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A period-ring-valued gamma function and a refinement of the reciprocity law on Stark units

By

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Abstract

This is an announcement of the results of the paper "On a common refinement of Stark units and Gross-Stark units". We study a relation between CM-periods, multiple gamma functions, the rank one abelian Stark conjecture, and their p-adic analogues. The main results are as follows. First we construct two kinds of period-ring valued functions under a slight generalization of Hiroyuki Yoshida's conjecture on "Absolute CM-periods". Here the period ring is in the sense of p-adic Hodge theory. Then we conjecture a reciprocity law on their special values concerning the absolute Frobenius action on Fontaine's period ring B_{cris}. We show that our conjecture implies a part of Stark's conjecture and a refinement of Gross' p-adic analogue simultaneously. We also provide some partial results for our conjecture.

§ 1. Introduction

First we recall some key words in order to explain the results in [10].

CM-periods. Let K be a CM-field, A an abelian variety defined over $\overline{\mathbb{Q}}$ with CM by K. Namely, we may identify

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = K.$$

We take its non-zero differential form ω_σ of the second kind where $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = K$ acts as $\sigma(K)$ for each $\sigma \in \mathrm{Hom}(K, \mathbb{C})$. Then the integrated value

$$\int_\gamma \omega_\sigma \in \mathbb{C}$$

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is called a CM-period for an arbitrary closed path γ whenever $\int_\gamma \omega_\sigma \neq 0$. Roughly speaking, Shimura's period symbol [13]

$$p_K(\sigma,\sigma') \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times \quad (\sigma,\sigma' \in \mathrm{Hom}(K,\mathbb{C}))$$

provides "generators" of the group generated by all CM-periods of K mod $\overline{\mathbb{Q}}^\times$. More precisely, let Φ be the CM-type of A, that is, $\Phi := \{\sigma \in \mathrm{Hom}(K,\mathbb{C}) \mid \omega_\sigma \text{ is holomorphic}\}$. Then we have

$$\prod_{\sigma' \in \Phi} p_K(\sigma,\sigma') \equiv \begin{cases} \pi^{-1}\int_\gamma \omega_\sigma & (\sigma \in \Phi) \\ \int_\gamma \omega_\sigma & (\sigma \notin \Phi) \end{cases} \mod \overline{\mathbb{Q}}^\times. \tag{1.1}$$

Stark units (with respect to real places). Let F be a totally real number field, K its abelian extension where only one real place of F splits, excepting the case $K/F = \mathbb{Q}/\mathbb{Q}$. We consider the partial zeta function

$$\zeta(s,\tau) := \sum_{(\frac{K/F}{\mathfrak{a}})=\tau} N\mathfrak{a}^{-s} \quad (\tau \in \mathrm{Gal}(K/F))$$

where $\mathfrak{a}$ runs over all integral ideals of F whose images under the Artin symbol are τ. Note that the assumption implies

$$\mathrm{ord}_{s=0}\zeta(s,\tau) = 1.$$

Then "the rank one abelian Stark conjecture" [15] implies

$$\exp(2\zeta'(0,\tau)) \in K^\times,$$

which satisfies the reciprocity law:

$$\tau'(\exp(2\zeta'(0,\tau))) = \exp(2\zeta'(0,\tau'\tau)) \quad (\tau,\tau' \in \mathrm{Gal}(K/F)).$$

Strictly speaking, $\exp(2\zeta'(0,\tau))$ is a real number and Stark's conjecture states that it is in the image $\iota(K^\times)$ under a real place $\iota\colon K \hookrightarrow \mathbb{R}$. Stark's conjecture also states $\exp(2\zeta'(0,\tau)) \in \mathcal{O}_K^\times$ in most cases, so $\exp(2\zeta'(0,\tau))$ is called a Stark unit.

The theme of the paper [10] is a "relation" between CM-periods and Stark units, in terms of classical or p-adic multiple gamma functions. As an example, we shall introduce an alternative (and partial) proof of Stark's conjecture in the case $F = \mathbb{Q}$, which was obtained in a previous paper [7]. Let $n \geq 3$, $\zeta_n := e^{\frac{2\pi i}{n}}$. Concerning CM-periods, we see that

- $\mathbb{Q}(\zeta_n)$ is a CM-field.

- (Each simple factor of) Jacobian variety $J(F_n)$ of Fermat curve $F_n\colon x^n+y^n=1$ has CM by $\mathbb{Q}(\zeta_n)$.
- $\eta_{r,s}:=x^r y^{n-s}\frac{dx}{x}$ $(0<r,s<n,\ r+s\neq n)$ are differential forms of the second kind.

