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A survey on long range scattering for Schrödinger equation and Klein-Gordon equation with critical nonlinearity of non-polynomial type

By

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Abstract

We summarize recent progress on long range scattering for nonlinear Schrödinger equation and nonlinear Klein-Gordon equation. We introduce a technique of extracting a resonant part, which has the same oscillation speed as its argument, from non-polynomial nonlinearities, and exhibit its two applications. Firstly, we consider nonlinear Schrödinger equation with a general nonlinearity of the critical order, and investigate the relation between the shape of the nonlinearity and a typical asymptotic behavior of small solutions. Secondly, we consider nonlinear Klein-Gordon equation with a gauge-invariant nonlinearity, and find an asymptotic behavior for both real-valued case and complex-valued case. A slight improvement is seen in the second application.

§ 1. Introduction

In this survey, we consider large time behavior of solutions to nonlinear Schrödinger equation

(NLS) $$i\partial_t u + \Delta u = F(u), \quad (t,x) \in \mathbb{R}^{1+d}$$

and nonlinear Klein-Gordon equation

(NLKG) $$(\Box + 1)u = F(u), \quad (t,x) \in \mathbb{R}^{1+d},$$

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where $d = 1, 2, 3$, $\Delta = \sum_{j=1}^{d} \partial_{x_j}^2$, $\Box = \partial_t^2 - \Delta$, u is an unknown, and $F(u)$ is a nonlinearity. For (NLS), the unknown u is $\mathbb{C}$-valued. For (NLKG), it is $\mathbb{R}$-valued or $\mathbb{C}$-valued. Throughout the paper, we suppose that the nonlinearity F is homogeneous of the critical order, i.e.,

$$F(\lambda u) = \lambda^{1+\frac{2}{d}} F(u) \tag{1.1}$$

holds for all $u \in \mathbb{C}$ and $\lambda > 0$. Typical nonlinearities in our mind are a gauge-invariant nonlinearity

$$F_{\mathrm{GI}}(u) = |u|^{\frac{2}{d}} u \tag{1.2}$$

and a real-part nonlinearity

$$F_{\mathrm{Re}}(u) = |\operatorname{Re} u|^{\frac{2}{d}} \operatorname{Re} u. \tag{1.3}$$

It will turn out that the study of the latter nonlinearity $F_{\mathrm{Re}}(u)$ is one of the main topic of the present survey.

There is a number of literature on the scattering problem for nonlinear Schrödinger equation and nonlinear Klein-Gordon equation. Roughly speaking, we say a solution *scatters* as $t \to \infty$ if the solution asymptotically behaves like a free solution, a solution to the corresponding linear equation, as $t \to \infty$ in a reasonable topology. The scattering takes place when the nonlinear interaction becomes negligible for large time.

As for the nonlinear Schrödinger equation and nonlinear Klein-Gordon equation with a power type gauge-invariant nonlinearity $|u|^{p-1}u$, it is known that the power $p = 1 + \frac{2}{d}$ is a critical exponent in such a sense that the equation admits no nontrivial scattering solution if $1 < p \leqslant 1 + \frac{2}{d}$ ([1, 9, 45, 56]).

When we consider a homogeneous nonlinearity F of the critical order, it is known that a possible behavior depends on the shape of the nonlinearity. A class of small solutions to (NLS) and (NLKG) with the gauge-invariant nonlinearity F_{GI} behaves like a free solution plus a phase correction term ([6, 8, 11, 15, 50]). On the other hand, if the nonlinearity is not gauge-invariant there are other possibilities. In particular, the equation may admits a nontrivial asymptotically free solution for a class of nonlinearities.

The question we address in this survey is as follows. Firstly, we consider nonlinear Schrödinger equation (NLS) with a general nonlinearity of the critical order, and investigate the relation between the shape of the nonlinearity and a typical asymptotic behavior of small solutions. This part is a summary of results in [37, 39]. Previous studies treat the case where a nonlinearity is a sum of the gauge-invariant nonlinearity and a polynomial of u and $\overline{u}$. We would like to handle a nonlinearity of non-polynomial type. One such example is the real-valued nonlinearity (1.3) for $d \geqslant 2$. Secondly, we consider nonlinear Klein-Gordon equation (NLKG) with a gauge-invariant nonlinearity, and find an asymptotic behavior for both real-valued case and complex-valued case. In the real-valued case, the gauge-invariant nonlinearity (1.2) is the same as real-valued nonlinearity (1.3). It will turn out that the analysis of (1.3) plays a crucial role. This part is a summary of results in [41, 42, 43] with a slight improvement.

We consider the final value problem: For a given asymptotic profile $u_{\mathrm{app}}(t,x)$, we seek a solutions to (NLS) and (NLKG) which asymptotically behaves like the profile. One may find a non-trivial solution to the final value problem only when the profile is nicely chosen. The final value problem is simpler than the initial value problem. In the both problems, an asymptotic profile is often given in terms of a *final state*, say $u_+(x)$, which is a function of the space variable. As for the initial value problem, we first construct u_+ from a given initial data by solving the equation globally in time. Hence, one has poor control on u_+ compared with the final value problem in which we are allowed to pick u_+ as we want. The main interest here is to find a way to construct $u_{\mathrm{app}}(t,x)$ from u_+. The final value problem is adequate to focus on this subject. We are generous to make any necessary assumption on u_+.

The survey is organized as follows. We first recall asymptotic behavior of solutions to the corresponding linear equations in the rest of Section 1. In Section 2, we introduce an expansion of a nonlinearity, which is a key ingredient in our analysis. Then, we consider nonlinear Schrödinger equation in Section 3. Sections 4 and 5 are devoted to the study of nonlinear Klein-Gordon equation. We first treat the real-valued case in Section 4, and then turn to the complex-valued case in Section 5. In Section 6, we give a few comment on applications to other related topics.

§ 1.1. Asymptotic behavior of linear solutions

Let us now briefly recall asymptotic behaviors for the corresponding linear equations, i.e., the equations without a nonlinearity. They can be deduced, for instance, by the stationary phase method. Our convention for the Fourier transform is as follows:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)dx.$$

As for the linear Schrödinger equation

$$\text{(S)} \qquad \begin{cases} i\partial_t u + \Delta u = 0, \\ u(0) = u_0, \end{cases}$$

it is known that

$$\text{(1.4)} \qquad u(t,x) \sim (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_0}\left(\frac{x}{2t}\right)$$

as $t \to \infty$ in a suitable topology. As for the linear Klein-Gordon equation

$$\text{(KG)} \qquad \begin{cases} (\Box + 1)u = 0, \\ u(0) = \phi_0, \quad \partial_t u(0) = \phi_1, \end{cases}$$

it is known that

$$u(t,x) \sim t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}}(t,x)(A(\mu)e^{i\theta} + B(\mu)e^{-i\theta}) \tag{1.5}$$

again in a suitable topology, where $\mathbf{1}_A(t,x)$ is a characteristic function of a set A,

$$\theta = \theta(t,x) := -\sqrt{t^2 - |x|^2}, \quad \mu = \mu(t,x) := \frac{x}{\sqrt{t^2 - |x|^2}},$$

and

$$\begin{aligned} A(\xi) &:= \frac{1}{2} e^{-i\frac{d}{4}\pi} \langle \xi \rangle^{\frac{d+2}{2}} \mathcal{F}[\phi_0 + i \langle \nabla \rangle^{-1} \phi_1](\xi), \\ B(\xi) &:= \frac{1}{2} e^{i\frac{d}{4}\pi} \langle \xi \rangle^{\frac{d+2}{2}} \mathcal{F}[\phi_0 - i \langle \nabla \rangle^{-1} \phi_1](-\xi). \end{aligned} \tag{1.6}$$

The cutoff $\mathbf{1}_{\{|x|<t\}}$ in the right hand side of (1.5) is put just for notational simplicity, as μ is not well-defined for $|x| \geqslant t$. Remark that we have $|\mu| \to \infty$ as $|x| \uparrow t$ for each fixed $t > 0$. Hence, at least for a good data $(u(0), \partial_t u(0))$, one has $|A(\mu)| + |B(\mu)| \to 0$ as $|x| \uparrow t$. In particular, if $(\phi_0, \phi_1) \in \mathcal{S}(\mathbb{R}^d)^2$ then $A, B \in \mathcal{S}(\mathbb{R}^d)$ and so the right hand side of (1.5) is smooth in $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

§ 2. Extracting a gauge-invariant part of the nonlinearity

We introduce an expansion of a nonlinearity, which plays a crucial role in our analysis. The expansion is due to [37] (see also [39]).

§ 2.1. A Fourier series expansion of a nonlinearity

Let us consider a nonlinearity F satisfying (1.1). By applying (1.1) with $\lambda = |u|^{-1}$, one has

$$F(u) = |u|^{\frac{2}{d}+1} F\left(\frac{u}{|u|}\right)$$

for $u \in \mathbb{C} \setminus \{0\}$. Now, let us introduce a variable $\eta \in \mathbb{R}$ by $e^{i\eta} = u/|u|$. Notice that $g(\eta) := F(e^{i\eta})$ is a 2π-periodic function and so, at least formally, we have a Fourier series expansion

$$g(\eta) = \sum_{n \in \mathbb{Z}} g_n e^{in\eta}, \quad g_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\eta} F(e^{in\eta}) d\eta \tag{2.1}$$

for $\eta \in \mathbb{R}$. Using the identity $e^{in\eta} = (e^{i\eta})^n = (u/|u|)^n$ and combining the above two identities, we reach to the expansion

$$F(u) = \sum_{n\in\mathbb{Z}} g_n F_n(u), \tag{2.2}$$

where

$$F_n(u) := |u|^{\frac{2}{d}+1-n} u^n.$$

Remark that the above expansion is with respect to oscillation. For instance, if u is a function given in a phase-amplitude from $u = \rho e^{i\phi}$ ($\rho \geqslant 0$, $\phi \in \mathbb{R}$), the summand is

$$g_n F_n(u) = g_n \rho^{\frac{2}{d}+1} e^{in\phi}.$$

The amplitude part is common and the phase part is multiplied by n.

It will turn out that the term which has the same oscillation as its argument u has, $g_1 F_1(u)$, possesses a significant effect on the large time behavior. The $F_1(u)$ is nothing but the gauge invariant nonlinearity $F_{\mathrm{GI}}(u)$. Thus, further extracting the gauge-invariant part $F_1(u)$, one also has

$$F(u) = g_1 F_1(u) + \sum_{n\neq 1} g_n F_n(u). \tag{2.3}$$

In what follows, $g_1 F_1(u)$ is referred to as a *resonant part* of F, and the other part

$$\mathcal{N}_F(u) := F(u) - g_1 F_1(u) = \sum_{n\neq 1} g_n F_n(u) \tag{2.4}$$

as a *non-resonant part* of F.

Remark 1. Extracting a resonant part by a Fourier coefficient or a contour integral is used in several previous results such as [27, 28, 32, 57]. However, they consider only polynomial type nonlinearities in these studies and so the extraction procedure can be replaced by an algebraic computation. In our setting, a polynomial nonlinearity implies $d = 1, 2$ and $g_n \neq 0$ only for $0 \leqslant n \leqslant 1 + 2/d$. Hence, what is new in the present study would be (a) the non-resonant term may contain terms corresponding to $n < 0$ and/or to $n > 1+2/d$ and (b) the non-resonant part is not necessarily a finite sum. The latter also means that we have to look at the decay order in n of the Fourier coefficient.

