_j \frac{dx_j}{dt} = \nabla_j^\perp H^{N,\Omega}(x_1, \cdots, x_N) \left(:= \left(\frac{\partial H^{N,\Omega}}{\partial x_{j,2}}, -\frac{\partial H^{N,\Omega}}{\partial x_{j,1}} \right) \right). \tag{2.2}$$

Here $\Omega = (\Omega_1, \cdots, \Omega_N) \in \mathbf{R}^N$ and $H^{N,\Omega}$ is the following function on $\Lambda^N \subset \mathbf{R}^{2N}$:

$$H^{N,\Omega}(x_1, \cdots, x_N) := \frac{1}{2} \sum_{k=1}^N \Omega_k^2 K(x_k, x_k) + \frac{1}{2} \sum_{1 \le k,l \le N,\, k \ne l} \Omega_k \Omega_l G(x_k, x_l), \tag{2.3}$$

where $G(x,y)$ is the Green function of $-\Delta$ with the Dirichlet boundary condition, i.e., $G(\cdot, y)$ satisfies

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Lambda, \qquad G(\cdot, y) = 0 \quad \text{on } \partial\Lambda,$$

and $K(x,y)$ is it's regular part defined as

$$G(x,y) = \frac{1}{2\pi} \log |x-y|^{-1} + K(x,y).$$

In this article, we do not take care of the validity of the theory of the vortices and we start with the system (2.2).

The function $H^{N,\Omega}$ is usually called *the Kirchhoff-Routh path function* and the value of $H^{N,\Omega}$ is constant along the solution of the vortex system (2.2) as long as it exists. Therefore, we are able to assume that the system of vortices (2.2) forms a Hamiltonian system with the Hamiltonian $H^{N,\Omega}$ and consequently we are able to develop the statistical mechanics for equilibrium vortices.

Here we introduce the (canonical) Gibbs distribution function

$$\mu^{N,\Omega}(x_1, \cdots, x_N) := \frac{e^{-\beta^N H^{N,\Omega}(x_1,\cdots,x_N)}}{\int_{\Lambda^N} e^{-\beta^N H^{N,\Omega}(x_1,\cdots,x_N)}},$$

where $\beta^N \in \mathbf{R}$ is a parameter called *the inverse temperature*. We are interested in the *negative temperature case* $\beta^N < 0$ according to the Onsager's observation [15]. We note that

$$Z_{\beta^N}(N) := \int_{\Lambda^N} e^{-\beta^N H^{N,\,\Omega}(x_1,\cdots,x_N)}$$

is called *the partition function* at β^N.

In the standard equilibrium statistical mechanics, the Gibbs distribution function represents the probability density function of finding the N vortices in the position $x_1, \cdots, x_N$ in *the equilibrium state* with the inverse of temperature of the system is β^N, see [5, 11] for example. We note that the meaning(definition) of *equilibrium* might need some discussion, but we start here, that is, we assume that the equilibrium is something determined by $\mu^{N,\,\Omega}$ and we consider the limit of $\mu^{N,\,\Omega}$ as $N \longrightarrow \infty$.

From $\mu^{N,\,\Omega}$, we are able to get a family of probability measures on Λ^j for $N \geq j$ as follows:

$$P_j^{N,\,\Omega}(x_1,\cdots,x_j) := \int_{\Lambda^{N-j}} \mu^{N,\,\Omega}(x_1,\cdots,x_j,x_{j+1},\cdots,x_N)dx_{j+1}\cdots dx_N,$$

which is called *the j-body distribution function* that represents the probability density function of finding the first j vortices in the position $x_1, \cdots, x_j$. Since $\overline{\Lambda}^j \subset \mathbf{R}^{2j}$ is compact for each $j \in \mathbf{N}$, we may extract a subsequence of $\{P_j^{N,\,\Omega}\}_{N=j}^{\infty}$ that converges weakly $*$ in $\mathcal{M}(\overline{\Lambda}^j) = C(\overline{\Lambda}^j)^*$. Using the diagonal argument, we are able to reach the following fact:

Proposition 2.1. *There exists a family $\{P_j^{\Omega}\}_{j\in\mathbf{N}}$ and a subsequence $\{\mu^{N_k,\,\Omega}\} \subset \{\mu^{N,\,\Omega}\}$ such that*

$$P_j^{N_k,\,\Omega} \longrightarrow P_j^{\Omega} \quad \textit{weakly} * \textit{in } \mathcal{M}(\overline{\Lambda}^j)$$

for each $j \in \mathbf{N}$.

In this situation, Caglioti et al. proved a structure theorem of P_j^{Ω} based on the argument of Messer-Spohn [14]. We note that the most important assumption to do that is

$$\Omega_1 = \Omega_2 = \cdots = \Omega_N,$$

that is, every vortex has the same intensity. Then the Hamiltonian is reduces to

$$H^{N,\,\Omega}(x_1,\cdots,x_N) = \Omega_1^2 \left\{ \frac{1}{2}\sum_{k=1}^{N} K(x_k,x_k) + \frac{1}{2}\sum_{1\leq k,l\leq N,\,k\neq l} G(x_k,x_l) \right\}$$

and is symmetric with respect to the permutation of $x_1, \cdots, x_N$. In this case, we omit Ω in $H^{N,\Omega}$, $P_j^{N,\,\Omega}$, etc. to simplify the presentation. In this case, we usually choose

$$\Omega_1 = \frac{1}{N}$$

in order to normalize the total vorticity of the system to 1.

