Semiconjugacies for skew products of interval maps

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Abstract

Distribution functions of non-atomic Gibbs measures on the unit interval define natural semiconjugacies between maps on [0, 1]. Using this method we extend a result of Milnor and Thurston in [3] about the semiconjugacy of unimodal maps to skew products with maps of the interval as fiber maps.

1 Introduction

In this note we use the existence of Gibbs measures for a discrete time dynamical system to define a semiconjugacy between the system and a piecewise linear map. In particular, we discuss the analogue of this construction in the case of skew products $(X \times Y, \tau, (T_x)_{x \in X})$ where $\tau : X \to X, T_x : Y \to Y (x \in X)$ and

$$T(x, y) = (\tau(x), T_x(y)).$$

In the latter case the notion of a Gibbs measure can be generalized to that of Gibbs families whose existence and uniqueness was discussed in [1]. Also recall that a dynamical system $T : Z \to Z$ is called semiconjugate to the system

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\( T' : Z' \to Z' \) if there is a continuous surjective map \( \Pi : Z \to Z' \) such that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{T} & Z \\
\downarrow \Pi & & \downarrow \Pi \\
Z' & \xrightarrow{T'} & Z'
\end{array}
\]

commutes, and we call \( \Pi : X \times Y \to X' \times Y' \) a semiconjugacy between the skew products \((X \times Y, T)\) and \((X' \times Y', T')\) if \( \Pi \) semiconjugates the dynamical systems and if \( \Pi \) maps fibers to fibers, i.e. if \( \Pi(\{x\} \times Y) \subset \{x'\} \times Y' \) for some \( x' \in X' \).

Consider the special case of a skew product where \( Y = [0, 1] \) and where each \( T_x \) is a piecewise continuous and monotone map of the interval \( Y \) with positive relative topological entropy \( h(T_x) \). Certain fiberwise expanding transformations \( T \) will be shown to be semiconjugate to a skew product where each fiber map is a continuous piecewise monotone map of the interval with slope \( \exp h(T_x) \).

Note that this result parallels the case of a map of the interval since the theory of skew products and their Gibbs families reduces to this case if \( X \) consists of a single point. For unimodal maps we rediscover a result of Mihno and Thurston in [3], where it has been shown in Theorem 7.4, that every unimodal map, for which the number of monotonicity intervals of \( T^n \) increases exponentially fast, is semiconjugate to a unimodal map with constant common slopes on each of the monotonicity branches. The proof given here is different.

The idea of the proof relies on the following simple fact. If \( \mu \) is a distribution on the unit interval, then its distribution function is monotone, surjective and even continuous if \( \mu \) has no atoms. Hence the Milnor-Thurston result is a statement of a piecewise scaling property of a distribution function without any atom. Such distributions are obtained as non-atomic Gibbs measures, in particular, as measures of maximal entropy.

In order to be more precise, let \( T : Z \to Z \) be a dynamical system and \( \varphi : Z \to \mathbb{R} \) be a function. Recall that a measure \( \mu \) is called a Gibbs measure for \( \varphi \) if the Jacobian \( d\mu \circ T/d\mu \) is defined \( \mu \)-a.e. and is given by

\[
\frac{d\mu \circ T}{d\mu} = e^{\varphi}.
\]

The following standard chain of arguments gives the existence of a Gibbs measure in the case of an open and expanding map \( T \) acting on a compact space \( Z \) and a continuous function \( \varphi \). By these assumptions, the map \( T \) has locally a constant number of preimages, which implies that \( T \) acts on continuous functions by its Perron-Frobenius operator

\[
V_\varphi f(y) = \sum_{T(y') = y} f(y') e^\varphi(y').
\]

Furthermore, its dual operator acts continuously on the space of signed measures on \( Z \). Therefore, by the Schauder-Tychonoff theorem there exists an eigenvalue
\( \lambda > 0 \) and a measure \( \mu \) such that \( d\mu \circ T/d\mu = \lambda \exp(-\varphi) \), where \( d\mu \circ T/d\mu \) refers to the Jacobian. In other words, \( \mu \) is a Gibbs measure for the potential \( \varphi + \log \lambda \).