Remark 2. By definition of g_1, we have an alternative integral expression for the non-resonant part $\mathcal{N}_F(u)$:

$$\mathcal{N}_F(u) = \frac{1}{2\pi} \int_0^{2\pi} (F(u) - e^{-i\theta} F(e^{i\theta} u)) d\theta. \tag{2.5}$$

The representation is less explicit compared with (2.4) but it holds with a weaker regularity assumption on F. It is used, for instance, in [44] to the study of non relativistic limit of a Klein-Gordon equation (see also [5]). We would remark that the regularity assumption for the representation (2.4) is not very restrictive. Indeed, in one hand, the series $\sum_{n\neq 1} g_n(u/|u|)^n$ converges absolutely and uniformly in $u \neq 0$ if $F(e^{i\eta})$ is Hölder continuous C^α ($\alpha > 1/2$), which is known as Bernstein's theorem. On the other hand, for a function F satisfying (1.1), $F(e^{it\eta})$ is Lipschitz continuous if and only if there exists a constant $C > 0$ such that

$$|F(u_1) - F(u_2)| \leqslant C(|u_1|^{\frac{2}{d}} + |u_2|^{\frac{2}{d}})|u_1 - u_2| \tag{2.6}$$

for all $u_1, u_2 \in \mathbb{C}$ ([37, Lemma A.1]). It is standard to assume the validity of (2.6).

§ 2.2. Expansion of $F_{\mathrm{Re}}(u)$

In this subsection, we recall the expansion of $F_{\mathrm{Re}}(u)$ given in [37, 39]. By the argument in the previous subsection, this is done by a Fourier series expansion of the corresponding 2π-periodic function

$$g_{\mathrm{Re}}(\eta) = |\cos\eta|^{\frac{2}{d}} \cos\eta.$$

As g_{Re} is an even function, the corresponding Fourier coefficients satisfies

$$g_n = g_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos\eta|^{\frac{2}{d}} \cos\eta \cos n\eta d\eta$$

for $n \geqslant 0$.

Example 2.1 (Two dimensional case)**.** When $d = 2$, one may compute g_n easily because $2/d$ is an integer. We have $g_n = 0$ for even n and

$$g_n = -\frac{4}{\pi} \frac{(-1)^{\frac{n-1}{2}}}{(n-2)n(n+2)}$$

for odd n. Thus, the expansion (2.3) reads as

$$|\operatorname{Re} u| \operatorname{Re} u = \frac{4}{3\pi}|u|u + \sum_{m\neq 1} \frac{4}{\pi} \frac{(-1)^m}{(2m-3)(2m-1)(2m+1)} |u|^{3-2m} u^{2m-1}. \tag{2.7}$$

Let us move on to three dimensions and higher. Note that the critical power $1 + \frac{2}{d}$ becomes fractional. The explicit value of the right hand side is given in [39, Proposition A.1].

Lemma 2.2 ([39]). *Let $\alpha \in (-1,\infty) \setminus (2\mathbb{Z}+1)$. Let*

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos\theta|^{\alpha-1}\cos\theta\cos n\theta d\theta$$

for $n \in \mathbb{Z}$. Then, $c_n = 0$ for even n and

$$c_n = \frac{(-1)^{\frac{n-1}{2}}\Gamma(\frac{\alpha+2}{2})\Gamma(\frac{n-\alpha}{2})}{\sqrt{\pi}\Gamma(-\frac{\alpha-1}{2})\Gamma(\frac{n+\alpha+2}{2})} \tag{2.8}$$

for odd n. In particular, $c_n = O(|n|^{-\alpha-1})$ as $|n| \to \infty$.

Outline of the proof. $c_n = 0$ for even n is obvious by symmetry. For odd n, an integration by parts gives us a recursive relation

$$c_{n+2} = -\frac{n-\alpha}{n+\alpha+2}c_n.$$

Since $c_1 = \frac{\Gamma(\frac{\alpha+2}{2})}{\sqrt{\pi}\Gamma(\frac{\alpha+3}{2})}$, we have the desired formula. □

Remark 3. In Lemma 2.2, the case where α is a positive odd integer is simpler because the case corresponds to the Fourier expansion of $(\cos\theta)^\alpha$. Hence, c_n is nonzero only for finite number of n and one obtains the explicit value of them by looking at the binomial expansion of $(\frac{e^{i\theta}+e^{-i\theta}}{2})^\alpha$, for instance.

Example 2.3 (Higher dimensional case). When $d \geqslant 3$, the expansion (2.3) reads as

$$|\operatorname{Re} u|^{\frac{2}{d}}\operatorname{Re} u = \frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})}|u|^{\frac{2}{d}}u + \sum_{m\neq 1}\frac{(-1)^{m-1}\Gamma(\frac{3}{2}+\frac{1}{d})\Gamma(m-1-\frac{1}{d})}{\sqrt{\pi}\Gamma(-\frac{1}{d})\Gamma(m+1+\frac{1}{d})}|u|^{2+\frac{2}{d}-2m}u^{2m-1}. \tag{2.9}$$

Remark that this formula holds true also for $d = 2$.

§ 3. Application to nonlinear Schrödinger equation

Let us introduce an example of the application of the expansion in Section 2. We consider nonlinear Schrödinger equation with a nonlinearity F which satisfies (1.1). Suppose that $F(e^{i\eta})$ is Lipschitz continuous in η. A detailed assumption will be specified later. Note that the assumption ensures that the Fourier coefficients g_n given in (2.1) is well-defined and the expansion (2.3) makes sense. As pointed out in Remark 2, Lipschitz continuous is equivalent to the existence of a constant $C > 0$ such that

$$|F(u_1) - F(u_2)| \leqslant C(|u_1|^{\frac{2}{d}} + |u_2|^{\frac{2}{d}})|u_1 - u_2|$$

for all $u_1, u_2 \in \mathbb{C}$.

§ 3.1. Relation between the resonant part and asymptotic behavior

Before stating our result, let us briefly review previous results on the asymptotic behavior. In [8, 50], it is revealed that one can construct a nontrivial solution to

$$i\partial_t u + \Delta u = \mu |u|^{\frac{2}{d}} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

which asymptotically behaves as

$$(2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_+}\left(\frac{x}{2t}\right) e^{-i\frac{\mu}{2}|\widehat{u_+}(\frac{x}{2t})|^{\frac{2}{d}} \log t} \tag{3.1}$$

for a suitable given small final state u_+, where $d = 1, 2, 3$ and μ is a given real constant (see also [3]). Remark that the phase correction part contains the constant μ, which indicates that the correction term is due to the nonlinearity. Recall that there is no non-trivial scattering solution ([1, 56]), which means that we may not construct a non-trivial solution without the phase correction term. The corresponding initial value problem is solved by Hayashi and Naumkin [11] (see also [33]). Recently, the validity of the above asymptotics is extended to the higher dimensions $d \geqslant 4$ in [4, 10] by imposing some assumptions that controls the decay of solutions. In [47], Moriyama, Tonegawa, and Tsutusmi consider[1]

$$i\partial_t u + \Delta u = u^{1+\frac{2}{d}}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d$$

for $d = 1, 2$ and show that it admits a nontrivial asymptotic free solution, that is, a solution which behaves like

$$(2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_+}\left(\frac{x}{2t}\right)$$

for large t.

These results are generalized in [21, 22, 54]. They consider (NLS) for $d = 1, 2$ with

$$F(u) = \mu |u|^{\frac{2}{d}} u + N_d(u), \tag{3.2}$$

where

$$N_d(u) = \begin{cases} \lambda_1 u^3 + \lambda_2 |u|^2 \overline{u} + \lambda_3 \overline{u}^3 & d = 1, \\ \lambda_1 u^2 + \lambda_2 \overline{u}^2 & d = 2, \end{cases}$$

$\lambda_j \in \mathbb{C}$, and $\mu \in \mathbb{R}$. Then, one can construct a solution behaves like (3.1) with the constant μ in the nonlinearity. This would suggest that the the resonant part, or the gauge-invariant part, decides the behavior and the non-resonant part makes no difference.

[1] Remark that the coefficient 1 in the nonlinearity is just because of a normalization. A simple change of variable enables us to treat a nonlinearity $\mu u^{1+\frac{2}{d}}$ with a complex constant $\mu \in \mathbb{C} \setminus \{0\}$.

§ 3.2. Main result on (NLS)

Our result extends the validity of the suggestion in the previous subsection to a larger class of nonlinearities: The expansion

$$F(u) = g_1 F_1(u) + \mathcal{N}_F(u) \tag{2.3}$$

is a generalization of (3.2) to a general homogeneous nonlinearity, where $\mathcal{N}_F$ is given by (2.4) (or by (2.5)). The conclusion is that the exponent g_1 decides a typical asymptotic behavior. (For the meaning of the "typicalness," see Remark 5, below.)

Let us introduce the precise statement of our result on (NLS). For this purpose. Set $\langle a\rangle = (1+|a|^2)^{1/2}$ for any $a \in \mathbb{C}^n$, where $|a|$ is the standard Euclidean norm. For $s, m \in \mathbb{R}$, the weighted Sobolev space on $\mathbb{R}^d$ is defined by $H^{s,m} = \{u \in \mathcal{S}'(\mathbb{R}^d); \langle x\rangle^m \langle\nabla\rangle^s u \in L^2\}$, and $\dot{H}^m = \{u \in \mathcal{S}'(\mathbb{R}^d); (-\Delta)^{m/2} u \in L^2\}$ denotes the homogeneous Sobolev space on $\mathbb{R}^d$. We simply write $H^s = H^{s,0}$. Let $\|g\|_{\mathrm{Lip}} := \sup_{\theta\neq\theta'} |g(\theta) - g(\theta')|/|\theta-\theta'|$ be the Lipschitz norm.

Assumption 3.1. Suppose that 2π-periodic function $g(\eta) = F(e^{i\eta})$ satisfies

1. $g_0 = 0$,
2. $g_1 \in \mathbb{R}$,
3. $\sum_{n\in\mathbb{Z}} |n|^{1+\eta}|g_n| < \infty$ for some $\eta > 0$,

where g_n is Fourier coefficient of $g(\eta)$ given in (2.1).