Here we may assume that there exists a measure μ on the infinitely many direct sum space $\overline{\Lambda}^\infty$ of compact space $\overline{\Lambda}$ and $P_j = \mu|_{\Lambda^j}$. We say that a measure ξ on Λ^∞ is symmetric if $\xi|_{\Lambda^j}$ is symmetric with respect to the permutation of $x_1, \cdots, x_j$ for each $j \in \mathbf{N}$. In this sense, μ is symmetric and we are able to use the Hewitt-Savage representation theorem for symmetric measures on $\overline{\Lambda}^\infty$, see [7] for the precise definition of symmetric measures and the details of the theory. The conclusion arranged for our purpose is as follows:

Proposition 2.2 (cf. [14, Lemma 3]). *Let $\Lambda \subset \mathbf{R}^d$ be a bounded domain, $P^N(x_1, \cdots, x_N) \in L^1(\Lambda^N) \subset \mathcal{M}(\overline{\Lambda}^N)$ be symmetric with respect to the permutation of $x_1, \cdots, x_N$. Suppose further that there exists $P_j \in \mathcal{M}(\overline{\Lambda}^j)$ such that*

$$P_j^{N_k} := \int_{\Lambda^{N_k - j}} P^{N_k} dx_{j+1} \cdots dx_{N_k} \longrightarrow P_j \tag{2.4}$$

weakly $*$ in $\mathcal{M}(\overline{\Lambda}^j)$ for each $j \in \mathbf{N}$. Then there exists a probability measure ν on $\mathcal{M}(\overline{\Lambda})$ independent of $j \in \mathbf{N}$ such that

$$P_j = \int_{\mathcal{M}(\overline{\Lambda})} \nu(d\rho)\rho(dx_1) \otimes \cdots \otimes \rho(dx_j) \left(=: \int_{\mathcal{M}(\overline{\Lambda})} \nu(d\rho)\rho^{\otimes j} \right)$$

for every $j \in \mathbf{N}$.

Then the problem reduces to know the structure of ν. We note that even for $j = 1$, the conclusion says that $P_1 = \int_{\mathcal{M}(\overline{\Lambda})} \nu(d\rho)\rho(dx_1)$. We also note that if

$$\nu \text{ is supported on } L^1(\Lambda) \subset \mathcal{M}(\overline{\Lambda}), \tag{2.5}$$

then $P_j \in L^1(\Lambda^j)$ and

$$P_j^N \longrightarrow P_j \quad \text{weakly in } L^1(\Lambda^j).$$

The most part of the efforts of Messer-Spohn, Caglioti et al., and Kiessling are devoted to establish (2.5) or, in other words, the absolutely continuity of P_j with respect to the Lebesgue measure. Actually they even show that $P_j \in L^\infty(\Lambda^j)$ and

$$P_j^N \longrightarrow P_j \quad \text{weakly in } L^p(\Lambda^j) \tag{2.6}$$

for every $p \in (1, \infty)$, see Proposition 3.3 and Proposition 4.3.

We end this section with some comments on our assumption of symmetry. One may feel that it seems too restrictive to assume that all the intensities are same. Indeed, they do not necessarily identical in a classical fluid. However, the intensities of vortices are *quantized* in a *quantum* fluid and it is natural to assume that they are identical, which is mentioned by Onsager [15]. We also note that the vortex system naturally appears in the theory of plasma confinement. In this case, the intensities are determined by the charges and masses of the particles in plasma. Therefore the same intensities occurs when a plasma consists of same particles. The non-neutral plasma represented by the pure electron plasma is an example of the case, see [4] for example.

§ 3. Messer-Spohn theory, revisited

Here we review the mean field theory of Messer-Spohn, which is the basic tool for Caglioti et al. and Kiessling. In this section, Λ is a bounded domain of $\mathbf{R}^d$ ($d \in \mathbf{N}$).

Messer-Spohn considered the canonical distribution of N classical point particles in Λ. The particles are assumed to follow the Hamiltonian

$$H^N(x_1, \cdots, x_N) = \frac{1}{2N^2} \sum_{1 \le k,l \le N,\, k \ne l} V(x_k, x_l),$$

where V is a function on $\Lambda \times \Lambda$ satisfying the following properties:

Condition (V1) (symmetric) $V(x,y) = V(y,x)$,

Condition (V2) (Lipschitz continuous) there exists $L > 0$ such that

$$|V(x,y) - V(x',y')| \le L(|x - x'| + |y - y'|) \quad \text{for all } x,\, x',\, y,\, y' \in \Lambda.$$

Since Λ is bounded, V is bounded on $\Lambda \times \Lambda$ from the above property 2.

Here we note again that H^N is symmetric with respect to the permutation of $x_1, \cdots, x_N$, which is the essential assumption in the Messer-Spohn theory. They consider the limit of the Gibbs distribution function for this H^N as $N \longrightarrow \infty$ taking the inverse temperature $b^N = N$.[1] We may assume μ_j^N weakly $*$ converges to $P_j = \int_{\mathcal{M}(\overline{\Lambda})} \nu(d\rho)\rho^{\otimes j} \in \mathcal{M}(\overline{\Lambda}^j)$ from Proposition 2.2. We study the structure of the probability measure ν on $\mathcal{M}(\overline{\Lambda})$.

In order to do this, we recall the variational principle for the Gibbs distribution function. Let us define the function space

$$P_{L\log L}(\Lambda^N) := \left\{ \mu \in L^1\left(\Lambda^N\right) \mid \mu \ge 0 \text{ a.e.}, \int_{\Lambda^N} \mu = 1,\ \int_{\Lambda^N} \mu \log \mu < \infty \right\}^2$$

and the quantities

$$U := \int_{\Lambda^N} H^N \mu, \quad , \quad S := -\int_{\Lambda^N} \mu \log \mu,$$

[1] In the Messer-Spohn paper [14], $H^N(x_1, \cdots, x_N) = \frac{1}{2N} \sum_{1 \le j,k \le N,\, j \ne k} V(x_j, x_k)$ and $\beta^N = 1$. We slightly change the setting to suit the case of vortices.

[2] In this paper, we set $0 \log 0 = 0$ and assume that the function $t \log t$ is defined for $t \ge 0$.

and

$$F^N_{\beta^N}(\mu) := U - TS = \int_{\Lambda^N} H^N \mu + \frac{1}{\beta^N} \int_{\Lambda^N} \mu \log \mu$$

which are called *the inertial energy*, *the entropy*, and *the Helmholtz free energy* of the state μ. We recall that β^N is the inverse of the temperature T. Since V is bounded, the functionals U, S, and $F^N_{\beta^N}$ are well defined over $P_{L\log L}(\Lambda^N)$.