Then Rohrlich's formula in [4] expresses CM-periods of $\mathbb{Q}(\zeta_n)$ explicitly, as

$$\int_\gamma \eta_{r,s}\equiv B\left(\frac{r}{n},\frac{s}{n}\right):=\frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}\mod \mathbb{Q}(\zeta_n)^\times.$$

Concerning Stark units, we see that

- $\mathbb{Q}(\zeta_n+\zeta_n^{-1})$ has a real place, so the unique real place of $\mathbb{Q}$ splits.
- Let $\sigma_{\pm a}\in \mathrm{Gal}(\mathbb{Q}(\zeta_n+\zeta_n^{-1})/\mathbb{Q})$ be defined by $\sigma_{\pm a}(\zeta_n+\zeta_n^{-1}):=\zeta_n^a+\zeta_n^{-a}$ for $0<a<n$, $(a,n)=1$. Then we have

  $$\zeta(s,\sigma_{\pm a})=\zeta(s,n,a)+\zeta(s,n,n-a),$$

  where $\zeta(s,n,a):=\sum_{k=0}^\infty(a+nk)^{-s}$ denotes the Hurwitz zeta function.

Therefore, by Lerch's formula (2.1), we obtain

$$\exp(2\zeta'(0,\sigma_{\pm a}))=\left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^2.$$

By Euler's formulas, we further see that $\exp(2\zeta'(0,\sigma_{\pm a}))=\dfrac{1}{2-\zeta_n^a-\zeta_n^{-a}}$, which is essentially a cyclotomic unit. This is the usual proof of Stark's conjecture in this case. On the other hand, in [7], we obtained a "relation" between CM-periods of $\mathbb{Q}(\zeta_n)$ and Stark units of $\mathbb{Q}(\zeta_n+\zeta_n^{-1})/\mathbb{Q}$ via the Gamma function, as follows.

- Rohrlich's formula and the cup product $H^1(F_n)\times H^1(F_n)\to H^2(F_n)=\mathbb{Q}(-1)$ induce "monomial relations"

  $$B\left(\frac{r}{n},\frac{s}{n}\right)B\left(\frac{n-r}{n},\frac{n-s}{n}\right)\equiv\int_\gamma\eta_{r,s}\int_{\gamma'}\eta_{n-r,n-s}\equiv 2\pi i\mod\overline{\mathbb{Q}}^\times$$

  since the period of the Lefschetz motive $\mathbb{Q}(-1)$ is $2\pi i$. Moreover, noting that $\Gamma(\frac{r}{n})^n=\Gamma(r)\prod_{k=1}^{n-1}B(\frac{r}{n},\frac{kr}{n})$, we obtain

  $$\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)\in 2\pi i\cdot\overline{\mathbb{Q}}^\times.$$

  It follows that $\exp(2\zeta'(0,\sigma_{\pm a}))\in\overline{\mathbb{Q}}^\times$ by Lerch's formula, without Euler's formulas.

- Furthermore, we showed that Coleman's formula in [2] on the absolute Frobenius action on F_n implies $\tilde{\sigma}_{\pm b}\left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right) \equiv \frac{\Gamma(\frac{ab}{n})\Gamma(\frac{n-ab}{n})}{2\pi}$, at least modulo the group μ_∞ of all roots of unity. Here $\tilde{\sigma}_{\pm b} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is an arbitrary lift of $\sigma_{\pm b} \in \mathrm{Gal}(\mathbb{Q}(\zeta_n + \zeta_n^{-1})/\mathbb{Q})$.

Summarizing the above, we provided an alternative proof of a part of Stark's conjecture when $F = \mathbb{Q}$, by using Rohrlich's formula on CM-periods of $\mathbb{Q}(\zeta_n)$ and Coleman's formula on the absolute Frobenius action on Fermat curves.

In [10], we formulated some conjectures which are generalizations of Rohrlich's formula and Coleman's formula, from $F = \mathbb{Q}$ to general totally real fields F. Furthermore, we clarified the relation to Stark's conjecture and Gross' p-adic analogue. In this paper we shall survey these results as follows. In §2, we introduce Conjecture 2.1 which is a generalization of Rohrlich's formula. This concerns a relation between monomial relations of CM-periods and the algebraicity of Stark units. In §3, we introduce Conjectures 3.1, 3.2 which are generalizations of Coleman's formula. In §4, we see that these concern a relation between the absolute Frobenius actions on p-adic CM-periods and the reciprocity law on Stark units. Some partial results are also provided. We note that these conjectures are (at least Conjecture 2.1 is) generalizations of Hiyoruki Yoshida's conjecture on "absolute CM-periods" in [17].

§ 2. Yoshida's conjecture and its refinement

§ 2.1. Multiple gamma functions

Recall Lerch's formula:

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp\left(\frac{d}{ds}\left[\sum_{m=0}^{\infty}(x+m)^{-s}\right]_{s=0}\right). \tag{2.1}$$

For a "good" countable subset $Z \subset \mathbb{R}$, we put

$$\Gamma_{\mathrm{mult}}(Z) := \exp\left(\frac{d}{ds}\left[\sum_{z\in Z} z^{-s}\right]_{s=0}\right).$$

Here we say Z is "good" if $\sum_{z\in Z} z^{-s}$ converges for $\Re(s) >> 0$, has a meromorphic continuation, is analytic at $s = 0$. (We use this notation only in this paper, in order to provide a rough idea.) In particular, for $x, \omega_1, \ldots, \omega_r > 0$, we see that

$$L_{x,(\omega_1,\ldots,\omega_r)} := \{x + m_1\omega_1 + \cdots + m_r\omega_r \mid 0 \le m_1, \ldots, m_r \in \mathbb{Z}\}$$

is "good" and $\Gamma_{\mathrm{mult}}(L_{x,(\omega_1,\ldots,\omega_r)})$ is called Barnes' multiple gamma function.