Theorem 3.2 ([37, 39]). *Let $d = 1, 2, 3$. Suppose that the nonlinearity F satisfies Assumption 3.1 for $\eta > 0$. Fix $\delta \in (d/2, \min(2, 1+2/d))$ so that $\delta - d/2 < 2\eta$. Then, there exists $\varepsilon_0 = \varepsilon_0(\|g\|_{\mathrm{Lip}})$ such that for any $u_+ \in H^{0,2} \cap \dot{H}^{-\delta}$ satisfying $\|\widehat{u_+}\|_{L^\infty} < \varepsilon_0$ there exists $T > 0$ and a solution $u \in C([T,\infty); L^2(\mathbb{R}^d))$ of* (NLS) *which satisfies*

$$\sup_{t\in[T,\infty)} t^b \left(\|u(t) - u_{\mathrm{app}}(t)\|_{L^2} + \|u - u_{\mathrm{app}}\|_{X_d(t)} \right) < \infty \tag{3.3}$$

for any $b < \delta/2$, where

$$u_{\mathrm{app}}(t) := (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_+}\left(\frac{x}{2t}\right) \exp\left(-i\frac{g_1}{2}\left|\widehat{u_+}\left(\frac{x}{2t}\right)\right|^{\frac{2}{d}} \log t\right) \tag{3.4}$$

and

$$\|f\|_{X_d(t)} = \begin{cases} \|f\|_{L^4_t([t,\infty); L^\infty_x(\mathbb{R}))} & d = 1, \\ \|f\|_{L^4_t([t,\infty); L^4_x(\mathbb{R}^2))} & d = 2, \\ 0 & d = 3. \end{cases} \tag{3.5}$$

The solution is unique in the following sense: If $\tilde{u} \in C([\tilde{T}, \infty); L^2(\mathbb{R}^d))$ solves (NLS) *and satisfies* (3.3) *for some $\tilde{T}$ and $b > d/4$ then $\tilde{u} = u$.*

Remark 4. When $g_0 \neq 0$, the behavior of the solution is different from (3.4) ([53, 55, 25, 38]).

Remark 5. The assumption $u_+ \in \dot{H}^{-\delta}$ means that the low frequency part of u_+ is considerably small. The restriction is essential because otherwise other kinds of asymptotic behavior take place ([12, 13, 16, 18, 20, 48, 49]). See also [21, Remark 1.6] for discussion with an example.

§ 3.3. Outline of the proof

The strategy of the proof is to solve an integral equation around a prescribed profile. Then, it turns out that the matter is reduced to the choice of an asymptotic profile.

Let us derive the integral equation in an abstract setting. Let $\tilde{u}(t)$ is a given profile to be specified later. We introduce a new unknown w by

$$w := u - \tilde{u}.$$

Then, we have

$$i\partial_t w + \Delta w = F(u) - (i\partial_t \tilde{u} + \Delta \tilde{u}) = (F(\tilde{u} + w) - F(\tilde{u})) - (i\partial_t \tilde{u} + \Delta \tilde{u} - F(\tilde{u})).$$

As $\tilde{u}(t)$ is expected to be an asymptotic profile of $u(t)$, we may suppose that

$$\|w(t)\|_{L^2} \to 0$$

holds as $t \to \infty$. Hence, by the Duhamel principle, $w(t)$ is a solution

$$w(t) = i \int_t^\infty e^{i(t-s)\Delta} (F(\tilde{u} + w) - F(\tilde{u}))(s) ds + E(t), \tag{3.6}$$

where $E(t)$ is an error term defined by

$$E(t) := -i \int_t^\infty e^{i(t-s)\Delta} (i\partial_t \tilde{u} + \Delta \tilde{u} - F(\tilde{u})) ds. \tag{3.7}$$

Conversely, if we find a solution $w(t)$ to (3.6) in a suitable sense then $u := \tilde{u} + w$ is the solution we look for.

We now state an existence result for the equation (3.6). We define $\|f\|_{X_d(t)} := 0$ for $d \geqslant 4$ and as in (3.5) for $1 \leqslant d \leqslant 3$. Then, one has the following.

Proposition 3.3. *Let $d \geqslant 1$. Suppose that the nonlinearity F satisfies* (1.1). *There exists $\varepsilon_0 = \varepsilon_0(d, \|g\|_{\mathrm{Lip}}) > 0$ with the following property: If $\tilde{u}(t)$ satisfies*

$$\sup_{t \geqslant T} t^{\frac{d}{2}} \|\tilde{u}(t)\|_{L^\infty} \leqslant \varepsilon_0 \tag{3.8}$$

and if an error term $E(t)$, given by (3.7), *satisfies*

$$\sup_{t \geqslant T} t^b (\|E\|_{L^\infty([t,\infty);L^2)} + \|E\|_{X_d(t)}) \leqslant \varepsilon \tag{3.9}$$

for some $T \geqslant 1$, $b \geqslant \frac{d}{4}$, and $0 < \varepsilon \leqslant \varepsilon_0$, then there exists a unique solution $w(t) \in C([T,\infty); L^2)$ to (3.6) *which satisfies*

$$\sup_{t \geqslant T} t^b (\|w\|_{L^\infty([t,\infty);L^2)} + \|w\|_{X_d(t)}) \leqslant 2\varepsilon.$$

Furthermore, $u := \tilde{u} + w$ is an L^2-solution to (NLS) *if $\tilde{u} \in C(\mathbb{R}; L^2)$.*

This proposition is an immediate consequence of the contraction mapping principle and Strichartz' estimate. We remark that the inequalities

$$|F(\tilde{u} + w) - F(\tilde{u})| \lesssim \|g\|_{\mathrm{Lip}} (|w|^{\frac{2}{d}} + |\tilde{u}|^{\frac{2}{d}})|w|$$

and

$$|F(\tilde{u} + w_1) - F(\tilde{u} + w_2)| \lesssim \|g\|_{\mathrm{Lip}} (|w_1|^{1+\frac{2}{d}} + |w_2|^{1+\frac{2}{d}} + |\tilde{u}|^{\frac{2}{d}})|w_1 - w_2|$$

hold if F satisfies (1.1). Further, a use of end-point Strichartz' estimate enables us to take $\|f\|_{X_d(t)} := 0$ for $d \geqslant 3$. For details, see [37, 39]. We prove a similar result for the Klein-Gordon equation in Appendix A.

The above proposition reduces the matter to finding an asymptotic profile $\tilde{u}(t)$ so that the assumption of the proposition is fulfilled. Hence, to prove Theorem 3.2, it suffices to show that the assumption of Proposition 3.3 is satisfied with the choice $\tilde{u}(t) = u_{\mathrm{app}}(t)$, defined by (3.4), under the assumption of Theorem 3.2. Note that it is immediate to see that

$$\sup_{t \geqslant T} t^{\frac{d}{2}} \|u_{\mathrm{app}}(t)\|_{L^\infty} \lesssim \|\widehat{u_+}\|_{L^\infty}$$

for any $T > 0$. This shows (3.8).

For the estimates of the error term E defined by (3.7), the following representation will be useful.

Lemma 3.4 ([21, 22, 54]). *Take $u_+ \in \mathcal{S}(\mathbb{R}^d)$. It holds that*

$$E(t) = \mathcal{R}(t)\widehat{w} - ig_1 \int_t^\infty e^{i(t-s)\Delta} \mathcal{R}(s) F_1(\widehat{w}) \frac{ds}{2s} + i \int_t^\infty e^{i(t-s)\Delta} \mathcal{N}_F(u_{\mathrm{app}}(s)) ds$$

for $t>0$, *where* F_1 *and* $\mathcal{N}_F$ *are as in* (2.3), $\widehat{w}(t)=\widehat{u_+}e^{-i\frac{g_1}{2}|\widehat{u_+}|^{\frac{2}{d}}\log t}$, *and* $\mathcal{R}(t)=M(t)D(t)(e^{-i\frac{1}{4t}\Delta}-1)$ *with*

$$(M(t)f)(x)=e^{i\frac{|x|^2}{4t}}f(x)$$

and

$$(D(t)f)(x)=(2it)^{-\frac{d}{2}}f((2t)^{-1}x).$$

Proof. By means of the expansion (2.3), it suffices to show that

$$\begin{aligned}(3.10)\quad -i\int_t^\infty & e^{i(t-s)\Delta}((i\partial_t+\Delta)u_{\mathrm{app}}-g_1F_1(u_{\mathrm{app}}))(s)ds\\ &=\mathcal{R}(t)\widehat{w}-ig_1\int_t^\infty e^{i(t-s)\Delta}\mathcal{R}(s)F_1(\widehat{w})\frac{ds}{2s}.\end{aligned}$$

Denote that right hand side of (3.10) by $\mathcal{E}$. Remark that $\mathcal{E}$ is a solution to

$$i\partial_t\mathcal{E}+\Delta\mathcal{E}=-(i\partial_t+\Delta)u_{\mathrm{app}}+g_1F_1(u_{\mathrm{app}})$$

and $\mathcal{E}(t)\to 0$ as $t\to\infty$ in L^2. Operating $\mathcal{F}e^{-it\Delta}$ to the both sides, we have

$$(3.11)\qquad i\partial_t(\mathcal{F}e^{-it\Delta}\mathcal{E})=-i\partial_t(\mathcal{F}e^{-it\Delta}u_{\mathrm{app}})+g_1\mathcal{F}e^{-it\Delta}F_1(u_{\mathrm{app}}).$$

Remark that the identity

$$(3.12)\qquad \begin{aligned}\mathcal{F}e^{-it\Delta}(\mathcal{R}(t)+M(t)D(t))&=\mathcal{F}e^{-it\Delta}M(t)D(t)e^{-i\frac{1}{4t}\Delta}\\ &=\mathcal{F}e^{-it\Delta}(M(t)D(t)\mathcal{F}M(t))\mathcal{F}^{-1}\\ &=1\end{aligned}$$

holds. Hence,

$$\mathcal{F}e^{-it\Delta}u_{\mathrm{app}}=\mathcal{F}e^{-it\Delta}M(t)D(t)\widehat{w}=\widehat{w}-\mathcal{F}e^{-it\Delta}\mathcal{R}(t)\widehat{w}.$$

Plugging this to (3.11), we obtain

$$(3.13)\qquad i\partial_t(\mathcal{F}e^{-it\Delta}\mathcal{E})=-i\partial_t\widehat{w}+i\partial_t(\mathcal{F}e^{-it\Delta}\mathcal{R}(t)\widehat{w})+g_1\mathcal{F}e^{-it\Delta}F_1(u_{\mathrm{app}}).$$

On the other hand, $\widehat{w}$ solves the ODE

$$i\partial_t\widehat{w}=\frac{1}{2t}g_1F_1(\widehat{w}).$$

We use (3.12) again to obtain

$$(3.14)\qquad i\partial_t\widehat{w}=g_1\mathcal{F}e^{-it\Delta}\frac{1}{2t}\mathcal{R}(t)F_1(\widehat{w})+g_1\mathcal{F}e^{-it\Delta}F_1(u_{\mathrm{app}}),$$

where we have used $\frac{1}{2t}M(t)D(t)F_1(\widehat{w}) = F_1(u_{\mathrm{app}})$ for $t > 0$. Combining (3.13) and (3.14), one has

$$i\partial_t(\mathcal{F}e^{-it\Delta}\mathcal{E}) = i\partial_t(\mathcal{F}e^{-it\Delta}\mathcal{R}(t)\widehat{w}) - g_1\mathcal{F}e^{-it\Delta}\frac{1}{2t}\mathcal{R}(t)F_1(\widehat{w}).$$

Note that $\mathcal{E} \to 0$ and $\mathcal{R}(t)\widehat{w} \to 0$ as $t \to \infty$ in L^2. Hence, an integration of the above identity from t to infinity gives us

$$i\mathcal{F}e^{-it\Delta}\mathcal{E} = i\mathcal{F}e^{-it\Delta}\mathcal{R}(t)\widehat{w} + g_1\int_t^\infty \mathcal{F}e^{-is\Delta}\mathcal{R}(s)F_1(\widehat{w}(s))\frac{ds}{2s}.$$