In the case of skew products we use the notion of Gibbs families on skew products as a generalization of Gibbs measures. Recall that a family \( \{\mu_x : x \in X\} \) of probability measures on \( Y \) is called a Gibbs family for a measurable function \( \varphi : X \times Y \to \mathbb{R} \) if there exists a positive measurable function (called gauge function) \( A : X \to \mathbb{R} \), such that, for each \( x \in X \), the Jacobian of \( \mu_x \) is given by

\[
\frac{d\mu_{\varphi(x)} \circ T_x}{d\mu_x} = A(x) \exp(-\varphi).
\] (1)

Using the existence of Gibbs families we extend the result of semiconjugacies for maps of the interval to certain skew products where the maps \( T_x \) \( (x \in X) \) are maps of the interval.

## 2 Semiconjugacies for skew-products

In this section we prove our result about semiconjugacies. We begin with the case of piecewise monotone map \( T \) of a totally ordered Polish space \( X \). Recall that \( X \) is totally ordered if there exists an order relation \( \leq \) such that for each \( x, y \in X \) either \( x \leq y \) or \( y \leq x \) and \( x \leq y \leq x \) implies that \( x = y \). This gives rise to a further relation \( \prec \), where \( x \prec y \) if \( x \leq y \) and \( x \neq y \). With this setting, the notion of closed and open intervals can be easily extended to the space \( X \) and these intervals will be denoted by \([a, b]\) and \((a, b)\), respectively. The topology on \( X \) is assumed to be generated by the open intervals or in other words, the topology on \( X \) is the order topology.

The map \( T \) is referred to be piecewise continuous and monotone if there exists a finite partition \( \alpha \) of \( X \) into intervals such that for each \( a \in \alpha \) the restriction \( T|_a \) is continuous and monotone. Let \( m \) be a non-atomic and nonsingular probability measure on \( X \) and let \( \Pi : X \to [0, 1] \subset \mathbb{R} \) and \( S : [0, 1] \to [0, 1] \) be defined as follows.

\[
\Pi : X \to [0, 1], \quad x \mapsto m(\{z \in X \mid z \leq x\}),
\]

\[
S : [0, 1] \to [0, 1], \quad y \mapsto \Pi(Tx) \text{ where } x \in \Pi^{-1}(\{y\}).
\]

Note that, since \( m \) has no atoms and is nonsingular, the map \( \Pi \) is onto, and \( S \) is well defined. Moreover, we have that \( S \circ \Pi = \Pi \circ T \) and \( S \) is piecewise continuous and monotone on \( \Pi(a) \) for each \( a \in \alpha \). Furthermore, we obtain the following immediate result.

**Proposition 2.1.** The map \( \Pi \) is continuous and semiconjugates \( T \) and \( S \). Furthermore, \( S \) is continuous and monotone on the interior \((\Pi(a))^\circ \) of \( \Pi(a) \) for each \( a \in \alpha \), and \( \lambda = m \circ \Pi^{-1} \) where \( \lambda \) refers to the Lebesgue measure. Moreover, \( \Pi \) is a homeomorphism if and only if \( m((a, b]) \neq 0 \) for all \( a, b \in X, a \prec b \).
In case that \( m \) is a Gibbs measure for the potential \( \varphi \) the following proposition gives the relation between the derivative \( DS \) of \( S \) and \( \varphi \).

**Proposition 2.2.** Let \( m \) be a Gibbs measure for the potential \( \varphi \). Assume that \( y_0 \) belongs to the interior \( (\Pi(a))^o \) of \( \Pi(a) \) for some \( a \in \alpha \) and that \( \exp(\varphi) \) is constant on \( \Pi^{-1}\{y_0\} \) and continuous in \( \partial \Pi^{-1}\{y_0\} \). Then \( S \) is differentiable in \( y_0 \) and, for \( x \in \Pi^{-1}\{y_0\} \),

\[
DS(y_0) = \begin{cases}
  e^{\varphi(x)} & \text{if } T|_a \text{ is increasing} \\
  -e^{\varphi(x)} & \text{if } T|_a \text{ is decreasing}.
\end{cases}
\]

**Proof.** Assume without loss of generality that \( S \) is monotone increasing on \( a \in \alpha \).

For \( y, y_0 \in \Pi(a), y > y_0 \) and \( x, x_0 \in X \) such that \( \Pi(x) = y \) and \( \Pi(x_0) = y_0 \) we have that

\[
\frac{S(y) - S(y_0)}{y - y_0} = m([T(x_0), T(x)])
\]

