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<th>Title</th>
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Kyoto University
On Drinfeld modules with Rasmussen-Tamagawa type conditions: a resume

By

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Abstract

This is an announcement of the author’s results [7] on a non-existence problem of Drinfeld modules defined over global function fields with some arithmetic constraints. The motivation is a conjecture of Christopher Rasmussen and Akio Tamagawa related with abelian varieties over number fields with constrained $\ell$-power torsion points. In this paper, considering the arithmetic similarity between Drinfeld modules and elliptic curves, we show that a Drinfeld module analogue of the Rasmussen-Tamagawa conjecture holds in some cases. We also see that there is a counter-example of the Drinfeld module analogue of the conjecture in a special case.

§ 1. Introduction

Let $k$ be a finite extension of $\mathbb{Q}$ and $g$ a positive integer. For a prime number $\ell$, denote by $\tilde{k}_\ell$ the maximal pro-$\ell$ extension of $k(\mu_\ell)$ which is unramified outside $\ell$, where $\mu_\ell = \mu_\ell(k)$ is the set of $\ell$-th roots of unity in $k$. For an abelian variety $X$ over $k$, write $k(X^{[\ell\infty]}) := k(\bigcup_{n \geq 1} X^{[\ell^n]})$ for the field generated by all $\ell$-power torsion points of $X$. Define $\mathcal{A}(k, g, \ell)$ to be the set of isomorphism classes $[X]$ of $g$-dimensional abelian varieties over $k$ which satisfy the following equivalent conditions:

- $k(X^{[\ell\infty]}) \subseteq \tilde{k}_\ell$,
- $X$ has good reduction at any finite place of $k$ not lying above $\ell$ and $k(X^{[\ell]})/k(\mu_\ell)$ is an $\ell$-extension,
X has good reduction at any finite place of \( k \) not lying above \( \ell \) and the mod \( \ell \) representation \( \bar{\rho}_{X,\ell} : G_k \to \text{Aut}_{\mathbb{F}_\ell}(X[\ell]) \simeq \text{GL}_{2g}(\mathbb{F}_\ell) \) is of the form
\[
\begin{pmatrix}
\chi_\ell^{i_1} & * & \cdots & * \\
* & \chi_\ell^{i_2} & \cdots & \\
\vdots & \vdots & \ddots & \\
* & \cdots & \cdots & \chi_\ell^{i_{2g}}
\end{pmatrix},
\]
where \( \chi_\ell \) is the mod \( \ell \) cyclotomic character and \( i_1, \ldots, i_{2g} \) are positive integers.

The equivalence of these conditions follows from the Néron-Ogg-Shafarevich criterion and some group theoretic lemma. According to the Shafarevich conjecture proved by Faltings, the set \( \mathcal{A}(k, g, \ell) \) is always finite. Rasmussen and Tamagawa conjectured the following.

**Conjecture 1.1** ([10, Conjecture 1]). The set \( \mathcal{A}(k, g, \ell) \) is empty if \( \ell \) is large enough.

For example, the following cases are known:

- \( k = \mathbb{Q} \) and \( g = 1 \) [10, Theorem 2],
- \( k = \mathbb{Q} \) and \( g = 2, 3 \) [11, Theorem 7.1 and Theorem 7.2],
- for abelian varieties with everywhere semistable reduction [8, Corollary 4.5] if \( k/\mathbb{Q} \) has odd degree or the discriminant of \( K \) is not divisible by \( \ell \), and [11, Theorem 3.6] in the general case,
- for abelian varieties with abelian Galois representations [9, Corollary 1.3],
- for QM abelian surfaces over certain imaginary quadratic fields [1, Theorem 9.3].

We notice that, under the assumption of the Generalized Riemann Hypothesis (GRH) for Dedekind zeta functions of number fields, the conjecture is true in general [11, Theorem 5.1]. The key tool of this proof is the effective version of the Chebotarev density theorem for number fields, which holds under GRH.