Thus,

$$\mathcal{E}(t) = \mathcal{R}(t)\widehat{w} - ig_1\int_t^\infty e^{i(t-s)\Delta}\mathcal{R}(s)F_1(\widehat{w}(s))\frac{ds}{2s}.$$

This is nothing but (3.10). □

Let us resume the outline of the proof of Theorem 3.2. We want to prove (3.9). The first two terms in the representation of E have the factor $\mathcal{R}(t)$ from which one can obtain decay in time. Remark that this structure is due to the choice of the constant $\frac{g_1}{2}$ in the phase correction in (3.4). Indeed, one deduces from (3.13) and (3.14) that if the phase correction is given with another constant then $\partial_t\mathcal{F}e^{-it\Delta}\mathcal{E}$ contains one more factor

$$\mathcal{F}e^{-it\Delta}F_1(u_{\mathrm{app}}) = \frac{1}{2t}F_1(\widehat{w}) - \mathcal{F}e^{-it\Delta}\frac{1}{2t}\mathcal{R}(t)F_1(\widehat{w}),$$

which is not integrable in time. On the other hand, the smallness of the non-resonant term comes from the fact that the oscillation of each terms in $\mathcal{N}_F$ is different from that of a linear equation (or that of u_{app}). Hence, we can derive necessarily time decay by integration by parts. One respect to be noted in this part is that the integration-by-parts produces a term like $\Delta\mathcal{N}_F(u_{\mathrm{app}})$. The technical difference among [21, 22, 37, 39] lies in the treatment of this part. In [22, 37, 39], we reduce required order of differentiation by introducing a time dependent frequency cutoff and by applying the integration-by-parts only to the low frequency. This respect becomes crucial when $d = 3$ because the term $\Delta\mathcal{N}_F(u_{\mathrm{app}})$ is not acceptable because $\mathcal{N}_F(z) \notin C^2$ for $d = 3$. For further details of the estimate of $E(t)$, see [37, 39].

§ 4. Application to nonlinear Klein-Gordon equation – $\mathbb{R}$-valued case

Let us move on to the next topic. We look for an asymptotic behavior of solutions to nonlinear Klein-Gordon equation (NLKG) with the gauge-invariant nonlinearity (1.2) for $d = 2, 3$. This section is a summary of [41, 42].

Before going into details. Let us briefly mention previous results. Concerning the cubic nonlinear Klein-Gordon equation in one dimension the final value problem is studied in [17, 34] and the initial value problem is treated in [6, 15] (see also [26, 46, 58] for quasi-linear case). As for a closely related model, the initial value problem of the cubic Dirac equation is studied in [2].

Remark that the nonlinearity $F_{\mathrm{GI}}(u)$ is a polynomial in one dimension. As far as the author knows, there was no result on the asymptotic behavior of solutions to (NLKG) with $F_{\mathrm{GI}}(u)$ for $d \geqslant 2$, in which case the nonlinearity is not a polynomial any more for $d \geqslant 2$. The Klein-Gordon equation with a quadratic polynomial nonlinearity in two dimensions is also extensively studied (see [7, 27, 28, 51]).

§ 4.1. Two-wave structure and the half Klein-Gordon equation

Let us first look at a difference of the structure of solutions to the Klein-Gordon equation and to the Schrödinger equation. In this subsection we do not use the real-valued property.

A characteristic property of solutions to the Klein-Gordon equation is that it is written as a superposition of two waves. For simplicity, we first consider linear equation $F \equiv 0$. To see the property, we introduce new unknowns

$$v_\pm = \frac{1}{2}(u \pm i \langle \nabla \rangle^{-1} \partial_t u). \tag{4.1}$$

Then, one verifies that they solve the half Klein-Gordon equation

$$\partial_t v_\pm \pm i \langle \nabla \rangle v_\pm = 0. \tag{4.2}$$

By using the Fourier multiplier, we have

$$v_\pm(t) = e^{\mp it\langle \nabla \rangle} v_\pm(0).$$

Hence, we obtain the following formula for a solution to the Klein-Gordon equation:

$$u(t) = v_+(t) + v_-(t) = e^{-it\langle \nabla \rangle} v_+(0) + e^{it\langle \nabla \rangle} v_-(0). \tag{4.3}$$

Note that the right hand side is a superposition of two waves. By the stationary phase method, we have

$$v_+(t) \sim t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} e^{-i\frac{d}{4}\pi} \langle \mu \rangle^{\frac{d+2}{2}} \widehat{v_+(0)}(\mu) e^{i\theta} = t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} A(\mu) e^{i\theta} \tag{4.4}$$

and

$$v_-(t) \sim t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} e^{i\frac{d}{4}\pi} \langle \mu \rangle^{\frac{d+2}{2}} \widehat{v_-(0)}(-\mu) e^{-i\theta} = t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} B(\mu) e^{-i\theta} \tag{4.5}$$

in a suitable sense as $t \to \infty$, where θ, μ, A, and B are the same as in (1.5). This is another proof of the asymptotics

$$u(t,x) \sim t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}}(t,x)(A(\mu)e^{i\theta} + B(\mu)e^{-i\theta}) \tag{1.5}$$

as $t \to \infty$. This formula also suggests that the solution to the Klein-Gordon equation is a superposition of two waves.

The above structure is the same for the nonlinear Klein-Gordon equations. Let us now consider (NLKG) with the gauge-invariant nonlinearity F_{GI}. We introduce $v_\pm$ as in (4.1). Then, one has the system of half Klein-Gordon equations:

$$\text{(hKG)} \qquad \begin{cases} i\partial_t v_+ - \langle\nabla\rangle v_+ = 2^{-1}\langle\nabla\rangle^{-1} F_{\mathrm{GI}}(v_+ + v_-), \\ i\partial_t v_- + \langle\nabla\rangle v_- = -2^{-1}\langle\nabla\rangle^{-1} F_{\mathrm{GI}}(v_+ + v_-). \end{cases}$$

Once we obtain a solution (v_+, v_-) to the system, one can immediately obtain a solution to the original Klein-Gordon equation by the relation $u = v_+ + v_-$.

§ 4.2. Extraction of a resonant part for real-valued solutions

We now consider peculiarity of the case where $u(t)$ is real-valued. As seen in the previous subsection, a solution to the Klein-Gordon equation is a superposition of two waves. To keep the superposition real-valued, the two waves must balance each other so that the imaginary parts of them are canceled out. More specifically, in terms of $v_\pm(t)$ introduced in (4.1), $u(t)$ is real-valued if and only if the relation

$$v_- = \overline{v_+} \tag{4.6}$$

holds. Further, the both two equations in the system (hKG) become the same one,

$$i\partial_t v_+ - \langle\nabla\rangle v_+ = 2^{-1}\langle\nabla\rangle^{-1} F_{\mathrm{GI}}(2\operatorname{Re} v_+). \tag{4.7}$$

When $u(t)$ is not real-valued, the relation (4.6) is not true any more, which implies the system (hKG) is not reduced a single equation. This respect increases the complexity of the situation. This case is addressed in the forthcoming section.

Let us restrict our attention to the real-valued case and see how our expansion in Section 2 is involved. In the real-valued case, we have the relation (4.6) and so the nonlinearity for (4.7) is

$$\begin{aligned} F_{\mathrm{GI}}(2\operatorname{Re} v_+) &= 2^{1+\frac{2}{d}}|\operatorname{Re} v_+|^{\frac{2}{d}}\operatorname{Re} v_+ \\ &= 2^{1+\frac{2}{d}} F_{\mathrm{Re}}(v_+). \end{aligned}$$

Hence, the equation (4.7) reads as

$$i\partial_t v_+ - \langle\nabla\rangle v_+ = 2^{\frac{2}{d}}\langle\nabla\rangle^{-1} F_{\mathrm{Re}}(v_+).$$

As in the NLS case, we see from the expansion (2.9) of F_{Re} that the gauge-invariant part of the nonlinearity of (4.9) is

$$2^{\frac{2}{d}}\frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\,\Gamma(2+\frac{1}{d})}\langle\nabla\rangle^{-1}(|v_+|^{\frac{2}{d}}v_+). \tag{4.8}$$

Thus, we reach to

$$i\partial_t v_+ - \langle\nabla\rangle v_+ = 2^{\frac{2}{d}} \frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})} \langle\nabla\rangle^{-1} (|v_+|^{\frac{2}{d}} v_+) + \mathcal{N}(v_+), \tag{4.9}$$

where $\mathcal{N}$ corresponds to the non-resonant part.

The above is a heuristic argument which explains the choice of our phase correction term (4.11) and asymptotic profile (4.10), below. The asymptotic profile involves a phase correction term with the coefficients in the resonant part of (4.9). We work rather with the original variable u in the actual proof. Nevertheless, the expansion of the nonlinearity $F_{\mathrm{Re}}(u)$ plays an essential roll in the proof, of course.

Remark 6. There is a slight difference between applications of the expansion discussed in Section 2 to Schödinger equation and that to Klein-Gordon equation. In the Schrödinger case, we directly expand the nonlinearity. In contrast, in the Klein-Gordon case, the nonlinearity is the gauge-invariant one. However, because of the two-wave structure and the symmetry (4.6), we encounter the real-valued nonlinearity F_{Re}.

§ 4.3. Main result on (NLKG) in the real-valued case

We now introduce our main result in this section. The notations are the same as in Subsection 3.2.

Theorem 4.1 ([41, 42]). *Let $d = 2, 3$. Let $\phi_0 \in H^{4,2}(\mathbb{R}^d)$ and $\phi_1 \in H^{3,1}(\mathbb{R}^d)$ be real-valued functions. Let*

$$A(\xi) := \frac{1}{2} e^{-i\frac{d}{4}\pi} \langle\xi\rangle^{\frac{d+2}{2}} \mathcal{F}[\phi_0 + i\langle\nabla\rangle^{-1}\phi_1](\xi).$$

Define the asymptotic profile $u_{\mathrm{app}}(t,x)$ by

$$u_{\mathrm{app}}(t,x) = t^{-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}}(t,x) 2\,\mathrm{Re}(A(\mu) e^{i\theta + iS(\mu)\log t}), \tag{4.10}$$

where $\theta = -\sqrt{t^2 - |x|^2}$, $\mu = x/\sqrt{t^2-|x|^2}$, and

$$S(z) = -2^{\frac{2}{d}} \frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})} \langle z\rangle^{-1} |A(z)|^{\frac{2}{d}}. \tag{4.11}$$

Then, for any $b \in (0, \frac{1}{2}(1+\frac{2}{d}))$, there exists $\varepsilon_0 = \varepsilon_0(d,b) > 0$ such that if $\|A\|_{L^\infty(\mathbb{R}^d)} \leqslant \varepsilon_0$ then there exist $T \geqslant 3$ and a unique solution $u \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2) \cap L_t^{\frac{2(d+2)}{d}}([T,\infty); W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$ to (NLKG) such that

$$\|u(t) - u_{\mathrm{app}}(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(t) - \partial_t u_{\mathrm{app}}(t)\|_{L^2(\mathbb{R}^d)} + \left\| \langle\nabla\rangle^{\frac{1}{2}} (u - u_{\mathrm{app}}) \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}((t,\infty)\times\mathbb{R}^d)} \lesssim t^{-b} \tag{4.12}$$

for $t \geqslant T$*. The solution is unique in the following sense: If* $\tilde{u} \in C([\tilde{T},\infty);H^1) \cap C^1([\tilde{T},\infty);L^2) \cap L_t^{\frac{2(d+2)}{d}}([\tilde{T},\infty);W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$ *solves* (NLKG) *and satisfies* (4.12) *for some* $\tilde{T} > 0$ *and* $b > d/4$ *then* $\tilde{T} \leqslant T$ *and* $\tilde{u} = u$ *for* $t \geqslant T$*.*

Remark 7. We have an improvement in Theorem 4.1 compared with [41, 42]. More precisely, we put the same assumption as in [41, 42] and obtain a stronger result: Firstly, the solution belongs to a better class. In particular, $u(t)$ is an $(H^1 \times L^2)$-solution. Secondly, the asymptotics (4.12) holds in a stronger topology. The improvement is due to a better perturbation result (Proposition 4.2, below).