§ 2.2. Shintani's formula and Yoshida's class invariants

Let F be a totally real field, $\mathfrak{f}$ an integral ideal of F, $C_{\mathfrak{f}}$ the narrow ray class group modulo $\mathfrak{f}$. Let D be Shintani's fundamental domain of $F_+/\mathcal{O}_{F,+}^{\times}$. Here $+$ denotes their totally positive parts: in particular, $\mathcal{O}_{F,+}^{\times}$ denotes the group of all totally positive units. For each $c \in C_{\mathfrak{f}}$, we take a representative integral ideal $\mathfrak{a}$ of $\overline{c} \in C_{(1)}$ satisfying $\mathfrak{f} \mid \mathfrak{a}$ ($\overline{c}$ denotes the image under the natural projection $C_{\mathfrak{f}} \to C_{(1)}$). We consider a subset

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\} \subset F_+.$$

Then Yoshida [17] defined a class invariant for $c \in C_{\mathfrak{f}}$, $\iota \in \mathrm{Hom}(F, \mathbb{R})$

$$\exp(X(c, \iota)) := \exp(X(c, \iota; D, \mathfrak{a})) := \Gamma_{\mathrm{mult}}(\iota(Z_c)) \times \prod_i \iota(a_i)^{\iota(b_i)}$$

for suitable $a_i, b_i \in F$. When we fix $D, \mathfrak{a}$, we drop them from the symbol. Although Z_c, a_i, b_i depend on the choices of $D, \mathfrak{a}$, we have

- Shintani's formula in [14] states that $\exp(\zeta'(0, c)) = \prod_{\iota \in \mathrm{Hom}(F,\mathbb{R})} \exp(X(c, \iota))$.
- $\exp(X(c, \iota)) \bmod \iota(\mathcal{O}_{F,+}^{\times})^{\mathbb{Q}}$ does not depend on $D, \mathfrak{a}$ ([17, Chap. III], [8, Lemma 3.11]). That is, there exist $\epsilon \in \mathcal{O}_{F,+}^{\times}$, $N \in \mathbb{N}$ satisfying

$$\exp(X(c, \iota; D, \mathfrak{a}))/\exp(X(c, \iota; D', \mathfrak{a}')) = \iota(\epsilon)^{\frac{1}{N}}.$$

We fix $\mathrm{id}\colon F \hookrightarrow \mathbb{R}$ and put $\exp(X(c)) := \exp(X(c, \mathrm{id}))$.

Remark. Strictly speaking, Shintani provided a fundamental domain D of $(F \otimes_{\mathbb{Q}} \mathbb{R})_+/\mathcal{O}_{F,+}^{\times}$. He expressed D as a finite disjoint union of cones and provided an expression $Z_c = \coprod_{i=1}^{k} L_{x_i, \boldsymbol{\omega}_i}$ with $x_i \in F_+$, $\boldsymbol{\omega}_i \in F_+^{r_i}$, $r_i \in \mathbb{N}$. In particular $\iota(Z_c)$ is "good" and $\Gamma_{\mathrm{mult}}(\iota(Z_c))$ is a finite product of Barnes' multiple gamma functions for any $\iota \in \mathrm{Hom}(F, \mathbb{R})$. Shintani's formula in [14] expresses $\exp(\zeta'(0, c))$ as a finite product of Barnes' multiple gamma functions and some elementary terms. Yoshida found a "canonical decomposition" of this product and defined the above invariant $\exp(X(c, \iota))$.

§ 2.3. Shimura's period symbol (a restatement)

Let K be a CM-field, $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$. We take an algebraic Hecke character χ of K^{τ} whose infinite type is $l \cdot (\tau^{-1} - \rho \circ \tau^{-1})$. Here we consider K as a subfield of $\mathbb{C}$, so $\tau^{-1} \in \mathrm{Hom}(K^{\tau}, \mathbb{C})$ has a meaning. By taking l large enough, we may assume that χ takes values in K. We consider the associated motive $M(\chi)$: $M(\chi)$ is a motive defined over K^{τ}, with coefficients in K of rank 1, whose L-function is equal to that of χ. We define $P(\chi) \in (K \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$ by the de Rham isomorphism

$$\begin{array}{ccc} H_B(M(\chi)) \otimes_{\mathbb{Q}} \mathbb{C} & \cong & H_{\mathrm{dR}}(M(\chi)) \otimes_{K^{\tau}} \mathbb{C}, \\ P(\chi)(c_B \otimes 1) & \mapsto & c_{\mathrm{dR}} \otimes 1, \end{array}$$

where c_B is a K-basis of $H_B(M(\chi))$, c_{dR} is a $K \otimes_{\mathbb{Q}} K^\tau$-basis of $H_{\mathrm{dR}}(M(\chi))$. We further decompose it as

$$\begin{aligned} K \otimes_{\mathbb{Q}} \mathbb{C} \cong & \quad \textstyle\bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{C}, \\ P(\chi) & \mapsto (P(\sigma,\chi))_{\sigma \in \mathrm{Hom}(K,\mathbb{C})}. \end{aligned}$$

