TITLE:

Hasse-Weil zeta functions of character varieties of hyperbolic 3-manifolds (Algebraic Number Theory and Related Topics 2017)

AUTHOR(S):

Harada, Shinya

CITATION:

Harada, Shinya. Hasse-Weil zeta functions of character varieties of hyperbolic 3-manifolds (Algebraic Number Theory and Related Topics 2017). 数理解析研究所講究録別冊 2020, B83: 3-10

ISSUE DATE:

2020-10

URL:

http://hdl.handle.net/2433/260685

RIMS Kôkyûroku Bessatsu
B83 (2020), 003–010

Hasse-Weil zeta functions of character varieties of hyperbolic 3-manifolds

By

Shinya Harada*

Abstract

The $\mathrm{SL}_2(\mathbb{C})$-character variety of a 3-manifold plays an important role in the study of 3-dimensional topology, which is known to be an algebraic set over the rational number field. This is a survey of the study of SL_2-character varieties over $\mathbb{Q}$ and their zeta functions. In particular, there is an explicit relationship between special values of zeta functions at $s = 2$ and hyperbolic volumes for closed arithmetic hyperbolic 3-manifolds.

§ 1. Introduction

The $\mathrm{SL}_2(\mathbb{C})$-character variety of a 3-manifold is known to be a powerful tool for the study of 3-manifolds. It was clarified by the work of Culler-Shalen in [4] in the 1980s that it can be used to construct essential surfaces in the 3-manifolds. Recently the importance of the study of character varieties of higher dimensional representations, namely the $\mathrm{SL}_n(\mathbb{C})$-character variety of a 3-manifold has been revealed (see [7],[6]). Hence it is still an active area of research in Topology.

$\mathrm{SL}_2(\mathbb{C})$-character varieties of 3-manifolds are algebraic varieties defined by a finite number of polynomials with rational coefficients (precisely they are affine algebraic sets). Therefore it is natural to study the structure of $\mathrm{SL}_2(\mathbb{C})$-character varieties in algebraic geometric/number theoretic way. However it is complicated in general. For instance, the dimension of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of knot complements in the 3-sphere is not bounded.

Received April 9, 2018. Revised January 18, 2019.
2010 Mathematics Subject Classification(s): Primary 57M27; Secondary 57M05, 11R42.
Key Words: *Hyperbolic 3-manifold,* $\mathrm{SL}_2(\mathbb{C})$*-character variety, zeta function, special value, hyperbolic volume.*
*Tokyo Denki University, Tokyo 120-8551, Japan.
e-mail: harada@mail.dendai.ac.jp

When M is an orientable complete hyperbolic 3-manifold of finite volume, the $\mathrm{SL}_2(\mathbb{C})$-character variety of M has a special irreducible component (canonical component) containing the character of (a lift of) the holonomy representation of M. In fact, the dimension of the canonical component is equal to the number of cusps of M. Hence we can expect that the canonical component would have encode considerable amount of information on topology/geometry of M.

In this note we give a survey of the study of the $\mathrm{SL}_2(\mathbb{C})$-character varieties and the Hasse-Weil zeta functions of them for some classes of hyperbolic 3-manifolds.

This is based on the talk given by the author at the workshop "Algebraic Number Theory and Related Topics 2017". He would like to appreciate the organizers of the workshop for giving this opportunity. He is grateful to the referee for his/her careful reading of the original manuscript, which made the article more accessible and sophisticated.

§ 2. Character varieties

§ 2.1. SL_2-character variety

Let M be a compact 3-manifold. The $\mathrm{SL}_2(\mathbb{C})$-character variety $X(M)(\mathbb{C})$ of M is the set of characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M)$:

$$X(M)(\mathbb{C}) = \{\text{characters of } \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}.$$

(For basic properties and applications of $\mathrm{SL}_2(\mathbb{C})$-character varieties, see Shalen's article in [5].) It is known that $X(M)(\mathbb{C})$ is an algebraic set defined over $\mathbb{Q}$. Hence there exists a unique affine reduced scheme of finite type over $\mathbb{Q}$ such that the set of its $\mathbb{C}$-rational points is equal to $X(M)(\mathbb{C})$. In this note we denote this affine reduced scheme by $X(M)$ and call it the SL_2-character variety of M.