§ 4.4. Outline of the proof

The proof of Theorem 4.1 is done in the same spirit as in the NLS case. As mentioned in Subsection 4.2, we work with the original variable $u(t)$. Let $\tilde{u}(t)$ be a given profile. Introduce a new unknown w by

$$w := u - \tilde{u}.$$

We want to seek a solution $w(t)$ to

$$(\Box + 1)w = (F_{\mathrm{GI}}(\tilde{u} + w) - F_{\mathrm{GI}}(\tilde{u})) - ((\Box + 1)\tilde{u} - F_{\mathrm{GI}}(\tilde{u}))$$

with $(w(t), \partial_t w(t)) \to 0$ as $t \to \infty$ in $H^1 \times L^2$. Note that, for a given function $g \in C(\mathbb{R}; L^2)$, a solution to the linear solution

$$(\Box + 1)u = g$$

with $(u(t), \partial_t u(t)) \to 0$ as $t \to \infty$ in $H^1 \times L^2$ is written as

$$\mathcal{G}[g](t) := \langle \nabla \rangle^{-1} \int_t^\infty \sin((t-s)\langle \nabla \rangle) g(s) ds. \tag{4.13}$$

Hence, by the Duhamel principle, the equation for w is written as

$$w(t) = \mathcal{G}[F_{\mathrm{GI}}(\tilde{u} + w) - F_{\mathrm{GI}}(\tilde{u})] + E, \tag{4.14}$$

where

$$E(t) := \mathcal{G}[(\Box + 1)\tilde{u} - F_{\mathrm{GI}}(\tilde{u})](t). \tag{4.15}$$

We have the following. This is an improvement of [41, Proposition 2.1] and [42, Proposition 3.1].

Proposition 4.2. *Let $d \geqslant 1$. There exists a constant $\varepsilon_0 = \varepsilon_0(d) > 0$ with the following property: If a given profile $\widetilde{u}(t,x) \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2)$ satisfies*

$$\sup_{t \geqslant T} t^{\frac{d}{2}} \|\widetilde{u}(t)\|_{L^\infty_x} \leqslant \varepsilon_0, \tag{4.16}$$

and

$$\sup_{t\geqslant T} t^b (\|E\|_{L^\infty([t,\infty);H^1_x)} + \|\partial_t E\|_{L^\infty([t,\infty);L^2_x)} + \| \langle \nabla \rangle^{\frac{1}{2}} E\|_{L^{\frac{2(d+2)}{d}}_{t,x}([t,\infty)\times\mathbb{R}^d)}) \leqslant \varepsilon \tag{4.17}$$

for some $T \geqslant 1$, $b \geqslant d/4$, and $0 < \varepsilon \leqslant \varepsilon_0$, where E is the error term defined in (4.15), then there exists a unique solution $w \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2) \cap L_t^{\frac{2(d+2)}{d}}([T,\infty); W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$ to the equation (4.14) satisfying

$$\sup_{t\geqslant T} t^b (\|w\|_{L^\infty([t,\infty);H^1_x)} + \|\partial_t w\|_{L^\infty([t,\infty);L^2_x)} + \| \langle \nabla \rangle^{\frac{1}{2}} w\|_{L^{\frac{2(d+2)}{d}}_{t,x}([t,\infty)\times\mathbb{R}^d)}) \leqslant 2\varepsilon. \tag{4.18}$$

Moreover, $u = \tilde{u} + w$ is an $(H^1 \times L^2)$-solution on $[T,\infty)$ to (NLKG).

Although it is shown by a standard argument, we give a full proof in Appendix A because this gives us improvements in Theorems 4.1 and 5.2.

The rest of the proof goes as follows. We want to apply Proposition 4.2 with the choice $\tilde{u}(t) = u_{\mathrm{app}}(t)$. (4.16) is easy to check. To see (4.17), we show a stronger bound

$$\sup_{t\geqslant T} t^b \, \|(\Box + 1)\tilde{u} - F_{\mathrm{GI}}(\tilde{u})\|_{L^1([t,\infty);L^2)} \lesssim_d 1 \tag{4.19}$$

for $b > d/4$ by a direct computation. A benefit with the condition (4.19) is that the nonlocal operator $\langle \nabla \rangle^{1/2}$ is excluded. However, it turns out that the choice $\tilde{u}(t) = u_{\mathrm{app}}(t)$ does not satisfies (4.19). More precisely, by the decomposition (2.9),

$$\begin{aligned}
&F_{\mathrm{GI}}(u_{\mathrm{app}}) \\
&= t^{-1-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} 2^{1+\frac{2}{d}} F_{\mathrm{Re}}(A(\mu) e^{i\theta + iS(\mu)\log t}) \\
&= t^{-1-\frac{d}{2}} \mathbf{1}_{\{|x|<t\}} 2^{1+\frac{2}{d}} \left(\frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})} F_{\mathrm{GI}}(A(\mu) e^{i\theta + iS(\mu)\log t}) + \mathcal{N}_{\mathrm{Re}}(A(\mu) e^{i\theta + iS(\mu)\log t}) \right) \\
&=: \mathrm{I} + \mathrm{II},
\end{aligned}$$

where $\mathcal{N}_{\mathrm{Re}}$ is the non-resonant part of the decomposition (2.9). One then verifies that $(\Box + 1)u_{\mathrm{app}} - \mathrm{I}$ has the desired decay in time but II does not. A remedy is to construct a "second" profile, say $v_{\mathrm{app}}(t)$, so that it is of order $O(t^{-1}u_{\mathrm{app}})$ and $(\Box+1)v_{\mathrm{app}} - \mathrm{II}$ has the desired decay in time. Then, $\tilde{u}(t) = u_{\mathrm{app}}(t) + v_{\mathrm{app}}(t)$ satisfies (4.19). This argument is inspired by [23] (see also [47, 54]). When $d = 3$, we further use an argument in [8] to handle the second derivative of $|u|^{\frac{2}{3}}u$ (see [41]).

§ 5. Application to nonlinear Klein-Gordon equation – $\mathbb{C}$-valued case

In this section, we consider nonlinear Klein-Gordon equation (NLKG) in the complex-valued case. This part is based on [43].

Asymptotic behavior of a complex-valued solution is less studied. Notice that splitting into the real part and the imaginary part enables us to rewrite the complex-valued equation into a system of Klein-Gordon equation if the considering nonlinearity is a polynomial. Hence, some models are handled by a study of real-valued system. As for the equation with the gauge-invariant nonlinearity, Sunagawa [57] derived the sharp decay estimate of the solution in one-dimensional case. Recently, Segata studied the initial value problem [52]. When $d \geqslant 2$, the nonlinearity is non a polynomial. Hence, the above reduction does not work well.

§ 5.1. The two-wave structure in the complex-valued case

As discussed in Subsection 4.1, by introducing the new unknowns

$$v_\pm = \frac{1}{2}(u \pm i \langle \nabla \rangle^{-1} \partial_t u). \tag{4.1}$$

the nonlinear Klein-Gordon equation (NLKG) with he gauge-invariant nonlinearity F_{GI} is written as the system of half Klein-Gordon equations:

$$\begin{cases} i\partial_t v_+ - \langle \nabla \rangle v_+ = \langle \nabla \rangle^{-1} F_{\mathrm{GI}}(v_+ + v_-), \\ i\partial_t v_- + \langle \nabla \rangle v_- = - \langle \nabla \rangle^{-1} F_{\mathrm{GI}}(v_+ + v_-), \end{cases} \tag{hKG}$$

In the real-valued case, the system (hKG) is reduced to a single equation (4.9) in light of the symmetry (4.6). However, in the complex-valued case the system (hKG) is not reduced any more.

§ 5.2. Extraction of resonant parts of the nonlinearity

Invoking the argument in Sections 3 and 4, the main step to find a correct asymptotic profile would be to find the resonant part of the nonlinearity which has the same oscillation as its argument has. In this subsection, we demonstrate the extraction with a heuristic argument.

Recalling the fact that we should regard (v_+, v_-) as a pair of unknowns, let us assume that they are written individually in the phase-amplitude form as follows:

$$v_\pm = \rho_\pm e^{i\theta_\pm},$$

where $\rho_\pm \geqslant 0$ and $\theta_\pm \in \mathbb{R}$. Then, what we do here is to extract an $e^{i\theta_\pm}$-oscillatory part from the function $F_{\mathrm{GI}}(v_+ + v_-)$.

To this end, we first use the gauge-invariant property of F_{GI} to obtain

$$F_{\mathrm{GI}}(v_+ + v_-) = F_{\mathrm{GI}}(\rho_+ e^{i\theta_+} + \rho_- e^{i\theta_-}) \\ = e^{i\theta_+} F_{\mathrm{GI}}(\rho_+ + \rho_- e^{i(\theta_- - \theta_+)}). \tag{5.1}$$

We regard $F_{\mathrm{GI}}(\rho_+ + \rho_- e^{i(\theta_- - \theta_+)})$ as a 2π-periodic function of

$$\Theta := \theta_- - \theta_+. \tag{5.2}$$

Then, using the Fourier series expansion, we have the relation

$$F_{\mathrm{GI}}(\rho_+ + \rho_- e^{i\Theta}) = \sum_{n\in\mathbb{Z}} G_n(\rho_+, \rho_-) e^{in\Theta}, \tag{5.3}$$

at least formally. Here the coefficient

$$G_n(\rho_+, \rho_-) := \frac{1}{2\pi} \int_0^{2\pi} F_{\mathrm{GI}}(\rho_+ + \rho_- e^{i\theta}) e^{-in\theta} d\theta \tag{5.4}$$

is a function of (ρ_+, ρ_-).

