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A trial to construct specific self-similar solutions to non-linear wave equations

By

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Abstract

An attempt to construct self-similar solutions to nonlinear wave equations $\Box u = |u|^p$ is explained. The existence of self-similar solutions has been already established by Pecher [8, 9], Kato-Ozawa [5, 6], etc, based on the standard fixed point theorem. In this note, we will discuss it by a constructive method using the theory of hypergeometric differential equations.

§ 1. Self-similar solution of nonlinear wave equations

We consider nonlinear wave equations of power type nonlineality

$$(\partial_t^2 - \Delta_x)u(t,x) = |u(t,x)|^p, \quad (t,x) \in \mathbf{R}_t \times \mathbf{R}_x^n \tag{NLW}$$

with $1 < p < \infty$. If $u(t,x)$ is a solution to (NLW), then the scaled function

$$u_\lambda(t,x) = \lambda^{2/(p-1)} u(\lambda t, \lambda x)$$

is also a solution to (NLW) whenever $\lambda \neq 0$. A solution $u(t,x)$ to (NLW) is said to be *self-similar* if

$$u(t,x) \equiv u_\lambda(t,x)$$

for any $\lambda \neq 0$.

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We discuss the existence of self-similar solution for as larger range of p as possible. To state some known results, we introduce three critical indices:

- $p_k(n) = \dfrac{n+1}{n-1}$

- $p_{str}(n) = \dfrac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$

- $p_{conf}(n) = \dfrac{n+3}{n-1}$

We note that $p_k(n) < p_{str}(n) < p_{conf}(n)$ and $p_k(n)$ is the lower bound of p for the existence of weak global solutions ([7]) while $p_{str}(n)$ is for strong one ([2], [4], [12], [13], [14], [15]). The index $p_{conf}(n)$ is so-called *conformal critical* power.

In the case $n = 3$, Pecher [9] showed the existence of self-similar solution for $p_{str}(n) < p < p_{conf}(n)$. Hidano [3], Kato-Ozawa [5, 6] generalize Pecher's result to the case $n \geq 2$. There are some attempts for $p_{conf} < p$ by Pecher [8], Planchon [10], Ribaud-Youssfi [11], de Almeida-Ferreira [1]. Their basic strategy is to show the existence and the uniqueness of solution $u(t,x)$ to (NLW) with the initial data

$$u(0,x) = \varepsilon|x|^{-2/(p-1)}, \quad \partial_t u(0,x) = \varepsilon|x|^{-2/(p-1)-1}$$

for small $\varepsilon > 0$ by the fixed point theorem on appropriate function spaces. Then the scaled function $u_\lambda(t,x)$ has the same initial data, and as a result of the uniqueness, we conclude that $u(t,x) \equiv u_\lambda(t,x)$.

§ 2. Reduction to hypergeometric differential euations

Our goal is to give a constructive proof of the existence of self-similar solutions. The definition of self-similarity

$$u(t,x) \equiv u_\lambda(t,x) := \lambda^{2/(p-1)} u(\lambda t, \lambda x)$$

with $\lambda = 1/t$ implies

$$u(t,x) = t^{-2/(p-1)} u(1, x/t).$$

Hence $u(t,x)$ has to be of the form

$$u(t,x) = t^{-\beta}\varphi(x/t), \quad \beta = 2/(p-1).$$

Conversely, such $u(t,x)$ is self-similar. Plugging it into (NLW), we have

$$\beta(\beta+1)\varphi(y) + 2(\beta+1)y\cdot\nabla\varphi(y) - \Delta\varphi(y) + \langle\varphi'' y, y\rangle = |\varphi(y)|^p,$$

hence for radially symmetric solution $\varphi(y) = \psi(|y|)$ we have

$$(r^2 - 1)\psi_{rr} + \left(2(\beta+1)r - \frac{n-1}{r}\right)\psi_r + \beta(\beta+1)\psi = |\psi|^p.$$

Then by further transformation $\psi(r) = f(r^2)$, we have reached the equation

$$4s(s-1)f_{ss} + 2\{(2\beta+3)s - n\}f_s + \beta(\beta+1)f = |f|^p$$

or equivalently

$$\text{(NLHG)} \qquad L_{a,b,c}f(s) = \frac{1}{4}|f(s)|^p$$

where $s(= r^2) \geq 0$ and

$$L_{a,b,c} = s(s-1)\frac{d^2}{ds^2} + \{(a+b+1)s - c\}\frac{d}{ds} + ab$$

with

$$\begin{aligned} a &= \beta/2 = 1/(p-1), \\ b &= (\beta+1)/2 = 1/(p-1) + 1/2, \\ c &= n/2. \end{aligned}$$

This ordinary differential equation can be regarded as a nonliner perturbation of *hypergeometric differential equation*

$$\text{(HG)} \qquad L_{a,b,c}h = 0$$

We note that *hypergeometric function*

$$h_{a,b,c}(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!}$$

is a solution to (HG) with $h_{a,b,c}(0) = 1$ and $h_{a,b,c}(s) > 0$ for $s \geq 0$.

§ 3. Symmetric property of (NLHG)

Let us find self-similar solutions to (NLHG) of the form

$$f(s) = \varepsilon h(s)g(s),$$

where h is a solution to (HG), e.g. $h = h_{a,b,c}$, and $\varepsilon > 0$. Plugging it into (NLHG), we have

$$\text{(G)} \qquad s(s-1)g_{ss} + \left\{(a+b+1)s - c + 2s(s-1)\frac{h_s}{h}\right\}g_s = \frac{1}{4}(\varepsilon h)^{p-1}|g|^p.$$

When $\varepsilon = 0$, apparently all constant functions are solutions to (G). Then it is natural to expect the existence of a bounded solution $g(s) > 0$ for sufficiently small $\varepsilon > 0$.