Then we have [10, Proposition 2-(ii)]

$$p_K(\sigma,\tau) \equiv (2\pi i)^{-\frac{\delta_{\sigma\tau}}{2}} P(\sigma,\chi)^{\frac{1}{2l}} \bmod \overline{\mathbb{Q}}^\times, \tag{2.2}$$

where we put $\delta_{\sigma\tau} := 1, -1, 0$ if $\sigma = \tau, \rho \circ \tau$, otherwise, respectively.

§ 2.4. A refinement of Yoshida's conjecture

Yoshida formulated a conjecture which expresses Shimura's period symbol p_K (§2.3) as a finite product of rational powers of Yoshida's class invariant $\exp(X(c))$ (§2.2). Here we introduce a "reverse version" [10, Conjecture 3]: note that this is not just a restatement but a refinement in the sense of Remark (ii) below. Recall that F is a totally real field and $C_{\mathfrak{f}}$ is the narrow ray class group modulo $\mathfrak{f}$.

Conjecture 2.1. *Assume that the narrow ray class field $H_{\mathfrak{f}}$ modulo $\mathfrak{f}$ contains a CM-field. Let K be the maximal CM-subfield of $H_{\mathfrak{f}}$. Then we have*

$$\exp(X(c)) \equiv \pi^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}} \bmod \overline{\mathbb{Q}}^\times \quad (c \in C_{\mathfrak{f}}).$$

Here c, c' in $p_K(\ )$ denotes their images under the Artin map $\mathrm{Art}\colon C_{\mathfrak{f}} \to \mathrm{Gal}(K/F)$*.*

Remark.

(i) When $F = \mathbb{Q}$, this conjecture holds true by Rohrlich's formula.

(ii) The original conjecture [17, Chap. III, Conjecture 3.9] is equivalent to

$$\prod_{c \in \mathrm{Art}^{-1}(\sigma)} \exp(X(c)) \equiv \prod_{c \in \mathrm{Art}^{-1}(\sigma)} \left(\pi^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}} \right) \quad \bmod \overline{\mathbb{Q}}^\times.$$

for $\sigma \in \mathrm{Gal}(K/F)$.

(iii) We focus on the algebraicity of Stark units:

"$\exp(\zeta'(0,\sigma)) \in \overline{\mathbb{Q}}^\times$ for $\sigma \in \mathrm{Gal}(H/F)$ if F is a totally real field, H has a real place, H/F is abelian, except for $H/F = \mathbb{Q}/\mathbb{Q}$."

Then Conjecture 2.1 states that this algebraicity follows from the monomial relation $p_K(\sigma,\sigma')p_K(\sigma,\rho\circ\sigma') \equiv 1 \bmod \overline{\mathbb{Q}}^\times$. Moreover, we can show that

(a) Conjecture 2.1 implies this algebraicity.

(b) "The original conjecture in [17] + this algebraicity" implies Conjecture 2.1.

For details, see [8].

§ 2.5. A numerical example [10, Examples 1, 2]

Let $K := \mathbb{Q}(\sqrt{2\sqrt{5}-26})$, $[\sigma\colon \sqrt{2\sqrt{5}-26} \mapsto -\sqrt{-2\sqrt{5}-26}] \in \mathrm{Hom}(K,\mathbb{C})$, ρ the complex conjugation. Then $\mathrm{Hom}(K,\mathbb{C}) = \{\mathrm{id},\rho,\sigma,\rho\circ\sigma\}$. Let

$$C\colon y^2 = \tfrac{7+\sqrt{41}}{2}x^6 + (-10-2\sqrt{41})x^5 + 10x^4 + \tfrac{41+\sqrt{41}}{2}x^3 + (3-2\sqrt{41})x^2 + \tfrac{7-\sqrt{41}}{2}x + 1.$$

Then $J(C)$ has CM, of CM-type $(K,\{\mathrm{id},\sigma\})$ [1, Table 2B, $DAB = [5,13,41], DAB^r = [41,11,20]$]. In fact, $\omega_{\mathrm{id}} = \frac{2dx}{y} + \frac{(\sqrt{5}-1)xdx}{y}$, $\omega_\sigma = \frac{(-\sqrt{5}+\sqrt{41})xdx}{y}$ are holomorphic differential forms where K acts via id,σ respectively. Numerically we have (by Maple's command `periodmatrix`)

$$\int \omega_{\mathrm{id}} = -0.4929421793\ldots - 0.8116152991\ldots i,$$
$$\int \omega_\sigma = -0.1395619319\ldots + 0.1323795194\ldots i.$$