Combining (5.1) and (5.3), one sees that the $e^{i\theta_+}$-oscillatory part is

$$G_0(\rho_+, \rho_-) e^{i\theta_+}, \tag{5.5}$$

which corresponds to term with $n = 0$. Similarly, using (5.2) also, one finds that the $e^{i\theta_-}$-oscillatory part is the term with $n = 1$,

$$G_1(\rho_+, \rho_-) e^{i\theta_+} e^{i(\theta_- - \theta_+)} = G_1(\rho_+, \rho_-) e^{i\theta_-}.$$

By definition, we have an identity

$$G_1(\rho_+, \rho_-) = G_0(\rho_-, \rho_+).$$

Hence, the $e^{i\theta_-}$-oscillatory part is written as

$$G_0(\rho_-, \rho_+) e^{i\theta_-}. \tag{5.6}$$

This is symmetric to (5.5).

Remark 8. In [44], the above argument is introduced to study the non relativistic limit of (NLKG).

Now we suppose that $\zeta := \rho_-/\rho_+ \in \mathbb{R}$ makes sense. Then, we have

$$G_0(\rho_+, \rho_-) e^{i\theta_+} = (L_0(\zeta) \rho_+^{1+\frac{2}{d}}) e^{i\theta_+} = (L_0(\zeta) \rho_+^{\frac{2}{d}}) v_+,$$

where

$$L_0(\zeta) := \frac{1}{2\pi}\int_0^{2\pi} F_{\mathrm{GI}}(1+\zeta e^{i\theta})d\theta. \tag{5.7}$$

Similarly, if we suppose $1/\zeta := \rho_+/\rho_- \in \mathbb{R}$ also makes sense then

$$G_0(\rho_-,\rho_+)e^{i\theta_-} = (L_0(1/\zeta)\rho_-^{\frac{2}{d}})v_-.$$

Thus, we finally reach to the following form

$$\begin{cases} i\partial_t v_+ - \langle\nabla\rangle v_+ = \langle\nabla\rangle^{-1}((L_0(\zeta)|v_+|^{\frac{2}{d}})v_+) + N_+, \\ i\partial_t v_- + \langle\nabla\rangle v_- = -\langle\nabla\rangle^{-1}((L_0(1/\zeta)|v_-|^{\frac{2}{d}})v_-) + N_-, \end{cases}$$

where $N_\pm$ are non-resonant terms. By looking at the resonant terms, one finds, at least formally, the phase correction terms of the form (5.11), below.

§ 5.3. Main result on (NLKG) in the complex-valued case

Let us introduce the precise statement of our result in this case. We only consider the two dimensional case. A similar result holds true for in one dimensional case (see [43, 52]). As discussed in the previous subsection, we would need to suppose that $|v_-|/|v_+|$ and $|v_+|/|v_-|$ make sense in order to extract the resonant parts. In view of (4.4) and (4.5), this corresponds to

$$\frac{|A(\xi)|}{|B(\xi)|} \quad \text{and} \quad \frac{|B(\xi)|}{|A(\xi)|}$$

make sense. Here we state our assumption in terms of A and B. Notice that it is easy to reconstruct ϕ_0 and ϕ_1 from given A and B by (1.6). For simplicity, we make a stronger assumption than that in [43].

Assumption 5.1. Let $A, B \in H^{2,2}(\mathbb{R}^2)$. Suppose that there exists a constant $\rho_0 \geqslant 1$ such that the identity

$$\tfrac{1}{\rho_0}|A(\xi)| \leqslant |B(\xi)| \leqslant \rho_0|A(\xi)| \tag{5.8}$$

holds true for all $\xi \in \mathbb{R}^2$. Suppose further that

$$\zeta(\xi) := \begin{cases} \frac{|B(\xi)|}{|A(\xi)|} & (A(\xi)\neq 0), \\ \lim\limits_{z\to\xi;\ A(z)\neq 0} \frac{|B(z)|}{|A(z)|} & (A(\xi)=0) \end{cases} \tag{5.9}$$

is well-defined for $\xi \in \operatorname{supp} A$. In addition, suppose that the above $\zeta(\xi)$ can be extended to a function on $\mathbb{R}^2$ so that it satisfies $\partial_{z_j}\zeta \in L^\infty(\mathbb{R}^2)$ and $\partial_{z_j}\partial_{z_k}\zeta \in L^\infty(\mathbb{R}^2)$ for $j,k=1,2$, and the inequality

$$\rho_0^{-1} \leqslant \inf_{\xi\in\mathbb{R}^2}\zeta(\xi) \leqslant \sup_{\xi\in\mathbb{R}^2}\zeta(\xi) \leqslant \rho_0$$

holds for the same constant $\rho_0 \geqslant 1$ as in (5.8). We abbreviate $\zeta(\xi) = |B(\xi)|/|A(\xi)|$.

Remark 9. We would emphasize that if one chooses real-valued data (ϕ_0, ϕ_1) satisfying the assumption of Theorem 4.1 then the pair (A, B) defined by (1.6) satisfies the above assumption with $\zeta \equiv 1$.

Now we are ready to sate our main result in this case.

Theorem 5.2. *Let A, B be functions satisfying Assumption 5.1 and let $\zeta(z)$ and $\rho_0 \geqslant 1$ be a ratio function and a constant, respectively, given in the assumption. Let $L_0(\zeta)$ be as in* (5.7). *For $t \geqslant 1$, define the profile $u_{\mathrm{app}}(t,x)$ by*

$$u_{\mathrm{app}}(t,x) = t^{-1}\mathbf{1}_{\{|x|<t\}}(t,x)[A(\mu)e^{i\theta+iS_A(\mu)\log t} + B(\mu)e^{-i\theta+iS_B(\mu)\log t}] \tag{5.10}$$

where $\theta = -\sqrt{t^2-|x|^2}$ and $\mu = x/\sqrt{t^2-|x|^2}$ and

$$\begin{aligned} S_A(z) &= -\tfrac{1}{2}\langle z\rangle^{-1} L_0(\zeta(z))|A(z)|, \\ S_B(z) &= \tfrac{1}{2}\langle z\rangle^{-1} L_0(1/\zeta(z))|B(z)|. \end{aligned} \tag{5.11}$$

Then, for any $b \in (0,1)$ there exists $\varepsilon_0 = \varepsilon_0(b, \rho_0) > 0$ such that if

$$\|A\|_{L^\infty(\mathbb{R}^2)} + \|B\|_{L^\infty(\mathbb{R}^2)} \leqslant \varepsilon_0 \tag{5.12}$$

then there exist $T \geqslant 1$ and a solution $u \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2) \cap L^4_t([T,\infty); W^{\frac12,4}_x)$ to (NLKG) *such that*

$$\|u(t) - u_{\mathrm{app}}(t)\|_{H^1(\mathbb{R}^2)} + \|\partial_t u(t) - \partial_t u_{\mathrm{app}}(t)\|_{L^2(\mathbb{R}^2)} + \|\langle\nabla\rangle^{\frac12}(u-u_{\mathrm{app}})\|_{L^4_{t,x}(\mathbb{R}^2)} \lesssim t^{-b}$$

for $t \geqslant T$. The solution is unique in the same sense as in Theorem 4.1. Furthermore, if the set $\{\xi \in \mathbb{R}^2 \mid |A(\xi)| \neq |B(\xi)|\}$ has positive measure, then the corresponding solution $u(t,x)$ is not real-valued.

Remark 10. If a pair of functions (A, B) satisfies Assumption 5.1 then for any constant $c \in \mathbb{C}\setminus\{0\}$ the pair (cA, cB) also does so with the exactly same constant $\rho_0 \geqslant 1$ and ratio function $\zeta(\xi)$. This ensures that if we find a data (A, B) satisfying Assumption 5.1 then we may have a data for which both Assumption 5.1 and the smallness condition (5.12) are fulfilled without changing the constant ρ_0. This respect is important because ε_0 depends on the constant ρ_0.

Let us give a comment on the proof of Theorem 5.2. The basic strategy is the same as in the real-valued case. We prove that $u_{\mathrm{app}}(t)$ defined in (5.10) satisfies the assumption of Proposition 4.2. Since (4.16) is obvious, the main part is to show (4.19). To this end, we extract the resonant parts by a similar argument as in Subsection 5.2. To handle the non-resonant part, we construct a second approximate solution $v_{\mathrm{app}}(t)$ just as in the real-valued case. For details, see [43].

§ 5.4. Representation of $L_0(\zeta)$ via hypergeometric functions

In this subsection, we give a representation of $L_0(\zeta)$ different from that given in [43] and collect basic properties. We treat general dimension $d \geqslant 1$ because the discussion in Subsection 5.2 suggests that $L_0(\zeta)$ defined by (5.7) will be useful to describes the phase modification also for $d \geqslant 3$. In the one dimensional case, we have the explicit form

$$L_0(\zeta) = 2\zeta^2 + 1. \tag{5.13}$$

Hence, in the sequel, we restrict our attention to the case $d \geqslant 2$. It will turn out there is a significant difference between $d = 1$ and $d \geqslant 2$.

For $d \geqslant 2$, we have the following formula:

Proposition 5.3. *Let $d \geqslant 2$. Let $L_0(\zeta)$ be as in (5.7) with (1.2). Then,*

$$L_0(\zeta) = \begin{cases} {}_2F_1[-1-\frac{1}{d}, -\frac{1}{d}; 1; \zeta^2] & 0 \leqslant \zeta < 1, \\ (1+\frac{1}{d})\zeta^{\frac{2}{d}}\, {}_2F_1[-\frac{1}{d}, -\frac{1}{d}; 2; \zeta^{-2}] & \zeta > 1, \end{cases} \tag{5.14}$$

where ${}_2F_1[a, b; c; x]$ $(a, b, c \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0})$ is the hypergeometric function defined by

$${}_2F_1[a, b; c; x] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$$

for $|x| < 1$. Here $(a)_k := \Gamma(a + k)/\Gamma(a)$ is the rising factorial.