Indeed, it can be justified by the symmetric property of (NLHG), that is, if we set

$$S : f(s) \mapsto -s^{-a} f(1/s), \qquad T : f(s) \mapsto f(1-s)$$

then we have the commutative diagram

$$\begin{array}{ccc}
L_{a,b,c} f = \frac{1}{4}|f|^p & \xrightarrow{S} & L_{a,a-c+1,a-b+1} f = \frac{1}{4}|f|^p \\
{\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle R=S^{-1}TS} \\
L_{a,b,a+b-c+1} f = \frac{1}{4}|f|^p & \xrightarrow{S} & L_{a,c-b,a-b+1} f = \frac{1}{4}|f|^p.
\end{array}$$

We note that S and T causes the interval shift $[0,1] \overset{S}{\longleftrightarrow} [1,\infty]$, and singular points shift $s = 0 \overset{T}{\longleftrightarrow} s = 1$, respectively. Then by this symmetry, it suffices to construct the solution to (G) only near $s = 0$, and the solution can be given as a power series.

§ 4. Tentative self-similar solution

Summarising the argument so far, a self-similar solution is *tentatively* constructed in the form of

$$u(t,x) = \varepsilon\, t^{-2/(p-1)} h(|x|^2/t^2)\, g(|x|^2/t^2),$$

where h is a solution to (HG) and g a bounded function. The final thing to be justified is that the nonlinear term $|u(t,x)|^p$ has a meaning as a distribution, e.g. $u \in L^p_{loc}(\mathbf{R}^{n+1})$. For example, when $n - 2 - 2/(p-1) > 0$ or equivalently $p > n/(n-2)$, the *stable* self-similar solution

$$u(t,x) = c_{n,p}|x|^{-2/(p-1)},$$
$$c_{n,p} = \{4a(a-c+1)\}^a = \left\{\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right\}^{1/(p-1)}$$

to (NLW) satisfies $u \in L^p_{loc}(\mathbf{R}^{n+1})$ since $-2p/(p-1) > -n$. Then it can be understood as a distribution.

Unfortunately, it is not always the case for our construction. Indeed, we have a tentative solution of the form

$$u(t,x) = d_{n,p}|t^2 - |x|^2|^{-1/(p-1)-1}(t^2 - |x|^2),$$
$$d_{n,p} = \{4a(c-b)\}^a = \left\{\frac{2}{p-1}\left(n-1-\frac{2}{p-1}\right)\right\}^{1/(p-1)}$$

when $n-1-2/(p-1)>0$ or equivalently $p>(n+1)/(n-1)$. It can be obtained from the stable self-similar solution above if we use the symmetric property of (NLHG) again. But $u \in L^p_{loc}(\mathbf{R}^{n+1})$ is equivalent to $-p/(p-1)>-1$, which is imposible.

§ 5. The case $n = 3$

In the case when $n=3$, we can justify that our tentative self-similar solutions make sense as distribution. Indeed in this case, the corresponding hypergeometric function is given by an elementary function, that is

$$h(s) = \begin{cases} \frac{(1+\sqrt{s})^{\frac{p-3}{p-1}} - (1-\sqrt{s})^{\frac{p-3}{p-1}}}{\sqrt{s}} & \text{for } 0<s<1 \\ \\ \frac{(\sqrt{s}+1)^{\frac{p-3}{p-1}} + (\sqrt{s}-1)^{\frac{p-3}{p-1}}}{\sqrt{s}} & \text{for } s>1 \end{cases}$$

is a solution to (HG) with $a=1/(p-1)$, $b=1/(p-1)+1/2$, $c=n/2$ for $p \neq 3$. For $p=3$, we need a modification:

$$h(s) = \begin{cases} \frac{\log(1+\sqrt{s}) - \log(1-\sqrt{s})}{\sqrt{s}} & \text{for } 0<s<1 \\ \\ \frac{\log(\sqrt{s}+1) + \log(\sqrt{s}-1)}{\sqrt{s}} & \text{for } s>1 \end{cases}$$

We remark that $h(s)$ is singular only at $s=1$ for $1<p\leq 3$ while $h(s)$ has no singularity for $p>3$. Then a self-similar solution is constructed by

$$u(t,x) = \varepsilon\, t^{-2/(p-1)} \underbrace{h(|x|^2/t^2)}_{sing:|x|=|t|} \underbrace{g(|x|^2/t^2)}_{bdd}.$$

If we require $u \in L^p_{loc}(\mathbf{R}^{n+1})$, it is equivalent to

$$\begin{aligned} h(|x|^2) \in L^p_{loc}(\mathbf{R}^n) &\Longleftrightarrow p\,\frac{p-3}{p-1} > -1 \\ &\Longleftrightarrow p > p_{str}(3) = 1+\sqrt{2}. \end{aligned}$$

If we further require $u(t,\cdot) \in L^p(\mathbf{R}^n)$, we also need

$$p\left(1-\frac{p-3}{p-1}\right) > 3 \Longleftrightarrow p < p_{conf}(3) = 3.$$

This is exactly the self-similar solution which Pecher [9] captured!

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