We note that $F^N(\mu)$ is bounded below and convex, we are able to see the following fact from the standard argument of calculus of variations:

Proposition 3.1. *The variational problem*

$$\inf_{\mu \in P_{L \log L}(\Lambda^N)} F^N_{\beta^N}(\mu), \tag{3.1}$$

is attained by the Gibbs distribution function

$$\mu^N(x_1, \cdots, x_N) := \frac{e^{-\beta^N H^N(x_1,\cdots,x_N)}}{\int_{\Lambda^N} e^{-\beta^N H^N(x_1,\cdots,x_N)}}. \tag{3.2}$$

Here we assume (2.4) and the conclusion of Hewitt-Savage representation theorem (Proposition 2.2) for a while.

From the symmetry of H^N and μ^N, it holds that

$$U(\mu^N) = \int_{\Lambda^N} H^N \mu^N = \frac{N-1}{2N} \int_{\Lambda \times \Lambda} V(x_1, x_2) P^N_2(x_1, x_2) dx_1 dx_2$$

and consequently

$$\begin{aligned} \lim_{N \longrightarrow \infty} U(\mu^N) &= \frac{1}{2} \int_{\Lambda \times \Lambda} V(x_1, x_2) P_2(dx_1 \otimes dx_2) \\ &= \int_{\mathcal{M}(\overline{\Lambda})} \nu(d\rho) < \frac{1}{2} V(x_1, x_2), \rho(dx_1) \otimes \rho(dx_2) > \\ &=: u(\nu) \quad \text{(the mean inertial energy)} \end{aligned}$$

since V is bounded and Lipschitz continuous in $\Lambda \times \Lambda$.

On the other hand, the entropy functional satisfies the following sub-additivity property[34]:

$$S(\mu) \le S(\mu_j) + S(\mu_{,N-j}),$$

where

$$\mu_j(x_1, \cdots, x_j) = \int_{\Lambda^{N-j}} \mu(x_1, \cdots, x_N) dx_{j+1} \cdots dx_N,$$

$$\mu_{,N-j}(x_{j+1}, \cdots, x_N) = \int_{\Lambda^j} \mu(x_1, \cdots, x_N) dx_1 \cdots dx_j$$

[3] Use $\log t \le t - 1$.

[4] We note that $S(\mu_{,N-j}) = S(\mu_{N-j})$ holds from the symmetry of μ with respect to the permutation of $x_1, \cdots, x_N$ and the notation $\mu_{,N-j}$ might not be a standard one. The author, however, think that it might be helpful for readers to distinguish μ_{N-j} and $\mu_{,N-j}$.

for $\mu \in P_{L\log L}(\Lambda^N)$, that is,

$$-\int_{\Lambda^N} \mu \log \mu \le -\int_{\Lambda^j} \mu_j \log \mu_j - \int_{\Lambda^{N-j}} \mu_{,N-j} \log \mu_{,N-j}.$$

Especially for the Gibbs distribution function μ^N, we have

$$S(\mu^N) \le S(\mu_1^N) + S(\mu_{,N-1}^N) \le \cdots \le S(\mu_1^N) + \cdots + S(\mu_1^N) = NS(P_1^N)$$

from the symmetry of μ^N, where $P_1^N(x_1) \equiv \mu_1^N(x_1)$ is the 1-body distribution function for μ^N.

Here we define

$$s(\nu) := \limsup_{N \longrightarrow \infty} \frac{1}{N} S(\mu^N) \quad \text{(the mean entropy)}$$

and

$$f_1(\nu) := u(\nu) - s(\nu) = \liminf_{N \longrightarrow \infty} F_{\beta^N}^N(\mu^N) \quad \text{(the mean free energy).}^5$$

We note that we assumed $\beta^N = N$. Obviously we have the following estimate:

$$s(\nu) \le \limsup_{N \longrightarrow \infty} S(P_1^N) (\le e|\Lambda|).$$

Here we further assume that (2.6) holds for some $p > 1$ and $\nu = \delta_{P_1}$. Then, since $S(\rho)$ is concave and weakly upper semi-continuous in $L^p(\Lambda)$, we get

$$\limsup_{N \longrightarrow \infty} S(P_1^N) \le S(P_1)$$

and

$$f_1(\nu) \ge \frac{1}{2} \int_{\Lambda \times \Lambda} V(x_1, x_2) P_1(x_1) P_1(x_2) dx_1 dx_2 + \int_{\Lambda} P_1 \log P_1 =: F_1(P_1) \tag{3.3}$$

On the other hand, for every $\widetilde{\rho} \in P_{L\log L}(\Lambda)$, the product $\widetilde{\rho}(x_1) \cdots \widetilde{\rho}(x_N)$ belongs to $P_{L\log L}(\Lambda^N)$ and we get, since $\beta^N = N$,

$$F_{\beta^N}^N(\widetilde{\rho}(x_1) \cdots \widetilde{\rho}(x_N)) = \frac{N-1}{2N} \int_{\Lambda \times \Lambda} V(x_1, x_2) \widetilde{\rho}(x_1) \widetilde{\rho}(x_2) dx_1 dx_2 - S(\widetilde{\rho}) \ge F_{\beta^N}^N(\mu^N)$$

from Proposition 3.1. Consequently we get

$$F_1(\widetilde{\rho}) \ge F_1(P_1) \quad \text{for every } \widetilde{\rho} \in P_{L\log L}(\Lambda^N) \tag{3.4}$$

at the limit $N \longrightarrow \infty$. This means that P_1 attains the variational problem

$$\inf_{\rho \in P_{L\log L}(\Lambda^N)} F_1(\rho). \tag{3.5}$$

[5] The suffix 1 of f_1 means that the corresponding inverse temperature is 1.

If we assume only (2.6) holds for some $p > 1$, we get

$$f_1(\nu) \geq \int_{\mathcal{M}(\overline{\Lambda})} F_1(\rho)\nu(d\rho)$$

instead of (3.3) and

$$F_1(\widetilde{\rho}) = \int_{\mathcal{M}(\overline{\Lambda})} F_1(\widetilde{\rho})\nu(d\rho) \geq \int_{\mathcal{M}(\overline{\Lambda})} F_1(\rho)\nu(d\rho) \quad \text{for every } \widetilde{\rho} \in P_{L\log L}(\Lambda^N).$$

instead of (3.4). Consequently ν is supported on the solutions of (3.5).