In this paper, we would like to consider a function field analogue of this conjecture. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements of characteristic \( p \). Let \( A = \mathbb{F}_q[t] \) be the ring of polynomials in indeterminate \( t \) and \( F = \mathbb{F}_q(t) \) the fraction field of \( A \). Instead of abelian varieties over number field, we consider Drinfeld modules defined over a finite extension \( K \) of \( F \). There are deep arithmetic similarity between Drinfeld modules and elliptic curves. Under these analogy, for any positive integer \( r \) and monic irreducible element \( \pi \in A \), we construct the set \( \mathcal{D}(K, r, \pi) \) of isomorphism classes of rank-\( r \) Drinfeld modules over \( K \) satisfying Rasmussen-Tamagawa type conditions (see Proposition 2.5). We shall not distinguish between monic irreducible elements of \( A \) and finite places of \( F \).
The followings are main results in this article.

**Theorem 1.2.** Suppose that \( r = p^\nu \) for some integer \( \nu > 0 \) and does not divide the inseparable degree \([K : F]\). Then the set \( \mathcal{D}(K, r, \pi) \) is empty for any \( \pi \) whose degree \( \deg(\pi) \) is large enough.

**Theorem 1.3.** Suppose that \( r = r_0p^\nu \), where \( r_0 > 1 \) is an integer prime to \( p \) and \( \nu \geq 0 \). Then the set \( \mathcal{D}(K, r, \pi) \) is empty for any \( \pi \) whose degree \( \deg(\pi) \) is large enough.

Therefore a Drinfeld module analogue of Conjecture 1.1 holds if \( r \) does not divide \([K : F]\). Conversely, if \( r \) divides \([K : F]\), then \( \mathcal{D}(K, r, \pi) \) is never empty for any monic irreducible \( \pi \) (see §4).

### §2. Construction of \( \mathcal{D}(K, r, \pi) \)

First, we give a brief introduction of Drinfeld modules (see [3], [5] and [12] for details). Let \( K \) be a finite extension of \( F \). Denote by \( K\{\tau\} \) the non-commutative polynomial ring over \( K \) in variable \( \tau \) satisfying \( \tau c = c^q\tau \) for any \( c \in K \). Here \( K\{\tau\} \) is isomorphic to the ring \( \text{End}_{\mathbb{F}_q}(\mathbb{G}_{a/K}) \) of \( \mathbb{F}_q \)-linear endomorphisms of the additive group \( \mathbb{G}_{a/K} \) over \( K \). Let \( r \) be a positive integer.

**Definition 2.1.** A Drinfeld \( A \)-module (or Drinfeld module, for short) \( \phi \) of rank \( r \) over \( K \) is an \( \mathbb{F}_q \)-algebra homomorphism \( \phi : A \to K\{\tau\} ; a \mapsto \phi_a \) such that \( \phi_t = t + c_1\tau + \cdots + c_r\tau^r \in K\{\tau\} \) and \( c_r \neq 0 \).

**Remark.** In general, for any field \( \mathcal{F} \) equipped with an \( \mathbb{F}_q \)-algebra homomorphism \( \iota : A \to \mathcal{F} \), a rank-\( r \) Drinfeld module \( \phi \) over \( \mathcal{F} \) is defined to be an \( \mathbb{F}_q \)-algebra homomorphism \( \phi : A \to \mathcal{F}\{\tau\} \) such that \( \phi_t = \iota(t) + c_1\tau + \cdots + c_r\tau^r \) and \( c_r \neq 0 \).

**Definition 2.2.** A morphism \( \mu : \phi \to \psi \) between two Drinfeld modules over \( K \) is an element \( \mu \in K\{\tau\} \) such that \( \mu\phi_a = \psi_a\mu \) for any \( a \in A \). Namely \( \mu \) makes the following diagram commutative

\[
\begin{array}{ccc}
\mathbb{G}_{a/K} & \xrightarrow{\mu} & \mathbb{G}_{a/K} \\
\phi_a \downarrow & & \downarrow \psi_a \\
\mathbb{G}_{a/K} & \xrightarrow{\mu} & \mathbb{G}_{a/K}
\end{array}
\]

for any \( a \in A \). We say that \( \mu \) is an isomorphism if \( \mu \in K^\times \).