Similarly, we define C' where $J(C')$ has CM, of CM-type $(K,\{\mathrm{id},\rho\circ\sigma\})$, by replacing $\sqrt{41}$ with $-\sqrt{41}$. Then we have

$$\int \omega'_{\mathrm{id}} = -0.4443866005\ldots - 0.3099403507\ldots i,$$
$$\int \omega'_{\rho\circ\sigma} = -2.0247186165\ldots + 0.4533729269\ldots i,$$

where $\omega'_{\mathrm{id}} := \frac{2dx}{y} + \frac{(\sqrt{5}-1)xdx}{y}$, $\omega'_{\rho\circ\sigma} := \frac{(-\sqrt{5}+\sqrt{41})xdx}{y}$ are holomorphic differential forms on C'. By definition (1.1) and monomial relations of Shimura's period symbol, we see that

$$\begin{aligned}\pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\rho)^{-1} &\equiv \pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\sigma)p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\rho\circ\sigma)\\ &\equiv \pi^{-1}\int\omega_{\mathrm{id}}\int\omega'_{\mathrm{id}} \mod \overline{\mathbb{Q}}^\times.\end{aligned}$$

Let $F := \mathbb{Q}(\sqrt{5})$, $\mathfrak{f} := (\frac{13-\sqrt{5}}{2})$. We easily see that $C_\mathfrak{f} = \{c_1 := [(1)], c_2 := [(3)]\} \cong \mathrm{Gal}(K/F)$, $c_1 \leftrightarrow \mathrm{id}$, $c_2 \leftrightarrow \rho$, $\zeta(0,c_1) = 1$, $\zeta(0,c_2) = -1$. Here $[(*)]$ denotes the ideal class in $C_\mathfrak{f}$ of the principal ideal $(*)$. Then we obtain numerically

$$\pi^{-1}\int\omega_{\mathrm{id}}\int\omega'_{\mathrm{id}} = \exp(X(c_1))(\tfrac{\sqrt{5}-1}{2})^{\frac{14}{41}}\frac{\sqrt{-8\sqrt{5}+20+(\sqrt{5}+15)\sqrt{2\sqrt{5}-26}}}{80}. \tag{2.3}$$

Remark. We explain the above example in terms of §2.3. Let $F_0 := \mathbb{Q}(\sqrt{41})$, χ an algebraic Hecke character of K whose infinity type is $l \cdot (\mathrm{id} - \rho)$. Then we see that

$$M(\chi) \times_K KF_0 = (H^1(C \times_{F_0} KF_0) \otimes H^1(C' \times_{F_0} KF_0) \otimes \mathbb{Q}(1))^{\otimes l}.$$

Hence we have

$$P(\mathrm{id}, \chi) \equiv \left((2\pi i)^{-1} \int \omega_{\mathrm{id}} \int \omega'_{\mathrm{id}}\right)^l \mod K^\times,$$

which is consistent with (2.2). We note that $\omega_{\mathrm{id}} \otimes \omega'_{\mathrm{id}}$ is defined over K, i.e., $\omega_{\mathrm{id}} \otimes \omega'_{\mathrm{id}} \in H_{\mathrm{dR}}(M(\chi))$, although each $\omega_{\mathrm{id}}, \omega'_{\mathrm{id}}$ is defined only over KF_0. In particular, we see that (2.3) is a supporting evidence of [10, assumption (35)].

§ 3. p-adic analogues

Let F be a totally real field, $\mathfrak{f}$ an integral ideal, $C_{\mathfrak{f}}$ the narrow ray class group modulo $\mathfrak{f}$, and μ_∞ the group of all roots of unity, as in the previous sections. In this section, we introduce Conjectures 3.1, 3.2 which provide a generalization of Coleman's formula. Note that the p-adic analogue $\exp_p(X_p(c))$ of Yoshida's class invariant associated with $c \in C_{\mathfrak{f}}$ can be defined only when the prime ideal $\mathfrak{p}$ corresponding to the p-adic topology on F divides $\mathfrak{f}$. Therefore we formulate conjectures in the case $\mathfrak{p} \mid \mathfrak{f}$ or $\mathfrak{p} \nmid \mathfrak{f}$, separately.

§ 3.1. A p-adic analogue of Yoshida's class invariant

Assume that

the prime ideal $\mathfrak{p}$ corresponding to a fixed embedding $F \hookrightarrow \mathbb{C}_p$ divides $\mathfrak{f}$.