When M is an orientable complete hyperbolic 3-manifold (see [2] for the basics on hyperbolic 3-manifolds), there is a representation $\rho_M : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ associated with the complete hyperbolic structure of M (it is called the holonomy representation of M). Let $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be a lift of ρ_M and $\chi_\rho = \mathrm{Tr}\,\rho : \pi_1(M) \to \mathbb{C}$ its character in $X(M)(\mathbb{C})$.

Definition 2.1. A canonical component $X_0(M)$ of $X(M)$ is an irreducible component of $X(M)$ containing the point corresponding to χ_ρ.

In general, the behavior of the dimension of the SL_2-character variety of M is complicated. For example, when M is a knot complement in the 3-sphere S^3, the character variety $X(M)$ may have any dimension (cf. [16]). However the following is known for hyperbolic 3-manifolds of finite volume:

Theorem 2.2 (cf. [5] Chapter 19 §4.5, [17]). *Let M be an orientable complete hyperbolic 3-manifold of finite volume with n cusps. Then we have*

$$\dim X_0(M) = n.$$

§ 2.2. (Invariant) trace field

Let M be an orientable complete hyperbolic 3-manifold and $\rho_M : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ its holonomy representation. If we take a lift ρ of ρ_M, then we can consider the trace field

$$T_M := \mathbb{Q}(\mathrm{Tr}\,\rho) = \mathbb{Q}(\mathrm{Tr}\,\rho(g) \mid g \in \pi_1(M)),$$

the invariant trace field

$$\mathrm{Inv}T_M := \mathbb{Q}(\mathrm{Tr}\,\rho^2) = \mathbb{Q}(\mathrm{Tr}\,\rho(g^2) \mid g \in \pi_1(M)) = \mathbb{Q}(\mathrm{Tr}\,\rho(g)^2 \mid g \in \pi_1(M))$$

and the invariant quaternion algebra

$$A(M) := \mathrm{Inv}T_M[(\rho(\pi_1(M))^2] = \left\{ \sum_{\text{finite sum}} a_i\gamma_i \mid a_i \in \mathrm{Inv}T_M,\ \gamma_i \in \rho(\pi_1(M))^2 \right\}.$$

They are independent of the choice of a lift of ρ_M. Moreover, $\mathrm{Inv}T_M$ and $A(M)$ are commensurable invariants for hyperbolic 3-manifolds. Namely if M' is a finite cover of M, then $\mathrm{Inv}T_{M'}$ (resp. $A(M')$) is isomorphic to $\mathrm{Inv}T_M$ (resp. $A(M)$).

Moreover, when M is a hyperbolic 3-manifold of finite volume, the Mostow-Prasad Rigidity theorem implies that the trace field T_M is a finite extension field of $\mathbb{Q}$. (In fact, T_M is a topological invariant of M.) Here we list some properties of the trace field and the invariant trace field (for details, see [15], Chapter 3).

- $T_M/\mathrm{Inv}T_M$ is an elementary 2-abelian extension.
- A number field K is an invariant trace field of some hyperbolic 3-manifold if and only if K has exactly one complex place.

When M is a hyperbolic link complement in S^3, the invariant trace field $\mathrm{Inv}T_M$ is equal to the trace field T_M. However when M is a closed hyperbolic 3-manifold, they are different in general.

The invariant trace field and the associated invariant quaternion algebra of a hyperbolic 3-manifold are important tools for the study of arithmetic hyperbolic 3-manifolds. For details, see §2.4 and [15], Chapter 8–9.

Invariant trace fields are characterized as number fields with exactly one complex place. They also have the following characterization.