**Example 2.3.** The rank-one Drinfeld module \( C : A \to \mathcal{F}\{\tau\} \) determined by \( C_t = t + \tau \) is called the Carlitz module.
Let \( \phi \) be a rank-\( r \) Drinfeld module over \( K \) and \( K^{\text{sep}} \) the separable closure of \( K \). Then \( \phi \) endows \( K^{\text{sep}} \) with a new \( A \)-module structure defined by \( a \cdot \lambda := \phi_a(\lambda) \) for any \( a \in A \) and \( \lambda \in K^{\text{sep}} \). For a non-zero element \( a \in A \), the \( a \)-torsion points of \( \phi \) is \( \phi[a] = \{ \lambda \in K^{\text{sep}}; a \cdot \lambda = \phi_a(\lambda) = 0 \} \). It is a free \( A/aA \)-module of rank \( r \) on which the absolute Galois group \( G_K \) of \( K \) acts. Let \( \pi \) be a monic irreducible element of \( A \). Define \( F_\pi := A/\pi A \) and \( q_\pi := \#F_\pi = q^{\deg(\pi)} \). The action of \( G_K \) on \( \phi[\pi] \) defines an \( F_\pi \)-linear \( r \)-dimensional \( G_K \)-representation

\[
\tilde{\rho}_{\phi,\pi} : G_K \to \text{Aut}_{F_\pi}(\phi[\pi]) \simeq \text{GL}_r(F_\pi).
\]

The \( \pi \)-adic Tate module of \( \phi \) is \( T_\pi(\phi) := \varprojlim \phi[\pi^n] \), which is a free \( A_\pi := \varprojlim A/\pi^n A \)-module of rank \( r \) with continuous \( G_K \)-action.

Let \( v \) be a finite place of \( K \), that is, \( v \) is a place lying above a monic irreducible element \( \pi_0 \in A \). Denote by \( K_v, O_{K_v} \) and \( F_v \), the completion of \( K \) at \( v \), its ring of integers and its residue field, respectively. Suppose that there exists a rank-\( r \) Drinfeld module \( \psi \) over \( K_v \) such that \( \phi \) is isomorphic to \( \psi \) over \( K_v \) and \( \psi_\ell = t + c'_1t^1 + \cdots + c'_rt^r \) for some \( c'_1, \ldots, c'_r \in O_{K_v} \). Then we say that \( \phi \) has stable reduction at \( v \) if \( c'_{r_0} \in O_{K_v}^\times \) for some \( 1 \leq r_0 \leq r \), and has good reduction at \( v \) if \( c'_r \in O_{K_v}^\times \). It is known that \( \phi \) has good reduction at \( v \) if and only if \( T_\pi(\phi) \) is unramified at \( v \) for any (in fact, for some) \( \pi \neq \pi_0 \), which is an analogue of the Néron-Ogg-Shafarevich criterion for abelian varieties.

\textbf{Example 2.4.} The action of \( G_F \) on the \( \pi \)-torsion points \( C[\pi] \) of the Carlitz module defines the character

\[
\chi_\pi : G_F \to F_\pi^\times,
\]

which is called the \textit{mod} \( \pi \) \textit{Carlitz character}. It is an analogue of the mod \( \ell \) cyclotomic character \( \chi_\ell : G_{Q_\ell} \to F_\ell^\times \). For instance, the extension \( F(C[\pi])/F \) is cyclic and its Galois group is isomorphic to \( F_\pi^\times \) via \( \chi_\pi \). Moreover, since \( C \) has good reduction at any finite place \( \pi_0 \) of \( F \), the action of Frobenius element \( \text{Frob}_{\pi_0} \) on \( C[\pi] \) is well-defined if \( \pi \neq \pi_0 \) and then we have \( \chi_\pi(\text{Frob}_{\pi_0}) \equiv \pi_0 \mod{\pi} \).

Under the above analogy, let us define the set \( \mathscr{D}(K, r, \pi) \). For a rank-\( r \) Drinfeld module \( \phi \) over \( K \) and a monic irreducible element \( \pi \in A \), set \( K(\phi[\pi^\infty]) := K(\cup_{n \geq 1} \phi[\pi^n]) \) and consider the subfield \( K_{\phi,\pi} := K(\phi[\pi]) \cap K(C[\pi]) \) of \( K(\phi[\pi]) \). Note that \( K(\phi[\pi]) \) may not contain \( K(C[\pi]) \). By similar arguments in the abelian variety case, we can prove the following.
Proposition 2.5. The following conditions are equivalent.