We need this assumption for the p-adic interpolation of the series $\sum_{z \in Z_c} z^{-s}$ defined in §2.1, §2.2. We define

$$\exp_p(X_p(c)) := \Gamma_{\mathrm{mult},p}(Z_c) \times \prod_i \exp_p(b_i \log_p a_i)$$

for the same $Z_c \subset F$, $a_i, b_i \in F$ as those in the definition of $\exp(X(c))$. Here we put

$$\Gamma_{\mathrm{mult},p}(Z) := \exp_p\left(\frac{d}{ds}\left[p\text{-adic interpolation of } \sum_{z \in Z} z^{-s}\right]_{s=0}\right).$$

When we do not fix a Shintani's fundamental domain D or a representative ideal $\mathfrak{a}$, we add them to the symbol: $\exp_p(X_p(c; D, \mathfrak{a}))$. Then we see that

(i) We obtained a p-adic analogue of Shintani's formula in [6].

(ii) $\exp_p(X_p(c)) \bmod (\mathcal{O}_{F,+}^{\times})^{\mathbb{Q}}$ does not depend on the choices of $D, \mathfrak{a}$ [9, Lemma 4].

(iii) The "ratio" $[\exp(X(c)) : \exp_p(X_p(c))] \bmod \mu_\infty$ does not depend on $\mathfrak{a}, D$ [9, Corollary 1-(i)]. That is, we have

$$\begin{aligned}&\exp(X(c; D, \mathfrak{a}))/\exp(X(c; D', \mathfrak{a}')) \\ &\equiv \exp_p(X_p(c; D, \mathfrak{a}))/\exp_p(X_p(c; D', \mathfrak{a}')) \mod \mu_\infty.\end{aligned}$$

§ 3.2. p-adic analogue of Shimura's period symbol

Let $B_{\mathrm{dR}}, B_{\mathrm{cris}}$ be Fontaine's p-adic period rings: note that B_{cris} (resp. $\overline{\mathbb{Q}_p}$) is a subring (resp. subfield) of B_{dR}. In [10, §5.1], we define

$$p_{K,p}(\sigma, \tau) \in B_{\mathrm{dR}}^{\times}$$

by replacing the de Rham isomorphism in §2.3 with comparison isomorphisms of p-adic Hodge theory. We also replace $2\pi i$ with the p-adic period $(2\pi i)_p$ of the Lefschetz motive. Since abelian varieties with CM have potentially good reductions, we see that

$$p_{K,p}(\sigma, \tau) \in (B_{\mathrm{cris}}\overline{\mathbb{Q}_p})^{\mathbb{Q}} := \{x \in B_{\mathrm{dR}} \mid \text{there exists } n \in \mathbb{N} \text{ satisfying } x^n \in B_{\mathrm{cris}}\overline{\mathbb{Q}_p}\}.$$

Moreover the "ratio"

$$[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \bmod \mu_\infty$$

is well-defined when we take the same basis c_B, c_{dR} of cohomology groups for $p_K, p_{K,p}$.

§ 3.3. Reciprocity laws concerning the absolute Frobenius action

Recall that $\mathfrak{p}$ is the prime ideal corresponding to the p-adic topology on F. We consider the completion $F_\mathfrak{p}$ of F and put $W_\mathfrak{p} \subset \mathrm{Gal}(\overline{F_\mathfrak{p}}/F_\mathfrak{p})$ to be the Weil group. That is, when $\tau \in W_\mathfrak{p}$, there exists $\deg_\mathfrak{p} \tau \in \mathbb{Z}$ satisfying

$$\tau|_{F_\mathfrak{p}^{\mathrm{ur}}} = \mathrm{Fr}_\mathfrak{p}^{\deg_\mathfrak{p} \tau}$$

where $*^{\mathrm{ur}}$ denotes the maximal unramified extension and $\mathrm{Fr}_\mathfrak{p}$ denotes the Frobenius automorphism at $\mathfrak{p}$. We consider a natural action $W_\mathfrak{p} \curvearrowright B_{\mathrm{cris}}\overline{\mathbb{Q}_p} = B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} \overline{\mathbb{Q}_p}$ defined by $\Phi_\tau := (\text{absolute Frobenius})^{\deg \mathfrak{p} \deg_\mathfrak{p} \tau} \otimes \tau$. The class invariants $\exp(X(c))$, $\exp_p(X_p(c))$ for $c \in C_\mathfrak{f}$ are introduced in §2.2, §3.1. Strictly speaking, these values depend on the choices of a Shintani's fundamental domain D and a representative ideal $\mathfrak{a}$.

Conjecture 3.1 ([10, Conjecture 4-(ii)]). *Assume that* $\mathfrak{p} \mid \mathfrak{f}$. *Under* Conjecture 2.1, *we define for* $c \in C_\mathfrak{f}$

$$\Gamma(c) := \frac{\exp(X(c))}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}} \frac{(2\pi i)_p^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}}{\exp_p(X_p(c))} \in (B_{\mathrm{cris}}\overline{\mathbb{Q}_p})^{\mathbb{Q}}/\mu_\infty,$$

which does not depend on the choices of $D, \mathfrak{a}$. *Then we have for* $\tau \in W_\mathfrak{p}$

$$\Phi_\tau(\Gamma(c)) \equiv \Gamma(c_\tau c) \bmod \mu_\infty \quad (c \in C_\mathfrak{f}),$$

where $c_\tau := \mathrm{Art}^{-1}(\tau|_{H_\mathfrak{f}}) \in C_\mathfrak{f}$.