Theorem 2.3 ([3], Corollary 1.2). *Let K, K' be number fields with exactly one complex place. Then K and K' are isomorphic if and only if $\zeta(K,s) = \zeta(K',s)$ (where $\zeta(K,s)$ is the Dedekind zeta function of K).*

§ 2.3. $\mathrm{PSL}_2(\mathbb{C})$-character variety

Let M be an orientable complete hyperbolic 3-manifold. Let $C_2 := \{\pm 1\}$ be the group of order 2. For any element ϵ of the cohomology group $H^1(\pi_1(M), C_2) = \mathrm{Hom}\,(\pi_1(M), C_2)$, ϵ acts on $\chi_\rho \in X(M)(\mathbb{C})$ by

$$\epsilon \cdot \chi_\rho(g) := \epsilon(g) \cdot \chi_\rho(g) = \chi_{\epsilon\cdot\rho}(g).$$

When ρ is a lift of the holonomy representation ρ_M of M, each $\epsilon \cdot \chi_\rho$ defines a different point in $X(M)(\mathbb{C})$ and corresponds to a lift of ρ_M.

There are some ways to define a $\mathrm{PSL}_2(\mathbb{C})$-character for a $\mathrm{PSL}_2(\mathbb{C})$-representation $\overline{\rho} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. In this note, for any $\mathrm{PSL}_2(\mathbb{C})$-representation $\overline{\rho} : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$, we define the $\mathrm{PSL}_2(\mathbb{C})$-character of $\overline{\rho}$ by

$$\overline{\chi}_{\overline{\rho}}(g) := \mathrm{Tr}\,\overline{\rho}(g)^2.$$

For basic properties of $\mathrm{PSL}_2(\mathbb{C})$-characters, see Heusener and Porti's article [11]. In particular, Heusener and Porti have showed the following

Theorem 2.4 ([11], Proposition 4.2). *There is a bijection*

$$X(M)(\mathbb{C})/H^1(\pi_1(M), C_2) \xrightarrow{\sim} \overline{X(M)}(\mathbb{C}).$$

Here the set $\overline{X(M)}(\mathbb{C})$ is defined by

$$\overline{X(M)}(\mathbb{C}) := \left\{ \begin{array}{c} \mathrm{PSL}_2(\mathbb{C})\textit{-characters of } \pi_1(M) \\ \textit{whose associated } \mathrm{PSL}_2(\mathbb{C})\textit{-representations lift to } \mathrm{SL}_2(\mathbb{C}) \end{array} \right\}.$$

It is well-known that the holonomy representation ρ_M of a hyperbolic 3-manifold always lifts to a $\mathrm{SL}_2(\mathbb{C})$-representation. However a representation of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbb{C})$ does not necessarily lift to $\mathrm{SL}_2(\mathbb{C})$. For instance, when M is a hyperbolic knot complement, any $\mathrm{PSL}_2(\mathbb{C})$-representation of $\pi_1(M)$ lifts to $\mathrm{SL}_2(\mathbb{C})$ (cf. [14], §2.1.2, [11], §4 Example 4.6). Hence $\overline{X(M)}(\mathbb{C})$ equals the set of all the $\mathrm{PSL}_2(\mathbb{C})$-characters of M. When M is a closed hyperbolic 3-manifold, they are different in general.

§ 2.4. Arithmetic hyperbolic 3-manifold

Let K be a number field with exactly one complex place and A a quaternion algebra over K which is ramified at all real places. Let $\rho : A \to \mathrm{M}(2, \mathbb{C})$ be a K-embedding of

A and $\mathcal{O} \subset A$ a (maximal) order of A. Denote by $\mathcal{O}^1$ the subgroup of the unit group $\mathcal{O}^\times$ with reduced norm 1. Finally, let $P : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ be the projection.

A complete orientable hyperbolic 3-manifold M of finite volume is called arithmetic when its fundamental group $\pi_1(M)$ (considered in $\mathrm{PSL}_2(\mathbb{C})$ by the holonomy representation $\rho_M : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$) is commensurable with such $P(\rho(\mathcal{O}^1))$.

For example, if the image of the holonomy representation ρ_M of a hyperbolic 3-manifold M is contained in $\mathrm{PSL}_2(\mathcal{O}_{\mathrm{Inv}T_M})$, then M is arithmetic.

Theorem 2.5 ([15], Theorem 8.4.1). *Arithmetic 3-manifolds are classified by the pairs (K, A) of invariant trace fields K and invariant quaternion algebras A/K (in fact, the pair $(\mathrm{Inv}T_M, A(M))$ is a complete commensurable invariant for arithmetic 3-manifolds).*

§ 3. Closed case

For closed orientable complete hyperbolic 3-manifolds of finite volume, we have obtained the structure of their canonical components and therefore their zeta functions.