Proof. For $\zeta \in [0, 1)$, we see from the generalized binomial theorem and the orthogonality of $\{e^{ik\theta}\}_{k\in\mathbb{Z}}$ that

$$\begin{aligned} L_0(\zeta) &= \frac{1}{2\pi} \int_0^{2\pi} (1 + \zeta e^{i\theta})^{1+\frac{1}{d}} (1 + \zeta e^{-i\theta})^{\frac{1}{d}}\, d\theta \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{1+\frac{1}{d}}{j} \binom{\frac{1}{d}}{k} \zeta^{j+k} \times \frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)\theta} d\theta \\ &= \sum_{k=0}^{\infty} \binom{1+\frac{1}{d}}{k} \binom{\frac{1}{d}}{k} \zeta^{2k}, \end{aligned}$$

where

$$\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$$

for $r \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$. For a non-integer r, we have

$$\binom{r}{k} = (-1)^k \frac{(-r)_k}{k!} = (-1)^k \frac{(-r)_k}{(1)_k}.$$

Thus,

$$L_0(\zeta) = \sum_{k=0}^{\infty} \frac{(-1-\frac{1}{d})_k(-\frac{1}{d})_k}{(1)_k k!}(\zeta^2)^k = {}_2F_1[-1-\tfrac{1}{d}, -\tfrac{1}{d}; 1; \zeta^2].$$

Similarly, for $\zeta > 1$,

$$\begin{aligned} L_0(\zeta) &= \zeta^{1+\frac{2}{d}} \frac{1}{2\pi} \int_0^{2\pi} |1+\zeta^{-1}e^{-i\theta}|^{\frac{2}{d}} (\zeta^{-1}e^{-i\theta}+1)e^{i\theta} d\theta \\ &= \zeta^{1+\frac{2}{d}} \frac{1}{2\pi} \int_0^{2\pi} (1+\zeta^{-1}e^{-i\theta})^{1+\frac{1}{d}} (1+\zeta^{-1}e^{i\theta})^{\frac{1}{d}} e^{i\theta} d\theta \\ &= \zeta^{1+\frac{2}{d}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{1+\frac{1}{d}}{j}\binom{\frac{1}{d}}{k} \zeta^{-(j+k)} \times \frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k-1)\theta} d\theta \\ &= \zeta^{\frac{2}{d}} \sum_{k=0}^{\infty} \binom{1+\frac{1}{d}}{k+1}\binom{\frac{1}{d}}{k} \zeta^{-2k}. \end{aligned}$$

Now we use the identity

$$\binom{1+\frac{1}{d}}{k+1} = (-1)^{k+1}\frac{(-1-\frac{1}{d})_{k+1}}{(1)_{k+1}} = \left(1+\frac{1}{d}\right) \times (-1)^k \frac{(-\frac{1}{d})_k}{(2)_k}$$

to conclude that

$$L_0(\zeta) = \left(1+\frac{1}{d}\right)\zeta^{\frac{2}{d}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{d})_k(-\frac{1}{d})_k}{(2)_k k!}\zeta^{-2k} = \left(1+\frac{1}{d}\right)\zeta^{\frac{2}{d}}{}_2F_1[-\tfrac{1}{d}, -\tfrac{1}{d}; 2; \zeta^{-2}]$$

as desired. $\square$

Proposition 5.4. *Let $d \geqslant 2$. Let $L_0(\zeta)$ be as in* (5.7) *with* (1.2). *Then, $L_0 \in (C^2(\mathbb{R}_+) \setminus C^3(\mathbb{R}_+)) \cap C^\infty(\mathbb{R}_+ \setminus \{1\})$. Further, $L_0(0) = 1$, $L_0(1) = 2^{1+\frac{2}{d}} \frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})}$, and the asymptotics*

$$L_0(\zeta) = \left(1+\frac{1}{d}\right)\zeta^{\frac{2}{d}} + O(\zeta^{\frac{2}{d}-2})$$

holds as $\zeta \to \infty$.

Proof. $L_0(0) = 1$ and the asymptotics for large ζ are obvious by the formula (5.14) and the series form definition of the hypergeometric function.

Next we consider continuity of $L_0(\zeta)$. By the smoothness of ${}_2F_1[a, b; c; x]$ for $|x| < 1$, the formula (5.14) immediately gives us $L_0 \in C^\infty(\mathbb{R}_+ \setminus \{1\})$. Let us consider the continuity at $\zeta = 1$. It is known that

$$\lim_{x\uparrow 1} {}_2F_1[a, b; c; x] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

if $c-a-b>0$ and if $c, c-a-b, c-a, c-b \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0}$. This shows

$$\lim_{\zeta \uparrow 1} L_0(\zeta) = \lim_{\zeta \downarrow 1} L_0(\zeta) = \frac{\Gamma(2+\frac{2}{d})}{\Gamma(2+\frac{1}{d})\Gamma(1+\frac{1}{d})} = 2^{1+\frac{2}{d}} \frac{\Gamma(\frac{3}{2}+\frac{1}{d})}{\sqrt{\pi}\Gamma(2+\frac{1}{d})},$$

where we have used the Legendre duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ with $z = 1+\frac{1}{d}$ to obtain the last identity. Hence, $L_0 \in C(\mathbb{R}_+)$. Further, making a use of the formula

$$\frac{d}{dx} {}_2F_1[a,b;c;x] = \frac{ab}{c} {}_2F_1[a+1,b+1;c+1;x],$$

we have

$$\lim_{\zeta \uparrow 1} L_0'(\zeta) = \frac{2}{d}\left(1+\frac{1}{d}\right) \frac{\Gamma(1+\frac{2}{d})}{\Gamma(2+\frac{1}{d})\Gamma(1+\frac{1}{d})} = \frac{2}{d} \frac{\Gamma(1+\frac{2}{d})}{\Gamma(1+\frac{1}{d})^2}$$

and

$$\lim_{\zeta \downarrow 1} L_0'(\zeta) = \frac{2}{d}\left(1+\frac{1}{d}\right) \frac{\Gamma(2+\frac{2}{d})}{\Gamma(2+\frac{1}{d})^2} - \frac{2}{d^2}\left(1+\frac{1}{d}\right) \frac{\Gamma(1+\frac{2}{d})}{\Gamma(2+\frac{1}{d})^2} = \frac{2}{d} \frac{\Gamma(1+\frac{2}{d})}{\Gamma(1+\frac{1}{d})^2},$$

which shows $L_0 \in C^1(\mathbb{R}_+)$. Similarly, a computation shows

$$\lim_{\zeta \uparrow 1} L_0''(\zeta) = \lim_{\zeta \downarrow 1} L_0''(\zeta) = \frac{4}{d^3} \frac{\Gamma(\frac{2}{d})}{\Gamma(1+\frac{1}{d})^2}$$

and so $L_0 \in C^2(\mathbb{R}_+)$. $L_0 \notin \mathbb{C}^3(\mathbb{R}_+)$ follows by definition (5.7), for instance. An alternative way to see this is to recall the fact

$${}_2F_1[a,b;c;x] \sim \begin{cases} \log \frac{1}{x} & c-b-a=0, \\ (1-x)^{c-a-b} & c-a-b<0 \end{cases}$$

as $x \uparrow 1$. By (5.14), one may see that $L_0'''(\zeta)$ contains a divergent factor ${}_2F_1[2-\frac{1}{d}, 3-\frac{1}{d}; 4; \zeta^2]$ if $\zeta < 1$ and ${}_2F_1[3-\frac{1}{d}, 3-\frac{1}{d}; 5; \zeta^{-2}]$ if $\zeta > 1$. □

Let us conclude this section by introducing another formula for $L_0(\zeta)$ for $d=2$ in terms of complete elliptic integrals.

Proposition 5.5 ([43]). *Let $d=2$. Let $L_0(\zeta)$ be as in (5.7) with (1.2). It holds that*

$$L_0(\zeta) = \frac{(1+\zeta)(7+\zeta^2)}{3\pi} E\left(\frac{2\sqrt{\zeta}}{1+\zeta}\right) - \frac{(1+\zeta)(1-\zeta)^2}{3\pi} K\left(\frac{2\sqrt{\zeta}}{1+\zeta}\right), \tag{5.15}$$

where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k) = \textstyle\int_0^{\pi/2} (1-k^2\sin^2\theta)^{-\frac{1}{2}}\, d\theta,$$

and $E(k)$ is the complete elliptic integral of the second kind defined by

$$E(k) = \textstyle\int_0^{\pi/2} (1-k^2\sin^2\theta)^{\frac{1}{2}}\, d\theta.$$

It is well-known that the complete elliptic integrals are written in terms of the hypergeometric functions as

$$K(k) = \frac{\pi}{2} {}_2F_1[\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2], \quad E(k) = \frac{\pi}{2} {}_2F_1[-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2].$$

By the Gauss transformation, one has

$$E\left(\frac{2\sqrt{\zeta}}{1+\zeta}\right) = (1+\zeta)^{-1}(2E(\zeta) + (\zeta^2-1)K(\zeta)) \quad \text{and} \quad K\left(\frac{2\sqrt{\zeta}}{1+\zeta}\right) = (1+\zeta)K(\zeta)$$

for $0 \leqslant \zeta < 1$. Notice that $\frac{2\sqrt{\zeta}}{1+\zeta}$ is invariant under $\mathbb{R}_+ \ni \zeta \mapsto 1/\zeta \in \mathbb{R}_+$. Combining these identities and the contiguous relations[2], we can reproduce the formula (5.15) from (5.14). We omit the details.

§ 6. Concluding remarks

§ 6.1. Applications to other type of problems

Extraction of a term with a specific oscillatory is one of fundamental techniques. It is useful in several problems in dispersive equations. One such example is scattering problem. Recently, sharp scattering criteria (or scattering/blowup dichotomies) are extensively studied via concentration/compactness type argument in the style of [29]. Roughly speaking, such a criterion is given by looking at a critical element, a counter example for scattering which is minimal in a suitable sense. This kinds of critical elements is often characterized as an optimizer of a minimization/maximization problem, and hence is constructed as a "limit" of an optimizing sequence. We sometimes encounter a singular limit problem as one possible scenario of the behavior of the optimizing sequence. The technique in Section 2 is used to analyze such singular limit problems. See [30, 40, 59] for isolation of NLS inside the mass-critical and -subcritical generalized KdV equation. See [5, 24, 31] for isolation of NLS inside mass-critical Klein-Gordon equation. It is also used to the study of non relativistic limit of Klein-Gordon equation in [44], as mentioned above.

§ 6.2. Related problems

We list open problems related to the topics in the present survey.

- Initial value problems. We introduce the final value problems. Some of the corresponding initial value problems are open up to now. In particular, nonlinear Klein-Gordon equation (NLKG) with the gauge-invariant nonlinearity (1.2) for $d \geqslant 2$

[2]The hypergeometric function ${}_2F_1[a, b; c; x]$ is written as a linear combination of any two of six contiguous functions ${}_2F_1[a \pm 1, b; c; x]$, ${}_2F_1[a, b \pm 1; c; x]$, and ${}_2F_1[a, b; c \pm 1; x]$.

would be an important problem. The regularity issue about the non-polynomial nonlinearity is one obvious obstacle.

- Large data problems. A Few result is available for final value problems with a large data (see [34], for instance). Large initial data would be more interesting and challenging. Especially, there must be a transition phenomenon when the equation admit a soliton. A sharp scattering criterion are given for NLS with a slightly above the critical power for $d = 1, 2$ by the author [35, 36].

- The exceptional case[3] $g_0 = 1$ for (NLS) with a general nonlinearity. The case $g_0 = 1$ is less understood. Several negative results are available [38, 53, 55]. There are also a result in the positive direction [14] (see also [19]).

§ Appendix A. Proof of Proposition 4.2

In this appendix, we prove Proposition 4.2 and its variant. Let us recall the statement, with a sight modification.