Now we state the main conclusion of Messer-Spohn.

Theorem 3.2 ([14, Theorem 2]). *The measure ν that appears in the limit of P_j^N as $N \longrightarrow \infty$ with $\beta^N = N$ is supported on the minimizer of* (3.5).

As the Euler-Lagrange equation of the variational problem (3.5), we get *the mean field equation*:

$$\rho(x) = \frac{e^{-\int_\Lambda V(x,y)\rho(y)dy}}{\int_\Lambda e^{-\int_\Lambda V(x,y)\rho(y)dy}dx}.$$

We note that when the minimizer of (3.5) is unique, then ν is supported only on the minimizer, but the weak limit P_1 might be a mixture of minimizers, in general. The uniqueness of the minimizer is also discussed in [14], see also [2].

The final task to get Theorem 3.2 is to show (2.5) for some $p > 1$. In order to do so, the following fact is sufficient:

Proposition 3.3 ([14, Lemma 1]). *Suppose $\beta^N = N$. Then for each $N \geq 2$ and $j \in \{1, \cdots, N-1\}$, it holds that*

$$0 \leq P_j^N(x_1, \cdots, x_j) \leq |\Lambda|^{-j} e^{2jM} e^{-\frac{j^2}{N}H^j(x_1,\cdots,x_j)}$$

for every $(x_1, \cdots, x_j) \in \Lambda^N$, where $M = \sup_{\Lambda\times\Lambda} |V(x,y)|$.

Obviously this gives that $\{P_j^N\}$ is bounded in $L^\infty(\Lambda)$, from which (2.6) for any $p \in (1, \infty)$ follows.

Proof. We divide the Hamiltonian H^N into three parts:

$$\begin{aligned} H^N &= \frac{1}{2N^2} \sum_{1\leq k,l\leq N,\, k\neq l} V(x_k, x_l) \\ &= \frac{j^2}{N^2} H^j(x_1, \cdots, x_j) + \frac{1}{N^2} W^{j,N-j}(x_1, \cdots, x_N) + \frac{(N-j)^2}{N^2} H^{N-j}(x_{j+1}, \cdots, x_N), \end{aligned}$$

where

$$W^{j,N-j}(x_1,\cdots,x_N)=\sum_{k=1}^{j}\sum_{l=j+1}^{N}V(x_k,x_l).$$

Then we get

$$\begin{aligned}\int_{\Lambda^{N-j}}e^{-\beta^N H^N}dx_{j+1}\cdots dx_N&=\int_{\Lambda^{N-j}}e^{-\frac{1}{2N}\sum_{1\le k,l\le N,\,k\ne l}V(x_k,x_l)}dx_{j+1}\cdots dx_N\\&\le e^{-\frac{j^2}{N}H^j}e^{\frac{j(N-j)}{N}M}\int_{\Lambda^{N-j}}e^{-\frac{(N-j)^2}{N}H^{N-j}}dx_{j+1}\cdots dx_N\\&\le e^{-\frac{j^2}{N}H^j}e^{jM}Z_{\frac{(N-j)^2}{N}}(N-j).\end{aligned}$$

On the other hand, from the Jensen inequality, we get

$$\begin{aligned}&\frac{\int_{\Lambda^N}e^{-\beta^N H^N}dx_1\cdots dx_N}{|\Lambda|^j Z_{\frac{(N-j)^2}{N}}(N-j)}\\&\ge\exp\left\{-|\Lambda|^{-j}Z_{\frac{(N-j)^2}{N}}(N-j)^{-1}\int_{\Lambda^N}(\frac{j^2}{N}H^j+\frac{1}{N}W^{j,N-j})e^{-\frac{(N-j)^2}{N}H^{N-j}}\right\}\\&\ge\exp(-jM).\end{aligned}$$

□

§ 4. On the case of vortices

Even if we assume that the all the intensities are equivalent, the Messer-Spohn theory is not directly applicable to the Hamiltonian of vortices (2.3). Indeed, the Green function $G(x,y)$ is symmetric but has logarithmic singularity at $x=y$ and there exist a potential term $\sum K(x_k,x_k)$, which is also singular (divergent to $-\infty$) as $x_k\longrightarrow\partial\Lambda$. We also note that we are interested in *the negative temperature* case $\beta^N<0$.

To simplify the presentation, we only review more complicated case $\beta^N<0$.

We have to start with the variational problem (3.1). Indeed, since H^N has singularity, it is not clear whether U and F_{β^N} is well defined on $P_{L\log L}(\Lambda^N)$.

We recall the elementary Young inequality

$$xy\le e^x+y\log y-y\quad\text{for } x\in\mathbf{R}\text{ and } y\ge 0.$$

(We set $0\log 0=0$ in this paper.) Using this inequality, we get

$$\int_{\Lambda^N}H^N\mu\le\int_{\Lambda^N}e^{-\beta^N H^N}-\frac{1}{\beta^N}\int_{\Lambda^N}\mu\log\mu+\frac{\log(-\beta^N)}{\beta^N}+\frac{1}{\beta^N}$$

for $\beta^N < 0$, that is, the inertial energy U and the free energy $F^N_{\beta^N}$ are well-defined if the partition function $Z_{\beta^N}(N) = \int_{\Lambda^N} e^{-\beta^N H^N}$ is finite. Therefore we have to start with the estimate of the partition function $Z_{\beta^N}(N)$.