- $K(\phi[\pi^\infty])/K_{\phi, \pi}$ is a pro-$p$ extension which is unramified at any finite place of $K_{\phi, \pi}$ not lying above $\pi$,
- $\phi$ has good reduction at any finite place of $K$ not lying above $\pi$ and $K(\phi[\pi])/K_{\phi, \pi}$ is a $p$-extension,
- $\phi$ has good reduction at any finite place of $K$ not lying above $\pi$ and the representation $\bar{\rho}_{\phi, \pi}$ is of the form

$$\bar{\rho}_{\phi, \pi} \simeq \begin{pmatrix}
\chi_{\pi}^{i_1} & \ast & \cdots & \ast \\
\chi_{\pi}^{i_2} & \ddots & & \\
& \ddots & \ast & \\
& & \chi_{\pi}^{i_r}
\end{pmatrix}.$$ 

Define $\mathcal{D}(K, r, \pi)$ to be the set of isomorphism classes $[\phi]$ of rank-$r$ Drinfeld modules over $K$ satisfying the above equivalent conditions, which is an analogue of $\mathcal{A}(k, g, \ell)$ in Section 1.

Remark. Although $\mathcal{A}(k, g, \ell)$ is always finite, the set $\mathcal{D}(K, r, \pi)$ may not be finite since the Drinfeld module analogue of the Shafarevich conjecture, which is a base of the finiteness of $\mathcal{A}(k, g, \ell)$, does not hold. For example, for any $a \in A$, consider the rank-two Drinfeld module $\phi^{(a)} : A \to F[\tau]$ defined by $\phi^{(a)}(t) = t + a \tau + \tau^2$. Then it is easily seen that $\phi^{(a)}$ has everywhere good reduction and the set $\{[\phi^{(a)}] ; a \in A\}$ of isomorphism classes is infinite [7, Example 5.11]. For this reason, the finiteness of $\mathcal{D}(K, r, \pi)$ is still unknown. In fact if $r \geq 2$ and $\pi = t$, then we can construct an infinite subset of $\mathcal{D}(K, r, t)$ [7, Proposition 5.15].

§ 3. Outline of proofs

Throughout this section, let $\phi$ be a rank-$r$ Drinfeld module over $K$ and suppose that $[\phi] \in \mathcal{D}(K, r, \pi)$ for a monic irreducible element $\pi \in A$. Using the strategy in [11] adapted to Drinfeld modules, we would like to show that there are some contradiction if $\deg(\pi)$ is sufficiently large.

§ 3.1. Sketch of the proof of Theorem 1.2

Suppose that $r = p^\nu$ for some positive integer $\nu > 0$ and $r$ does not divide $[K : F]_i$. We may assume that $\deg(\pi) > 1$.

Let $u$ be a finite place of $K$ lying above $\pi$. Recall that there exists a separable extension $(K', w)$ of $(K, u)$ such that $\phi$ has stable reduction at $w$ and $e_{w|u}$ divides
\( \prod_{s=1}^{r}(q^s - 1) \). Applying the Drinfeld-Tate uniformization [3] to \( \phi \) seen as a Drinfeld module over \( K'_w \), we obtain an exact sequence

\[
0 \rightarrow \psi[\pi] \rightarrow \phi[\pi] \rightarrow H_{\mathbb{F}_n} \rightarrow 0
\]

of \( \mathbb{F}_\pi[G_{K'_w}] \)-modules, where \( \psi \) is a Drinfeld module over \( K'_w \) with good reduction and \( H_{\mathbb{F}_n} \) is a finite \( \mathbb{F}_\pi \)-vector space which \( G_{K'_w} \) acts as a finite group. Denote by \( \tilde{\rho}_{\psi,\pi} \) and \( \tilde{\rho}_{H_{\mathbb{F}_n}} \) the \( G_{K'_w} \)-representations attached to \( \psi[\pi] \) and \( H_{\mathbb{F}_n} \), respectively. Then the sequence (3.1) means that the semisimplification \( \tilde{\rho}_{\phi,\pi}^{\text{ss}} \) of \( \tilde{\rho}_{\phi,\pi} \) is of the form \( \tilde{\rho}_{\phi,\pi}^{\text{ss}} = \tilde{\rho}_{\psi,\pi}^{\text{ss}} \oplus \tilde{\rho}_{H_{\mathbb{F}_n}}^{\text{ss}} \).