Proposition A (Proposition 4.2). *Let $d \geqslant 1$. There exists a constant $\varepsilon_0 = \varepsilon_0(d) > 0$ with the following property: If a given complex-valued profile $\widetilde{u}(t,x) \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2)$ satisfies*

$$\sup_{t \geqslant T} t^{\frac{d}{2}} \|\widetilde{u}(t)\|_{L^\infty_x} \leqslant \varepsilon_0, \tag{A.1}$$

and

$$\sup_{t \geqslant T} t^b (\|E\|_{L^\infty([t,\infty);H^1_x)} + \|\partial_t E\|_{L^\infty([t,\infty);L^2_x)} + \| \langle \nabla \rangle^{\frac{1}{2}} E\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t,\infty)\times\mathbb{R}^d)}) \leqslant \varepsilon \tag{A.2}$$

for some $T \geqslant 1$, $b \geqslant d/4$, $0 < \varepsilon \leqslant \varepsilon_0$, where E is the error term defined in (4.15)*, then there exists a unique solution $w \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2) \cap L_t^{\frac{2(d+2)}{d}}([T,\infty); W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$ to the equation* (4.14) *satisfying*

$$\sup_{t \geqslant T} t^b (\|w\|_{L^\infty([t,\infty);H^1_x)} + \|\partial_t w\|_{L^\infty([t,\infty);L^2_x)} + \| \langle \nabla \rangle^{\frac{1}{2}} w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t,\infty)\times\mathbb{R}^d)}) \leqslant 2\varepsilon. \tag{A.3}$$

Moreover, $u = \tilde{u} + w$ is an $(H^1 \times L^2)$-solution on $[T,\infty)$ to (NLKG)*. Furthermore, if $\widetilde{u}(t)$ satisfies*

$$\liminf_{t\to\infty} \|\mathrm{Im}\, \widetilde{u}(t)\|_{L^2_x(\mathbb{R}^2)} > 0 \tag{A.4}$$

in addition then $u(t)$ is not real-valued.

[3] If $g_0 \neq 0$ then we may let $g_0 = 1$ by a simple change of variable.

To prove the proposition we will recall Strichartz' estimate. We call a pair $(q,r) \in [2,\infty]^2$ is sharp Klein-Gordon admissible if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right)$$

and if $(q,r,d) \neq (2,\infty,2)$.

Lemma B. *Let T be a real number and let $\mathcal{G}[\cdot]$ be as in* (4.13). *It holds for any two sharp Klein-Gordon admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ that*

$$\left\| \langle \partial_t \rangle^{1+\frac{d+2}{2}(\frac{1}{r}-\frac{1}{2})} \mathcal{G}[f] \right\|_{L_t^q([T,\infty);L_x^r)} + \left\| \langle \nabla \rangle^{1+\frac{d+2}{2}(\frac{1}{r}-\frac{1}{2})} \mathcal{G}[f] \right\|_{L_t^q([T,\infty);L_x^r)} \lesssim_{d,q,\tilde{q}} \left\| \langle \nabla \rangle^{-\frac{d+2}{2}(\frac{1}{\tilde{r}}-\frac{1}{2})} f \right\|_{L_x^{\tilde{q}'}([T,\infty);L_x^{\tilde{r}'})}.$$

The implicit constant is independent of q and $\tilde{q}$ if $d \neq 2$.

These can be found, for instance, in [5, 31].

Proof of Proposition A. We will solve the equation

$$w(t) = \mathcal{G}[F_{\mathrm{GI}}(\tilde{u}+w) - F_{\mathrm{GI}}(\tilde{u})] + E \tag{4.14}$$

by a contraction mapping principle. Fix $T \geqslant 1$, $b \geqslant d/4$, and $\varepsilon > 0$ as numbers given by the assumption. Introduce a complete metric space X as follows: X is a set of functions $w \in C([T,\infty);H^1) \cap C^1([T,\infty);L^2) \cap L_t^{\frac{2(d+2)}{d}}([T,\infty);W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$ such that $\|w\|_X \leqslant 2\varepsilon$, where

$$\|w\|_X := \sup_{t\geqslant T} t^b (\|w\|_{L^\infty([t,\infty);H_x^1)} + \|\partial_t w\|_{L^\infty([t,\infty);L_x^2)} + \| \langle \nabla \rangle^{\frac{1}{2}} w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t,\infty)\times\mathbb{R}^d)}).$$

The metric of X is defined by $d(w_1,w_2) := \|w_1 - w_2\|_X$.

We regard the right hand side of (4.14) as a functional of w and denote by $\Phi[w]$. It suffices to show that Φ is a contraction map on X. Fix $w \in X$. Remark that

$$|F_{\mathrm{GI}}(\tilde{u}+w) - F_{\mathrm{GI}}(\tilde{u})| \lesssim |\tilde{u}|^{\frac{2}{d}}|w| + |w|^{1+\frac{2}{d}}.$$

For $d \geqslant 3$, Lemma B gives us

$$\|\mathcal{G}[F_{\mathrm{GI}}(\tilde{u}+w) - F_{\mathrm{GI}}(\tilde{u})]\|_{L^\infty([t,\infty);H^1)} \lesssim \left\| |\tilde{u}|^{\frac{2}{d}}|w| \right\|_{L_t^1([t;\infty);L_x^2)} + \left\| \langle \nabla \rangle^{\frac{d+2}{2d}} |w|^{1+\frac{2}{d}} \right\|_{L_t^2([t;\infty);L_x^{\frac{2d}{d+2}})}.$$

Remark that

$$\left\| |\tilde{u}|^{\frac{2}{d}}|w| \right\|_{L_t^1([t;\infty);L_x^2)} \lesssim \varepsilon_0^{\frac{2}{d}} \left(\sup_{t\geqslant T} t^b \|w(t)\|_{L^2} \right) \|t^{-1-b}\|_{L_t^1([t;\infty))} \leqslant \frac{1}{b} t^{-b} \varepsilon_0^{\frac{2}{d}} \|w\|_X \tag{A.5}$$

and

$$\left\|\langle\nabla\rangle^{\frac{d+2}{2d}}|w|^{1+\frac{2}{d}}\right\|_{L^2_t([t;\infty);L_x^{\frac{2d}{d+2}})} \lesssim \left(\sup_{t\geqslant T} t^b\|w(t)\|_{H^1}\right)^{1+\frac{2}{d}}\left\|t^{-b(1+\frac{2}{d})}\right\|_{L^2_t([t;\infty))}$$
$$\lesssim \left(b\left(1+\frac{2}{d}\right)-\frac{1}{2}\right)^{-\frac{1}{2}} t^{-b(1+\frac{2}{d})+\frac{1}{2}}\|w\|_X^{1+\frac{2}{d}} \tag{A.6}$$

for $t \geqslant T$. Similar estimates are valid for the rest two terms in the X norm. Hence, if $-b(1+\frac{2}{d})+\frac{1}{2}\leqslant -b \Leftrightarrow b\geqslant d/4$ then

$$\begin{aligned}\|\Phi[w]\|_X &\leqslant C_d(\varepsilon_0^{\frac{2}{d}}\|w\|_X+\|w\|_X^{1+\frac{2}{d}})+\|E\|_X\\ &\leqslant (1+2(1+2^{\frac{2}{d}})C_d\varepsilon_0^{\frac{2}{d}})\varepsilon.\end{aligned}$$

Hence, $\Phi[w]\in X$ if ε_0 is small. A similar argument shows $\Phi[w]$ is contraction for small ε_0.

For $d=1,2$, the end-point Strichartz estimate fails. So, we use another pair to handle the term $|w|^{1+\frac{2}{d}}$. In one dimension, we take $(\tilde{q},\tilde{r})=(4,\infty)$ and use the estimate

$$\begin{aligned}\left\|\langle\nabla\rangle^{\frac{3}{4}}|w|^3\right\|_{L_t^{\frac{4}{3}}([t;\infty);L^1_x)} &\lesssim \left(\sup_{t\geqslant T}t^b\|w(t)\|_{H^1}\right)^{\frac{3}{2}}\left\|t^{-\frac{3}{2}b}\right\|_{L^2_t([t;\infty))}\|w\|^{\frac{3}{2}}_{L^6_{t,x}([t,\infty)\times\mathbb{R})}\\ &\lesssim (3b-1)^{-\frac{1}{2}}t^{-3b+\frac{1}{2}}\|w\|_X^3,\end{aligned}$$

which is acceptable if $-3b+\frac{1}{2}\leqslant -b\Leftrightarrow b\geqslant 1/4$. In two dimensions, we take $(\tilde{q},\tilde{r})=(3,6)$ and use the estimate

$$\begin{aligned}\left\|\langle\nabla\rangle^{\frac{2}{3}}|w|^2\right\|_{L_t^{\frac{3}{2}}([t;\infty);L_x^{\frac{6}{5}})} &\lesssim \left(\sup_{t\geqslant T}t^b\|w(t)\|_{H^1}\right)^{\frac{4}{3}}\left\|t^{-\frac{4}{3}b}\right\|_{L^2_t([t;\infty))}\|w\|^{\frac{2}{3}}_{L^4_{t,x}([t,\infty)\times\mathbb{R})}\\ &\lesssim \left(b-\tfrac{3}{8}\right)^{-\frac{1}{2}}t^{-2b+\frac{1}{2}}\|w\|_X^2,\end{aligned}$$

which is acceptable if $-2b+\frac{1}{2}\leqslant -b\Leftrightarrow b\geqslant 1/2$.

The last criterion for being a non real-valued solution is obvious. □

One also have a version with which we can handle the large data case.

Proposition C. *Let $d\geqslant 1$. Let $T\geqslant 1$. Let $\widetilde{u}(t,x)\in C([T,\infty);H^1)\cap C^1([T,\infty);L^2)$ be a given complex-valued profile. If*

$$\sup_{t\geqslant T}t^{\frac{d}{2}}\|\widetilde{u}(t)\|_{L^\infty_x}\leqslant R \tag{A.7}$$

is true for some constant $R>0$ then there exists $b_0=b_0(R,d)\geqslant d/4$ with the following property: If the error term E defined in (4.15) *satisfies*

$$\sup_{t\geqslant T}t^{b_0}(\|E\|_{L^\infty([t,\infty);H^1_x)}+\|\partial_t E\|_{L^\infty([t,\infty);L^2_x)}+\|\langle\nabla\rangle^{\frac{1}{2}}E\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t,\infty)\times\mathbb{R}^d)})\leqslant 1 \tag{A.8}$$

then there exists a unique solution

$$w \in C([T,\infty); H^1) \cap C^1([T,\infty); L^2) \cap L_t^{\frac{2(d+2)}{d}}([T,\infty); W_x^{\frac{1}{2},\frac{2(d+2)}{d}})$$

to the equation (4.14) *satisfying*

$$\sup_{t\geqslant T} t^{b_0}(\|w\|_{L^\infty([t,\infty);H^1_x)} + \|\partial_t w\|_{L^\infty([t,\infty);L^2_x)} + \| \langle\nabla\rangle^{\frac{1}{2}} w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t,\infty)\times\mathbb{R}^d)}) \leqslant 2. \tag{A.9}$$

Moreover, $u = \tilde{u} + w$ *is an* $(H^1 \times L^2)$*-solution on* $[T,\infty)$ *to* (NLKG)*. Furthermore, if* $\widetilde{u}(t)$ *satisfies* (A.4) *in addition then* $u(t)$ *is not real-valued.*

Proof. The proof is essentially the same as in that of Proposition A. Notice that the right hand sides of (A.5) and (A.6) have the factor b^{-1} and $(b(1+\frac{2}{d})-\frac{1}{2})^{-1/2}$, respectively. Hence, we have the estimate of the form

$$\|\Phi[w]\|_X \leqslant C(b^{-1}R^{\frac{2}{d}} + b^{-\frac{1}{2}}\|w\|_X^{\frac{2}{d}})\|w\|_X + \|E\|_X$$

for $b \geqslant d/4$. Thus, we are able to find a unique solution by the contraction mapping principle if b is sufficiently large. □

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