Proposition 4.1 ([2, Lemma 2.1],[10, Lemma 1])**.** *Suppose that* $\beta^N = \beta N$ *and* $\beta \in (-8\pi, 0)$*. Then there exists a constant* $C = C(\beta, \Lambda)$ *independent of* N *such that*

$$Z_{\beta^N}(N) \leq C^N.$$

Proof. We know that $K(x, y)$ is bounded from the above on $\Lambda \times \Lambda$. It holds that

$$-\beta^N \cdot \frac{1}{N^2} \sum_{k=1}^{N} K(x_k, x_k) \leq -\beta \sup_{(x,y)\in\Lambda\times\Lambda} K(x, x) =: C_0.$$

On the other hand, using the Hölder inequality, we get

$$\begin{aligned}
&\int_{\Lambda^N} e^{-\beta^N H^N} \leq e^{C_0} \int_{\Lambda^N} \prod_{k=1}^{N} \prod_{l\neq k} e^{-\frac{\beta}{2N} G(x_k,x_l)} dx_1 \cdots dx_N \\
&\leq e^{C_0} \prod_{k=1}^{N} \left(\int_{\Lambda^N} \prod_{l\neq k} e^{-\frac{\beta}{2} G(x_k,x_l)} dx_1 \cdots dx_N \right)^{\frac{1}{N}} \\
&= e^{C_0} \int_{\Lambda} \left(\int_{\Lambda} e^{-\frac{\beta}{2} G(x_1,x_2)} dx_2 \right)^{N-1} dx_1 \leq e^{C_0 N} \int_{\Lambda} \left(\int_{\Lambda} |x_1 - x_2|^{\frac{\beta}{4\pi}} dx_2 \right)^{N-1} dx_1.
\end{aligned}$$

Therefore the conclusion follows if $\beta \in (-8\pi, 0)$. □

Thanks to Proposition 4.1, we see that

$$\sup_{\mu\in P_{L\log L}} F^N_{\beta^N}(\mu) < \infty \quad \text{if } \beta^N = \beta N \text{ for } \beta \in (-8\pi, 0).^6 \tag{4.1}$$

Similar to Proposition 3.1, we get the following fact:

Proposition 4.2. *The variational problem* (4.1) *for* $\beta \in (-8\pi, 0)$ *is attained by the Gibbs distribution function* (3.2)*.*

The another point that we have to take care is the definition of the mean inertial energy. For the Hamiltonian of vortices, it holds that

$$\begin{aligned}
U(\mu^N) &= \int_{\Lambda^N} H^N \mu^N \\
&= \frac{N-1}{2N} \int_{\Lambda\times\Lambda} G(x_1, x_2) P_2^N(x_1, x_2) dx_1 dx_2 + \frac{1}{2N} \int_{\Lambda} K(x_1, x_1) P_1^N(x_1) dx_1.
\end{aligned}$$

[6] It seems more natural to use *the Massieu function* $\Psi_\beta = S - \beta U = -\beta F_\beta$. Indeed both variational problems (3.1) and (4.1) become $\sup \Psi_\beta$. See [11], for example, for the Massieu function and other thermodynamic functions.

Since $G(x,y)$ and $K(x,x)$ have singularities, we are not able to handle this with the weak convergence of measures. Here we note that the coefficient of the latter term tends to 0 as $N \longrightarrow \infty$, which implies that $K(x_1,x_1)$ might be negligible at the limit.

First we prepare the similar results of Proposition 3.3.

Proposition 4.3 ([2, Theorem 3.1],[10, Lemma 4], see also [12, Theorem 3.2]). *For each $N \geq 2$, $j \in \{1,\cdots,N-1\}$, $\beta^N = \beta N$ for $\beta \in (-8\pi, 0)$, there exists a constant $C = C(\beta,\Lambda,j)$ independent of N such that*

$$0 \leq P_j^N(x_1,\cdots,x_j) \leq Ce^{-\frac{j^2}{N}H^j(x_1,\cdots,x_j)}$$

for every $(x_1,\cdots,x_j) \in \Lambda^N$.

Thanks to this estimate, we get $P_j \in L^\infty\left(\Lambda^j\right)$ and (2.6) for each $p \in (1,\infty)$ because $\frac{j^2}{N}H^j \longrightarrow 0$ almost everywhere. On the other hand, the singularities of $G(x,y)$ and $K(x,x)$ are logarithmic and they belong to $\left(L^p\left(\Lambda^2\right)\right)^*$ and $L^p\left(\Lambda\right)^*$ respectively. Consequently we are able to follow the argument of Messer-Spohn and finally we reach the following conclusion:

Theorem 4.4 ([2, Theorem 2.1]). *The measure ν that appears in the weak limit of P_j^N as $N \longrightarrow \infty$ with $\beta^N = N\beta$ for $\beta \in (-8\pi,0)$ is supported on the maximizer of*

$$\sup_{\rho \in P\log P(\Lambda)} F_\beta(\rho), \tag{4.2}$$

where

$$F_\beta(\rho) = \frac{1}{2}\int_{\Lambda\times\Lambda} G(x_1,x_2)\rho(x_1)\rho(x_2)dx_1dx_2 + \frac{1}{\beta}\int_\Lambda \rho\log\rho.$$

We note that the potential $K(x,x)$ does not affect on the limit $N \longrightarrow \infty$. We also note that we get *the mean field equation* as the Euler-Lagrange equation of the variational problem (4.2):

$$\rho(x) = \frac{e^{-\beta\int_\Lambda G(x,y)\rho(y)dy}}{\int_\Lambda e^{-\beta\int_\Lambda G(x,y)\rho(y)dy}dx}.$$

Proof of Proposition 4.3. Similar to the proof of Proposition 3.3, we divide the summation in the Hamiltonian H^N into three parts.

$$H^N = \frac{j^2}{N^2}H^j + \frac{1}{N^2}W^{j,N-j} + \frac{(N-j)^2}{N^2}H^{N-j}.$$

Then it holds that

$$\begin{aligned}(4.3)\qquad &\int_{\Lambda^{N-j}} e^{-\beta^N H^N} dx_{j+1}\cdots dx_N \\ &\le e^{-\frac{j^2\beta}{N}H^j}\int_{\Lambda^{N-j}} e^{-\frac{\beta}{N}W^{j,N-j}} e^{-\frac{(N-j)^2\beta}{N}H^{N-j}} dx_{j+1}\cdots dx_N \\ (4.4)\qquad &\le e^{-\frac{j^2\beta}{N}H^j}\left(\int_{\Lambda^{N-j}} e^{-\frac{p\beta}{N}W^{j,N-j}}\right)^{\frac{1}{p}}\left(\int_{\Lambda^{N-j}} e^{-\frac{p'(N-j)^2\beta}{N}H^{N-j}}\right)^{\frac{1}{p'}}\end{aligned}$$

for $p, p' \in (1,\infty)$ satisfying $\frac{1}{p}+\frac{1}{p'}=1$, which we choose later.