Estimating the ramification of \( H_{\mathbb{F}_n} \), we can take a finite separable extension \( L \) of \( K'_w \) satisfying the followings:

- the action of the inertia subgroup \( I_L \) on \( H_{\mathbb{F}_n} \) is trivial,
- for the maximal tamely ramified extension \( L_0 \) of \( K'_w \) in \( L \), the ramification index \( e(L_0/K'_w) \) divides the integer \( \prod_{s=1}^{r}(q^s - 1) \).

Write \( e_{u,\phi} := e(L_0/F_\pi) \) for the absolute ramification index of \( L_0 \), so that it divides the integer \( M(K, r) := [K : F](q^r - 1)\prod_{s=1}^{r}(q^s - 1)^2 \) by construction. Note that \( M(K, r) \) is prime to \( p \).

Now we see that \( \tilde{\rho}_{\phi,\pi}^{\text{ss}} (= \tilde{\rho}_{\psi,\pi}^{\text{ss}} \oplus \tilde{\rho}_{H_{\mathbb{F}_n}}^{\text{ss}}) \) is an irreducible factor of \( \tilde{\rho}_{\phi,\pi}^{\text{ss}} \). If it comes from \( \tilde{\rho}_{\psi,\pi}^{\text{ss}} \), then by Theorem 2.14 of [4] there exists an integer \( j_s \in [0, e_{u,\phi}] \cap \mathbb{Z} \) such that

\[
\chi^s_{\pi}|_{I_{L_0}} = \omega^j_1,
\]

where \( \omega^j_1 : I_{L_0} \rightarrow \mathbb{F}_\pi^\times \) is the level-one fundamental character. On the other hand, if \( \chi^s_{\pi} \) is a factor of \( \tilde{\rho}_{H_{\mathbb{F}_n}}^{\text{ss}} \), then we can show that \( \chi^s_{\pi}|_{I_{L_0}} = \omega^j_1 = 1 \) by Theorem 2.14 of [4] and a little computation. Since the relation \( \chi_{\pi}|_{I_{L_0}} = \omega^e_{u,\phi} \) is also known [6, Proposition 9.4.3], for any \( 1 \leq s \leq r \), we obtain

\[
(3.2) \quad i_se_{u,\phi} \equiv j_s \pmod{q^s - 1}
\]

for some integer \( j_s \in [0, e_{u,\phi}] \cap \mathbb{Z} \).

Let \( v \) be a finite place of \( K \) above \( t \). Since \( \phi \) has good reduction at \( v \), for any integer \( n \), the characteristic polynomial \( P_{v,n}(x) = \det(x - \text{Frob}_v^n|T_{\pi}(\phi)) \) of \( n \)-power of Frobenius element at \( v \) is well-defined and it has coefficients in \( A \). By the above relation (3.2), the congruence

\[
P_{v,e_{u,\phi}}(x) \equiv \prod_{s=1}^{r}(x - \chi_{\pi}(\text{Frob}_v)^{i_se_{u,\phi}}) \equiv \prod_{s=1}^{r}(x - \chi_{\pi}(\text{Frob}_v)^{j_s}) \equiv \prod_{s=1}^{r}(x - t^{f_v(i_j)}) \pmod{\pi}
\]
holds. We can check that the absolute values of all coefficients of $P_{v,e_{u,\phi}}(x)$ and $\prod_{s=1}^{r}(x-t_{i_{s}})$ are smaller than $q^{[K:F]M(K,r)}$. Assume that

$$(3.3)\quad \deg(\pi) > r[K:F]M(K,r).$$

Since the absolute value of $\pi$ is $|\pi| = q^{\deg(\pi)}$, the above congruence implies $P_{v,e_{u,\phi}}(x) = \prod_{s=1}^{r}(x-t_{i_{s}})$. Comparing the absolute values of roots of $P_{v,e_{u,\phi}}(x)$ and $\prod_{s=1}^{r}(x-t_{i_{s}})$, we see that $r$ divides $e_{u,\phi}$. In particular $r|M(K,r)$.