Conjecture 3.2 ([10, Conjecture 4-(i)]). *Assume that* $\mathfrak{p} \nmid \mathfrak{f}$. *Under* Conjecture 2.1, *we define for* $c \in C_\mathfrak{f}$

$$\Gamma(c; D, \mathfrak{a}) := \frac{\exp(X(c; D, \mathfrak{a}))}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}} (2\pi i)_p^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}} \in (B_{\mathrm{cris}}\overline{\mathbb{Q}_p})^{\mathbb{Q}}/\mu_\infty.$$

Then we have for $\tau \in W_\mathfrak{p}$ *with* $\deg_\mathfrak{p} \tau = 1$

$$\Phi_\tau(\Gamma(c; D, \mathfrak{a})) \equiv \frac{\pi_\mathfrak{p}^{\frac{\zeta(0,[\mathfrak{p}]c)}{h_F^+}} \Gamma([\mathfrak{p}]c; D, \mathfrak{p}\mathfrak{a})}{\prod_{\substack{\tilde{c} \in C_{\mathfrak{f}\mathfrak{p}} \\ \tilde{c} \mapsto [\mathfrak{p}]c \in C_\mathfrak{f}}} \exp_p(X_p(\tilde{c}; D, \mathfrak{p}\mathfrak{a}))} \bmod \mu_\infty \quad (c \in C_\mathfrak{f}),$$

where h_F^+ *is the narrow class number,* $\pi_\mathfrak{p}$ *is a suitable generator of* $\mathfrak{p}^{h_F^+}$, *and* $\tilde{c}$ *runs over all narrow ideal classes modulo* $\mathfrak{f}\mathfrak{p}$ *whose images under the natural projection are equal to* $[\mathfrak{p}]c \in C_\mathfrak{f}$. *In particular, since* $\tilde{c} \in C_{\mathfrak{f}\mathfrak{p}}$, $\exp_p(X_p(\tilde{c}; D, \mathfrak{p}\mathfrak{a}))$ *in the right-hand side is well-defined. Note that, although the definition of* $\Gamma(c; D, \mathfrak{a})$ *does depend on the choices of* $D, \mathfrak{a}$, *the validity of the conjecture does not.*

Remark.

(i) When $H_\mathfrak{f}$ does not contain any CM-field, we see that $\zeta(0,c) = 0$ [10, Proposition 3]. Hence we regard $p_K(\ ) = p_{K,p}(\ ) = 1$ in this case.

(ii) "mod μ_∞ ambiguity" occurs when we take rational powers of periods or consider $\exp_p$ outside of the convergence region. These may be avoidable by "S, T-modified version".

§ 4. Main results

Let F be a totally real field, $\mathfrak{p}$ the prime ideal corresponding to the p-adic topology on F, $C_{\mathfrak{f}}$ the narrow ray class group modulo $\mathfrak{f}$. We defined two kinds of period-ring valued Γ-functions $\Gamma(c)$ (when $\mathfrak{p} \mid \mathfrak{f}$), $\Gamma(c; D, \mathfrak{a})$ (when $\mathfrak{p} \nmid \mathfrak{f}$) for $c \in C_{\mathfrak{f}}$ in §3.3. Although the proof is elementary, the following proposition seems to be interesting: the class invariant $\Gamma(c)$ becomes an "Euler system" which takes values in the p-adic period ring.

Proposition 4.1 ([10, Proposition 7-(ii)]). *Let $\mathfrak{q}$ be a prime ideal, $\phi\colon C_{\mathfrak{f}\mathfrak{q}} \to C_{\mathfrak{f}}$ the natural projection, and $c \in C_{\mathfrak{f}}$. For simplicity assume that $\mathfrak{p} \mid \mathfrak{f}$. Then we have*

$$\prod_{\tilde{c} \in C_{\mathfrak{f}\mathfrak{q}},\ \phi(\tilde{c})=c} \Gamma(\tilde{c}) \equiv \begin{cases} \Gamma(c)\Gamma([\mathfrak{q}]c)^{-1} & (\mathfrak{q} \nmid \mathfrak{f}) \\ \Gamma(c) & (\mathfrak{q} \mid \mathfrak{f}) \end{cases} \mod \mu_\infty.$$

The following theorems are the main results in [10].

Theorem 4.2 ([10, Theorem 1]). Conjectures 2.1, 3.1, *and* 3.2 *imply the reciprocity law on Stark's units* (§1) *up to μ_∞:*

$$\tau'(\exp(2\zeta'(0, \tau))) \equiv \exp(2\zeta'(0, \tau'\tau)) \mod \mu_\infty \quad (\tau, \tau' \in \mathrm{Gal}(K/F))$$

for any abelian extension K of F having a real place.