Similar to Proposition 4.1, we get

$$\begin{aligned}\int_{\Lambda^{N-j}} e^{-\frac{p\beta}{N}W^{j,N-j}} &\le e^{\frac{pj(N-j)}{N}C_0}\int_{\Lambda^{N-j}}\prod_{k=1}^{j}\prod_{l=j+1}^{N}|x_k-x_l|^{\frac{p\beta}{2\pi N}} \\ &\le e^{\frac{pj(N-j)}{N}C_0}\prod_{k=1}^{j}\left(\int_{\Lambda^{N-j}}\prod_{l=j+1}^{N}|x_k-x_l|^{\frac{p\beta j}{2\pi N}}\right)^{\frac{1}{j}} \\ &= e^{\frac{pj(N-j)}{N}C_0}\prod_{k=1}^{j}\left(\int_{\Lambda}|x_k-y|^{\frac{p\beta j}{2\pi N}}dy\right)^{\frac{N-j}{j}}\end{aligned}$$

since $\beta < 0$. Here we choose

$$p := \frac{N}{2j}$$

for $N > 2j$. Then we have

$$\frac{p\beta j}{2\pi N} = \frac{\beta}{4\pi} > -2$$

for $\beta \in (-8\pi, 0)$ and consequently we get

$$(4.5)\qquad \int_{\Lambda^{N-j}} e^{-\frac{p\beta}{N}W^{j,N-j}} \le C^{N-j}$$

for some positive constant $C = C(\beta,\Lambda)$.

On the other hand, when $p = \frac{N}{2j}$, it holds that $p' = \frac{N}{N-2j}$. Here we set

$$\frac{p'(N-j)^2\beta}{N} = \frac{(N-j)}{N-2j}\beta(N-j) =: \beta'(N-j)$$

Then $\beta' \uparrow \beta$ as $N \longrightarrow \infty$ and $\beta' \in (-8\pi, 0)$ for sufficiently large N. Therefore we get

$$\int_{\Lambda^{N-j}} e^{-\frac{p'(N-j)^2\beta}{N}H^{N-j}} = Z_{\beta'(N-j)}(N-j) < \infty$$

from Proposition 4.1. More precisely, since $G(x,y) \geq 0$ and $K(x,x)$ is not bounded below, it holds that

$$\begin{aligned} Z_{\beta'(N-j)}(N-j) \int_{\Lambda^j} e^{-\frac{\beta'}{2(N-j)}\sum_{k=1}^{j} K(x_k,x_k)} dx_1 \cdots dx_j \\ = \int_{\Lambda^N} e^{-\beta'(N-j)H^{N-j}(x_{j+1},\cdots,x_N) - \frac{\beta'}{2(N-j)}\sum_{k=1}^{j} K(x_k,x_k)} dx_1 \cdots dx_N \\ \leq \int_{\Lambda^N} e^{-\frac{N\beta'}{N-j}NH^N} dx_1 \cdots dx_N = Z_{\widetilde{\beta}N}(N), \end{aligned}$$

where

$$\widetilde{\beta} := \frac{N\beta'}{N-j} = \frac{N\beta}{N-2j} = p'\beta,$$

that is,

$$Z_{\beta'(N-j)}(N-j) \leq C^j Z_{\widetilde{\beta}N}(N) \tag{4.6}$$

for some C independent of N satisfying

$$\int_{\Lambda} e^{-\frac{\beta'}{2(N-j)}K(x_1,x_1)} dx_1 = \int_{\Lambda} e^{-\frac{\widetilde{\beta}}{2N}K(x_1,x_1)} dx_1 \geq C^{-1}. \tag{4.7}$$

We note that this is possible because $-\frac{\widetilde{\beta}}{2N}K(x_1,x_1) \longrightarrow 0$ locally uniformly in Λ as $N \longrightarrow \infty$.

Combing (4.4), (4.5), and (4.6), we get

$$\begin{aligned} P_j^N(x_1,\cdots,x_j) &\leq Z_{\beta N}(N)^{-1} e^{-\frac{j^2\beta}{N}H^j} C^{\frac{N-j}{p}} C^{\frac{j}{p'}} Z_{\widetilde{\beta}N}(N)^{\frac{1}{p'}} \\ &\leq C^{3j} e^{-\frac{j^2\beta}{N}H^j} Z_{\beta N}(N)^{-1} Z_{p'\beta N}(N)^{\frac{1}{p'}}. \end{aligned}$$

We note that

$$Z_{p'\beta N}(N)^{\frac{1}{p'}} = \|e^{-\beta N H^N}\|_{p'}$$

and $p' = \frac{N}{N-2j} \downarrow 1$ as $N \longrightarrow \infty$. Therefore taking $q > p'$ independent of N, we get

$$\begin{aligned} Z_{p'\beta N}(N)^{\frac{1}{p'}} = \|e^{-\beta N H^N}\|_{L^{p'}(\Lambda^N)} &\leq \|e^{-\beta N H^N}\|_{L^1(\Lambda^N)}^{1-\theta} \|e^{-\beta N H^N}\|_{L^q(\Lambda^N)}^{\theta} \\ &= Z_{\beta N}(N)^{1-\theta} Z_{q\beta N}(N)^{\frac{\theta}{q}} \end{aligned}$$

for θ satisfying $1-\theta+\frac{\theta}{q} = \frac{1}{p'} = 1-\frac{2j}{N}$, that is, $\theta = \frac{q}{q-1}\cdot\frac{2j}{N}$. Consequently we have

$$Z_{\beta N}(N)^{-1} Z_{p'\beta N}(N)^{\frac{1}{p'}} \leq Z_{\beta N}(N)^{-\theta} Z_{q\beta N}(N)^{\frac{\theta}{q}} = Z_{\beta N}(N)^{-\theta} Z_{q\beta N}(N)^{\frac{2j}{N(q-1)}}.$$