Theorem 3.1 ([10], Theorem 2). *Let M be a closed orientable complete hyperbolic 3-manifold of finite volume. The canonical component $X_0(M)$ is unique as a closed subscheme of $X(M)$, which does not depend on the choice of a lift of the holonomy representation $\rho_M : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$, and $X_0(M)$ is isomorphic to $\mathrm{Spec}\, T_M$ of the trace field T_M. Therefore the Hasse-Weil zeta function $\zeta(X_0(M), s)$ is equal to the Dedekind zeta function $\zeta(T_M, s)$.*

Theorem 3.2 ([10], Theorem 3). *Let M be a closed orientable complete hyperbolic 3-manifolds of finite volume. Let $C_2 := \{\pm 1\}$ be the finite group of order 2 and $H^1(\pi_1(M), C_2) := \mathrm{Hom}\,(\pi_1(M), C_2)$. The group $H^1(\pi_1(M), C_2)$ acts on $X_0(M)$ and the quotient scheme $\overline{X}_0(M) := X_0(M)/H^1(\pi_1(M), C_2)$ is isomorphic to $\mathrm{Spec}\,(\mathrm{Inv}T_M)$ of the invariant trace field $\mathrm{Inv}T_M$. Therefore the Hasse-Weil zeta function $\zeta(\overline{X}_0(M), s)$ is equal to the Dedekind zeta function $\zeta(\mathrm{Inv}T_M, s)$.*

This result and the Borel's volume formula imply our main result which describes the relation between the special value of the zeta function of the above quotient scheme at $s = 2$ and the hyperbolic volume of an arithmetic closed 3-manifold.

Theorem 3.3 ([10], Theorem 5). *Let M be an arithmetic closed orientable complete hyperbolic 3-manifold of finite volume. Then the special value $\zeta(\overline{X}_0(M), 2)$ is expressed in terms of the hyperbolic volume $\mathrm{Vol}\,(M)$, discriminant $\Delta_{\mathrm{Inv}T_M}$ and π as follows:*

$$\zeta(\overline{X}_0(M), 2) \sim_{\mathbb{Q}^\times} \frac{(4\pi^2)^{[\mathrm{Inv}T_M:\mathbb{Q}]-1}\mathrm{Vol}\,(M)}{|\Delta_{\mathrm{Inv}T_M}|^{3/2}},$$

where $\sim_{\mathbb{Q}^\times}$ means the equality holds up to a rational number.

§ 4. Cusped case

There are several results on $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$-character varieties of cusped hyperbolic 3-manifolds (mainly on explicit descriptions of defining polynomials of character varieties). However it is still difficult to study their algebraic/arithmetic geometric properties. Here we list some known cases of infinite families of cusped hyperbolic 3-manifolds.

- A certain infinite family of two-bridge knot/link (called double twist knot/link) complements (contains twist knots) (Macasieb, Petersen and van Luijk [14]). They have studied the genus of the canonical component of the $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$-character varieties, the degree of the defining polynomials of the canonical components and some topological applications.
- A certain infinite family of once-punctured torus bundles (Baker and Petersen [1]). They have obtained similar results to those in the above case on the structure of the canonical components of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$-character varieties.

In particular, in [14] $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$-character varieties of hyperbolic twist knot complements were studied. Their canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character varieties are hyperelliptic curves (which may be $\mathbb{P}^1(\mathbb{C})$). On the other hands, canonical components of $\mathrm{PSL}_2(\mathbb{C})$-character varieties of hyperbolic twist knot complements are always isomorphic to $\mathbb{P}^1(\mathbb{C})$. Hence it is not expected to obtain some relations between hyperbolic volumes of hyperbolic twist knot complements and special values of zeta functions of $\mathrm{PSL}_2(\mathbb{C})$-character varieties of them unlike the closed hyperbolic 3-manifold case. Note that in the above two cases the defining polynomials of the canonical components of $\mathrm{SL}_2(\mathbb{C})$-character varieties are expressed in terms of Chebyshev polynomials.