Sketch of proof. Let H be the maximal subfield of the ray class field $H_{\mathfrak{f}}$ where the real place $\mathrm{id}\colon F \hookrightarrow \mathbb{R}$ splits. For simplicity, assume that $\mathfrak{p} \mid \mathfrak{f}$. Then we can show that

$$\prod_{c \mapsto \sigma} \Gamma(c) \equiv \exp(\zeta'(0, \sigma)) \mod \mu_\infty \quad (\sigma \in \mathrm{Gal}(H/F)).$$

Since Φ_τ is τ-semilinear, we obtain $\tau(\exp(\zeta'(0, \sigma))) \equiv \exp(\zeta'(0, \tau \circ \sigma)) \bmod \mu_\infty$ for $\tau \in W_{\mathfrak{p}}$. Then we vary $\mathfrak{p}$. □

Gross formulated a p-adic analogue of Stark's conjecture, which is called the rank one abelian Gross-Stark conjecture [5]. Dasgupta-Darmon-Pollack [3] and Ventullo [16] showed that this conjecture holds true. On the other hand, Yoshida and the author formulated its refinements in [11, 12].

Theorem 4.3 ([10, Theorem 2]). Conjecture 3.2 *implies refinements in* [11, 12] *of the rank one abelian Gross-Stark conjecture under an assumption* [10, assumption (35)].

Sketch of proof. We consider the product

$$\prod_{c \mapsto \sigma} \Gamma(c) \quad (\sigma \in \mathrm{Gal}(H/F))$$

for the maximal subfield H of $H_{\mathfrak{f}}$ where $\mathfrak{p}$ splits completely, instead of H in the above proof. $\square$

Theorem 4.4 ([10, Theorem 4-(iii)]). *When $H_{\mathfrak{f}}$ is abelian over $\mathbb{Q}$ and $\mathfrak{p} \nmid 2$,* Conjecture 3.1 *holds true.*

Sketch of proof. The case $F = \mathbb{Q}$ follows from Rohrlich's formula and Coleman's formula [10, Theorem 3]. In this case, the assumption $\mathfrak{p} \nmid 2$ is needed for using Coleman's results. We reduce the problem to this case, by well-known formula on L-functions:

$$L(s,\chi) = \prod_{\psi \in \widehat{G}, \psi|_H = \chi} L(s,\psi) \quad (\chi \in \widehat{G},\ G := \mathrm{Gal}(H_{\mathfrak{f}}/\mathbb{Q}) \supset H := \mathrm{Gal}(H_{\mathfrak{f}}/F)).$$

By this, we can express $\exp(\zeta'(0,c))$'s of F in terms of those of $\mathbb{Q}$. However, Shintani's formula states that for $c \in C_{\mathfrak{f}}$ of F

$$\exp(\zeta'(0,c)) = \prod_{\iota \in \mathrm{Hom}(F,\mathbb{R})} \exp(X(c,\iota))$$

and we need just $\exp(X(c,\iota))$, not their product. Recall the definition

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\},$$
$$\exp(X(c,\iota)) := \Gamma_{\mathrm{mult}}(\iota(Z_c)) \times \prod_i \iota(a_i)^{\iota(b_i)}.$$

Then we see that, roughly speaking, $\exp(X(c,\iota))$ depends on $\iota(c)$, rather than on c. When $H_{\mathfrak{f}}/\mathbb{Q}$ is abelian, $\iota(c) \in C_{\iota(\mathfrak{f})}$ (and $\iota(F)$, $\iota(\mathfrak{f})$) do not depend on $\iota \in \mathrm{Hom}(F,\mathbb{R})$. Hence, we obtain an expression like

$$\exp(\zeta'(0,c)) = \exp(X(c))^{[F:\mathbb{Q}]} \times \text{ explicit correction terms}$$

by Yoshida's technique concerning the replacement of $D, \mathfrak{a}$. Since the same holds true for $\exp_p(\zeta_p'(0,c))$, we have

$$[\exp(\zeta'(0,c)) : \exp_p(\zeta_p'(0,c))] \equiv [\exp(X(c))^{[F:\mathbb{Q}]} : \exp_p(X_p(c))^{[F:\mathbb{Q}]}] \bmod \mu_\infty.$$

Then we obtain an explicit formula on the ratios $[\exp(X(c)) : \exp_p(X_p(c))]$ of F in terms of those of $\mathbb{Q}$. A similar relation holds true for the "period part $[p_K(\ ) : p_{K,p}(\ )]$" by a simpler argument. $\square$

By a similar argument, we can show the following theorem. In this case, we need the assumption $p \neq 2$ for technical reasons, in order to rewrite Coleman's results in the proof of [10, Theorem 3].

Theorem 4.5 ([10, Theorem 4-(ii)]). *When $H_{\mathfrak{f}}$ is abelian over $\mathbb{Q}$ and $p \neq 2$ remains prime in F,* Conjecture 3.2 *holds true.*

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