Since we are able to choose $q > p' > 1$ sufficiently close to 1 such that $q\beta \in (-8\pi, 0)$, we are able to use Proposition 4.1 and get

$$Z_{q\beta N}(N)^{\frac{1}{N}} \leq C$$

for some constant $C > 0$ independent of N. On the other hand, since $G(x, y) \geq 0$ and $\beta < 0$, we get

$$\begin{aligned} Z_{\beta N}(N) &= \int_{\Lambda^N} e^{-\beta N H^N} \\ &\geq \int_{\Lambda^N} e^{-\frac{\beta}{2N}\sum_{k=1}^N K(x_k, x_k)} = \left(\int_\Lambda e^{-\frac{\beta}{2N} K(x_1, x_1)} dx_1 \right)^N \geq C^N, \end{aligned}$$

that is,

$$Z_{\beta N}(N)^{-\theta} \leq C^{-N\theta} = C^{\frac{2jq}{q-1}}$$

for some constant $C > 0$ independent of N as in (4.7). Then the conclusion follows. □

§ 5. On the case of vortices in an external field

In this final section, we confirm that the Messer-Spohn argument is applicable even if we slightly perturbed the Hamiltonian of vortices as follows:

$$\begin{aligned} H_c^N &:= H^N + \frac{c}{N} \sum_{k=1}^N \varphi(x_k) \\ &= \frac{1}{2N^2} \left\{ \sum_{k=1}^N K(x_k, x_k) + \sum_{1 \leq k, l \leq N,\, k \neq l} G(x_k, x_l) \right\} + \frac{c}{N} \sum_{k=1}^N \varphi(x_k), \end{aligned}$$

where $\varphi \in C(\overline{\Lambda})$ represents the profile of the background field that every particle (vortex) interacts with and $c \in \mathbf{R}$ is the perturbation parameter. We are interested in the asymptotic behavior of the corresponding Gibbs distribution μ_c^N as $c \longrightarrow 0$ for large N, which is the target of *the linear response theory.* The first step to establish the theory is to know *the mean field equation*, which we want to do in this paper. Since the perturbed Hamiltonian is also symmetric under the permutation of $(x_1, \cdots, x_N)$, the Messer-Spohn argument is applicable.

In the following, the perturbation of several concepts such as the Gibbs distribution function, the partition functions, and the j-body distribution function will be expressed with the suffix c such as μ_c^N, $Z_{c,\beta}(N)$, $P_{c,j}^N$, etc.

We start from the following estimates:

Proposition 5.1 (cf. Proposition 4.1)**.** *For each $\varphi \in C(\overline{\Lambda})$, $\bar{c} > 0$, $\beta^N = \beta N$, and $\beta \in (-8\pi, 0)$, there exists a constant $C = C(\beta, \Lambda, \|\varphi\|_{C(\overline{\Lambda})}, \bar{c})$ independent of N and $c \in [-\bar{c}, \bar{c}]$ such that*

$$Z_{c,\beta^N}(N) \leq C^N.$$

Proof. From the definition of H_c^N, it holds that

$$\begin{aligned} Z_{c,\beta^N}(N) &= \int_{\Lambda^N} e^{-\beta^N H_c^N} = \int_{\Lambda^N} e^{-\beta^N H^N - \beta c \sum_{j=1}^N \varphi(x_j)} \\ &\le e^{-\beta \bar{c} N \|\varphi\|_{C(\overline{\Lambda})}} Z_{\beta^N}(N). \end{aligned}$$

Then the conclusion follows from Proposition 4.1. □

Thanks to Proposition 5.1, it holds that

$$\sup_{\mu \in P_{L \log L}} F_{c,\beta^N}^N(\mu) < \infty \quad \text{if } \beta^N = \beta N \text{ with } \beta \in (-8\pi, 0) \tag{5.1}$$

and the following fact:

Proposition 5.2 (cf. Proposition 3.1). *The variational problem* (5.1) *for* $\beta^N = \beta N$ *with* $\beta \in (-8\pi, 0)$ *is attained by the perturbed Gibbs distribution function*

$$\mu_c^N(x_1, \cdots, x_N) := \frac{e^{-\beta^N H_c^N(x_1,\cdots,x_N)}}{\int_{\Lambda^N} e^{-\beta^N H_c^N(x_1,\cdots,x_N)}}.$$

For the perturbed Hamiltonian, it holds that

$$\begin{aligned} U_c(\mu^N) &:= \int_{\Lambda^N} H_c^N \mu^N \\ &= \frac{N-1}{2N} \int_{\Lambda\times\Lambda} G(x_1,x_2) P_2^N(x_1,x_2) dx_1 dx_2 + \frac{1}{2N} \int_\Lambda K(x_1,x_1) P_1^N(x_1) dx_1 \\ &\quad + c \int_\Lambda \varphi(x_1) P_1^N(x_1) dx_1, \end{aligned}$$

from which we are able to see that φ would survive in the mean field limit $N \longrightarrow \infty$. Naturally we define

$$F_{c,\beta}(\rho) = \frac{1}{2} \int_{\Lambda\times\Lambda} G(x_1,x_2)\rho(x_1)\rho(x_2) dx_1 dx_2 + c \int_\Lambda \varphi(x_1)\rho(x_1) dx_1 + \frac{1}{\beta} \int_\Lambda \rho \log \rho$$

and get the following conclusion:

Theorem 5.3 (cf. Theorem 3.2). *For each* $\varphi \in C(\overline{\Lambda})$ *and* $c \in \mathbf{R}$, *the measure* ν *that appears in the weak limit of* $P_{c,j}^N$ *as* $N \longrightarrow \infty$ *with* $\beta^N = N\beta$ *for* $\beta \in (-8\pi, 0)$ *is supported on the maximizer of*

$$\sup_{\rho \in P \log P(\Lambda)} F_{c,\beta}(\rho). \tag{5.2}$$

From this fact, we reach our main purpose, *the mean field equation*, as the Euler-Lagrange equation of the variational problem (5.2):

$$\rho(x) = \frac{e^{-\beta\{\int_\Lambda G(x,y)\rho(y)dy + c\varphi(x)\}}}{\int_\Lambda e^{-\beta\{\int_\Lambda G(x,y)\rho(y)dy + c\varphi(x)\}} dx}. \tag{5.3}$$

All what we have to do to prove Theorem 5.3 is to show the following similar estimate to Proposition 4.3. Then we get (2.6) for $P^N_{c,j}$ and the Messer-Spohn argument works as we observed in the previous sections.