Apply the same arguments to any place $u$ of $K$ lying above $\pi$ and set $e_{\phi} := \gcd\{e_{u,\phi}; u|\pi\}$. Denote by $K_{s}$ the separable closure of $F$ in $K$. There are only finitely many ramified places of $F$ in $K_{s}$, so that we may assume that $\pi$ is unramified in $K_{s}$. Note that if $K \neq K_{s}$, then $K/K_{s}$ is totally ramified at any places. Hence all $e_{u|\pi}$ must divide $[K:F]$. Since $r$ is a $p$-power, we see that $r|[K:F]$. This is a contradiction.

§ 3.2. Sketch of the proof of Theorem 1.3

Next, we consider the case where $r$ has a divisor $r_{0} \geq 2$ which is prime to $p$, so that $r = r_{0}p^{\nu}$ for some integer $\nu \geq 0$. We also define the index $e_{\phi}$ same as the previous subsection and assume (3.3).

Now let $i_{1}, \ldots, i_{r}$ be positive integers satisfying $\phi_{\phi,\pi}^{\pi_{0}} \simeq \chi_{\pi}^{i_{1}} \oplus \cdots \oplus \chi_{\pi}^{i_{r}}$. For any $r$-tuple $s = (s_{1}, \ldots, s_{r})$ of integers $1 \leq s_{1}, \ldots, s_{r} \leq r$, set $\varepsilon_{s} := \chi_{\pi}^{i_{s_{1}}+\cdots+i_{s_{r}}-1}$ and define

$$\epsilon := (\varepsilon_{s}) : G_{F} \to (\mathbb{F}_{\pi}^{\times})^{\oplus r}.$$ 

Set $m_{\phi} := \#\epsilon(G_{F})$, which is the least common multiple of the orders of $\varepsilon_{s}$. Since $\epsilon$ factors through $\mathbb{F}_{\pi}^{\times}$, the integer $m_{\phi}$ divides $q_{\pi} - 1$. Using the assumption (3.3), we can also prove that $m_{\phi}$ divides $e_{\phi}$, and hence $m_{\phi}|M(K,r)$. By the consequence of the effective Chebotrev density theorem [2, Corollary 3.4], we can prove the following.

**Proposition 3.1.** For any positive integer $m$ dividing $q_{\pi} - 1$, there exists a positive constant $C = C(K,m)$ depending only on $K$ and $m$ which satisfies the following: if $\deg(\pi) > C$, then there exist a monic irreducible element $\pi_{0} \in A$ distinct to $\pi$ and a place $v$ of $K$ above $\pi_{0}$ such that

- $\epsilon(Frob_{\pi_{0}}) = 1$,
- $\deg(\pi_{0}) < \deg(\pi)$,
- $f_{v|\pi_{0}} = 1$.

We also assume that $\deg(\pi) > \max\{C(K,m); m|M(K,r)\}$. Take $\pi_{0}$ and $v$ as in Proposition 3.1. Recall that $r$ is of the form $r = r_{0}p^{\nu}$. Under the above assumptions, we can prove that the existence of $[\phi] \in \mathcal{D}(K,r,\pi)$ contradicts to properties of $v|\pi_{0}$ as follows. For the characteristic polynomial

$$P_{v,1}(x) = a_{r} + a_{r-1}x + \cdots + a_{p^{\nu}}x^{r-p^{\nu}} + \cdots + x^{r} \in A[x]$$
of \( \text{Frob}_v \), since \( \epsilon(\text{Frob}_v) = \epsilon(\text{Frob}_{\pi_0}) = 1 \), we can check that the \( r_0 \)-power \( (a_{p^\nu})^{r_0} \) of the coefficient of \( x^{p^\nu} \) in \( P_{v,1}(x) \) satisfies

\[
(a_{p^\nu})^{r_0} \equiv (-1)^r \left( \frac{r}{p^\nu} \right)^{r_0} \pi_0 \pmod{\pi}.
\]

Since \( \left( \frac{r}{p^\nu} \right) \) is prime to \( p \), the right hand side of (3.4) is not zero. We see that \( j(a_{p^\nu})^{r_0} \) is prime to \( p \), the right hand side of (3.4) is not zero. We see that \( j = \deg(\pi_0) < \deg(\pi) \) imply that \( (a_{p^\nu})^{r_0} = \alpha \pi_0 \) for some \( \alpha \in \mathbb{F}_q^* \), which contradicts to the assumption \( r_0 \geq 2 \).