We have given explicit descriptions of SL_2-character varieties and zeta functions of the figure 8 knot complement and arithmetic hyperbolic two-bridge link complements in [8], [9]. In the description of their zeta functions, somehow their Dedekind zeta functions of real number fields appear. Since in these cases their trace fields (they are also invariant trace fields) are imaginary quadratic fields, there seems no direct relation between the trace fields and the real number fields appeared in the description of the zeta functions. (In fact, these number fields are real cyclotomic fields.) Hence the structure of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$-character varieties of cusped hyperbolic 3-manifolds seems quite different from those of closed hyperbolic 3-manifolds.

We note that there are also results on the number of SL_2-representations of $\pi_1(M)$ when M is a torus knot complement (hence M is not hyperbolic) over a finite field ([12], [13]).

§ 5. Questions

- The canonical component $X_0(M)/\mathbb{Q}$ is unique when M is closed. Is a canonical component $X_0(M)/\mathbb{Q}$ unique for a cusped hyperbolic 3-manifold M?

- There is a characterization of invariant trace fields as number fields. Characterize trace fields T_M as number fields.

- As well as invariant trace fields case, does the following equality hold for trace fields for hyperbolic closed 3-manifolds M, M'?

$$T_M \tilde{\to} T_{M'} \Longleftrightarrow \zeta(T_M, s) = \zeta(T_{M'}, s).$$

- What is the topological meaning of the special value of $\zeta(T_M, s)$ for a closed hyperbolic 3-manifold M?

- Study the quotient $\dfrac{\zeta(T_M, s)}{\zeta(\mathrm{Inv}T_M, s)}$ for a closed hyperbolic 3-manifold M.

References

[1] K. L. Baker and K. L. Petersen, *Character varieties of once-punctured torus bundles with tunnel number one*, Internat. J. Math. **24** (2013), 57 pp.

[2] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, 1992.

[3] T. Chinburg, E. Hamilton, D. D. Long, and A. W. Reid, *Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds*, Duke Math. J. **145** (2008), 25–44.

[4] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. Math. (2) **117** (1983), 109–146.

[5] R. J. Daverman and R. B. Sher (eds.), *Handbook of geometric topology*, North-Holland, 2002.

[6] S. Friedl, T. Kitayama, and M. Nagel, *Representation varieties detect essential surfaces*, Math. Res. Lett. **25** (2018), 803–817.

[7] T. Hara and T. Kitayama, *Character varieties of higher dimensional representations and splittings of 3-manifolds*, arXiv:1410.4295.

[8] S. Harada, *Hasse-Weil zeta function of absolutely irreducible* SL_2*-representations of the figure 8 knot group*, Proc. Amer. Math. Soc. **139** (2011), 3115–3125.

[9] ———, *Hasse-Weil zeta functions of* SL_2*-character varieties of arithmetic two-bridge link complements*, Finite Fields Appl. **27** (2014), 115–129.

[10] ———, *Hasse-Weil zeta functions of* SL_2*-character varieties of closed orientable hyperbolic* 3*-manifolds*, arXiv:1512.07747.

[11] M. Heusener and J. Porti, *The variety of characters in* $\mathrm{PSL}_2(\mathbb{C})$, Bol. Soc. Mat. Mexicana **10** (2004), 221–237.

[12] W. Li and L. Xu, *Counting* $\mathrm{SL}_2(\mathbf{F}_{2^s})$ *representations of torus knot groups*, Acta Math. Sin. **19** (2003), 233–244.

[13] ———, *Counting* $\mathrm{SL}_2(\mathbf{F}_q)$*-representations of torus knot groups*, J. Knot Theory Ramifications **13** (2004), 401–426.

[14] M. L. Macasieb, K. L. Petersen, and R. M. van Luijk, *On character varieties of two-bridge knot groups*, Proc. Lond. Math. Soc. (3) **103** (2011), 473–507.

[15] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Grad. Texts in Math., **219**, Springer-Verlag, 2003.

[16] L. Paoluzzi and J. Porti, *Non-standard components of the character variety for a family of Montesinos knots*, Proc. Lond. Math. Soc. (3) **107** (2013), 655–679.

[17] W. P. Thurston, *Three-dimensional geometry and topology*, Princeton Math. Series, **35**, Princeton Univ. Press, 1997.