Proposition 5.4 (cf. Proposition 4.3). *For each $N \geq 2$, $j \in \{1, \cdots, N-1\}$, $\beta^N = \beta N$ with $\beta \in (-8\pi, 0)$, $\varphi \in C(\overline{\Lambda})$, and $\bar{c} > 0$, there exists a constant $C = C(\beta, \Lambda, j, \|\varphi\|_{C(\overline{\Lambda})}, \bar{c})$ independent of N and $c \in [-\bar{c}, \bar{c}]$ such that*

$$0 \leq P^N_{c,j}(x_1, \cdots, x_j) \leq Ce^{-\frac{j^2}{N} H^j_c(x_1, \cdots, x_j)}$$

for every $(x_1, \cdots, x_j) \in \Lambda^N$.

Proof. The proof is almost the same as Proposition 4.3 but we have to take care for the coefficient of $c\varphi(x_j)$ in H^N_c is not $\frac{1}{N^2}$ but $\frac{1}{N}$. Actually it holds that

$$\begin{aligned} H^N_c &= \frac{j^2}{N^2} H^j + \frac{1}{N^2} W^{j,N-j} + \frac{(N-j)^2}{N^2} H^{N-j} + \frac{c}{N} \sum_{k=1}^N \varphi(x_k) \\ &= \frac{j^2}{N^2} H^j_c + \frac{1}{N^2} W^{j,N-j} + \frac{(N-j)^2}{N^2} H^{N-j}_c \\ &\qquad + \left(\frac{c}{N} - \frac{jc}{N^2}\right) \sum_{k=1}^j \varphi(x_k) + \left(\frac{c}{N} - \frac{(N-j)c}{N^2}\right) \sum_{k=j+1}^N \varphi(x_k). \end{aligned}$$

Let

$$-\beta \|\varphi\|_{C(\overline{\Lambda})} =: C_1.$$

Then we get

$$\begin{aligned} &\int_{\Lambda^{N-j}} e^{-\beta^N H^N_c} dx_{j+1} \cdots dx_N \qquad (5.4) \\ &\quad \leq e^{2j\bar{c}C_1} e^{-\frac{j^2\beta}{N} H^j_c} \int_{\Lambda^{N-j}} e^{-\frac{\beta}{N} W^{j,N-j}} e^{-\frac{(N-j)^2\beta}{N} H^{N-j}_c} dx_{j+1} \cdots dx_N \\ &\quad \leq e^{2j\bar{c}C_1} e^{-\frac{j^2\beta}{N} H^j_c} \left(\int_{\Lambda^{N-j}} e^{-\frac{p\beta}{N} W^{j,N-j}} \right)^{\frac{1}{p}} \left(\int_{\Lambda^{N-j}} e^{-\frac{p'(N-j)^2\beta}{N} H^{N-j}_c} \right)^{\frac{1}{p'}} \qquad (5.5) \end{aligned}$$

for $p = \frac{N}{2j}$ and $p' = \frac{N}{N-2j}$. We note that we are able to use (4.5).

Here we also set $\frac{p'(N-j)^2\beta}{N} =: \beta'(N-j)$ and get

$$\int_{\Lambda^{N-j}} e^{-\frac{p'(N-j)^2\beta}{N} H^{N-j}_c} = Z_{c,\beta'(N-j)}(N-j) < \infty$$

from Proposition 5.1 since $\beta' \uparrow \beta \in (-8\pi, 0)$ as $N \longrightarrow \infty$.

Similar to the proof of Proposition 4.3, it holds that

$$
\begin{aligned}
&Z_{c,\beta'(N-j)}(N-j)\int_{\Lambda^j} e^{-\frac{\beta'}{2(N-j)}\sum_{k=1}^{j}K(x_k,x_k)-\beta' c\sum_{k=1}^{j}\varphi(x_k)}dx_1\cdots dx_j \\
&= \int_{\Lambda^N} e^{-\beta'(N-j)H_c^{N-j}(x_{j+1},\cdots,x_N)-\frac{\beta'}{2(N-j)}\sum_{k=1}^{j}K(x_k,x_k)-\beta' c\sum_{k=1}^{j}\varphi(x_k)}dx_1\cdots dx_N \\
&\le \int_{\Lambda^N} e^{-\frac{N\beta'}{N-j}NH^N-\beta' c\sum_{k=1}^{N}\varphi(x_k)}dx_1\cdots dx_N \\
&= \int_{\Lambda^N} e^{-\frac{N\beta'}{N-j}NH_c^N+\left(\frac{N\beta'}{N-j}-\beta'\right)c\sum_{k=1}^{N}\varphi(x_k)}dx_1\cdots dx_N \\
&= e^{-\frac{jN}{N-j}\beta'\bar{c}\|\varphi\|_{C(\overline{\Lambda})}}Z_{c,\widetilde{\beta}N}(N),
\end{aligned}
$$

where $\widetilde{\beta}=p'\beta$ as before and we are able to conclude (4.6) with a parameter c.

Consequently we get

$$
\begin{aligned}
P_{c,j}^N(x_1,\cdots,x_j) &\le C^{3j}e^{-\frac{j^2\beta}{N}H_c^j}Z_{c,\beta N}(N)^{-1}Z_{c,p'\beta N}(N)^{\frac{1}{p'}} \\
&\le Ce^{-\frac{j^2\beta}{N}H_c^j}Z_{c,\beta N}(N)^{-\theta}
\end{aligned}
$$

for some fixed $q>p'$ and $\theta=\frac{q}{q-1}\cdot\frac{2j}{N}$.

Finally we note that

$$
\begin{aligned}
Z_{c,\beta N}(N) &= \int_{\Lambda^N} e^{-\beta NH_c^N} \ge \int_{\Lambda^N} e^{-\frac{\beta}{2N}\sum_{k=1}^{N}K(x_k,x_k)-\beta c\sum_{k=1}^{N}\varphi(x_k)} \\
&= \left(\int_{\Lambda} e^{-\frac{\beta}{2N}K(x_1,x_1)-\beta c\varphi(x_1)}dx_1\right)^N \ge C^N
\end{aligned}
$$

for some constant $C>0$ independent of N. This guarantees the conclusion. □

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