### § 4. Construction of a Drinfeld module contained in \( \mathcal{O}(K, r, \pi) \)

Suppose that \( r \) divides \( [K : F] \). Under this assumption, we can construct the rank-\( r \) Drinfeld module \( \Phi \) over \( K \) such that \( [\Phi] \in \mathcal{O}(K, r, \pi) \) for any \( \pi \). If \( r = 1 \), then the Carlitz module \( \mathcal{C} \) satisfies the properties in Proposition 2.5. Hence we may assume that \( r > 1 \).

First, we prepare some notations. Since \( r \) is a \( p \)-power, the map \( A \to A; a \mapsto a^r \) is a ring homomorphism. For any element \( a = \sum_{i=0}^{n} x_i \in A \) with \( x_i \in \mathbb{F}_q \), set \( \hat{a} := \sum_{i=0}^{n} x_i^{1/r} \). Then \( A \to A; a \mapsto \hat{a} \) is a ring automorphism and the composite \( A \to A; a \mapsto \hat{a}^r \) is an \( \mathbb{F}_q \)-algebra homomorphism.

By the general theory on function fields, the separable closure \( K_s \) of \( F \) in \( K \) coincides with the field \( K^{[K:F]} \) of all \( [K : F] \)-power elements of \( K \). Since \( r \) divides \( [K : F] \), the \( r \)-power root \( \sqrt[r]{t} \) of \( t \in F \) is contained in \( K \). Now consider the \( \mathbb{F}_q \)-algebra homomorphism \( \iota : A \to K \) given by \( \iota(t) = \sqrt[r]{t} \) and define the rank-one Drinfeld module \( \mathcal{C}' \) over \( (K, \iota) \) by \( \mathcal{C}' = \sqrt[r]{t} + \tau \). Then we can relate \( \mathcal{C}' \) with the Carlitz module \( \mathcal{C} \) as follows.

**Lemma 4.1.** For any element \( \lambda \in \mathcal{C}'[\pi] \), there exists a unique \( \delta \in \mathcal{C} [\pi] \) such that \( \lambda = \delta^{1/r} \).

Since \( \iota(\hat{a}^r) = a \) for any \( a \in A \) by construction, the composite

\[
\Phi : A \xrightarrow{(\hat{\cdot})} A \xrightarrow{(\cdot)^r} A \xrightarrow{\mathcal{C}'} K\{\tau\}
\]

is an \( \mathbb{F}_q \)-algebra homomorphism and

\[
\Phi_\iota = \mathcal{C}'_{\mathcal{O}^r} = (\sqrt[r]{t} + \tau)^r = t + \cdots + \tau^r.
\]

Therefore \( \Phi \) is a rank-\( r \) Drinfeld module over \( K \) and has good reduction at any finite place \( v \) of \( K \) since \( \sqrt[r]{t} \in \mathcal{O}_{K,v} \). Note that if \( r = 1 \), then \( \Phi \) is the Carlitz module \( \mathcal{C} \). Computing the \( G_K \)-action on \( \Phi[\pi] \) by using Lemma 4.1, we obtain the following.
Proposition 4.2. Let \( i \) be a positive integer satisfying \( ir \equiv 1 \pmod{q_\pi - 1} \). Suppose that \( r \) divides \( [K : F]_1 \). Then the representation \( \bar{\rho}_{\Phi, \pi} \) is of the form

\[
\begin{pmatrix}
\chi^i \\
* \\
\ddots \\
\chi^i
\end{pmatrix}
\]

In particular \( \mathcal{D}(K, r, \pi) \) is never empty for any \( \pi \).

References