If \( \exp(\varphi) \) is constant on \( \Pi^{-1}\{y_0\} \) and is continuous in \( \partial \Pi^{-1}\{y_0\} \) the limit as \( y \to y_0 \) is independent of the choice of the representatives of \( y_0 \) in \( X \). Hence,

\[
\lim_{y \to y_0} \frac{S(y) - S(y_0)}{y - y_0} = \frac{dm \circ T}{dm}(x_0) = e^{\varphi(x_0)}.
\]

\[\square\]

Note that the latter condition for the existence of \( DS \) can be reformulated as follows. If the assignment \( y \mapsto \exp(\varphi(\hat{x})) \), where \( y \in (\Pi(a))^o \) and \( \hat{x} \in \Pi^{-1}\{y\} \), is independent of the choice of \( \hat{x} \) and extends to a continuous function in \( y \) then \( DS(y) \) exists. Furthermore, there is a straightforward generalization of these results to skew products of the following class. Let \( X \) be a topological space, \( Y \) be a totally ordered space as above and \( T : X \times Y \to X \times Y_0 \), \( (x, y) \mapsto (\tau(x), T_x(y)) \) where each fiber map is monotone and continuous on each atom of the partition \( \alpha_x \) of \( Y \). Moreover, assume that \( \{\mu_x \mid x \in X\} \) is a family of non-atomic, nonsingular Borel probability measures on \( Y \) such that \( x \mapsto \mu_x \) is weak* continuous. We then have, for

\[
\Pi_x : Y \to [0,1], \quad y \mapsto (x, \mu_x(\{z \mid z \leq y\}))
\]

\[
S : X \times [0,1] \to X \times [0,1], \quad (x, y) \mapsto (\tau(x), \Pi_{\tau(x)}(T_x(y))
\]

where \( \hat{y} \in \Pi_x^{-1}\{x\} \).

**Proposition 2.3.** The map \( \Pi : X \times Y = X \times [0,1], (x, y) \mapsto (x, \Pi_x(y)) \) semi-conjugates the skew products \( T \) and \( S \), and \( S_{\hat{x}} \) is continuous and monotone on \( (\Pi_x(a))^o \) for each atom \( a \in \alpha_x \). The map \( \Pi \) is a homeomorphism if and only if \( \mu_x([a, b]) \neq 0 \) for all \( x \in X \), \( a, b \in Y \), \( a < b \).

Le \( \{\mu_x \mid x \in X\} \) be a weak* continuous Gibbs family for the continuous potential \( \varphi \) and continuous gauge function \( A : X \to \mathbb{R} \) having no atom on each
fiber. We then have for $x \in X$ and $y \in (\Pi_x(a))^o$ for $a \in \alpha_x$, such that the assignment $y \mapsto \exp(\varphi(y))$ is independent of the choice of $\hat{y} \in \Pi_x^{-1}\{y\}$ and continuous in $y$,

$$DS_x(y) = \begin{cases} e^{\varphi(x, \hat{y})} & : T_x|_a \text{ is increasing} \\ -e^{\varphi(x, \hat{y})} & : T_x|_a \text{ is decreasing.} \end{cases}$$

Proof. Since the assertions concerning the fiber maps follow by Propositions 2.1 and 2.2 it is left to show that $(x, y) \mapsto (x, \Pi_x(y))$ is continuous. So assume that $((x_n, y_n))$ is a sequence in $X \times Y$ converging to $(x, y)$. Since $\mu_{x_n}$ has no atoms for each $n \in \mathbb{N}$, $\lim_{m \to \infty} \Pi_x(\varphi(x_n)) = \Pi_x(y)$. Furthermore, the weak* continuity of $x \to \mu_x$ gives that $\lim_{m \to \infty} \mu_{x_n}(\{ z \mid z \leq y_m \}) = \mu_x(\{ z \mid z \leq y_m \})$ for all $m \in \mathbb{N}$. This essentially gives the assertion. \qed

Note that sufficient conditions for the existence of weak* continuous Gibbs families can be deduced from [1]. A skew product $T : X \times Y \to X \times Y$, where $X$ and $Y$ are compact metric spaces with metrics $d_X$ and $d_Y$, respectively, is called fiber expanding, if the fiber maps $T_x : \{ x \} \times Y \to \{ \tau(\varphi) \} \times Y$ are uniformly expanding in Ruelle’s sense. This means that there exists $a > 0$ and $\rho \in (0, 1)$ such that for $x \in X$ and $u, v' \in Y$ and $d_Y(T_x(u), v') < 2a$, then there exists a unique $v \in Y$ such that $T_x(v) = v'$ and $d_Y(u, v) < 2a$. Furthermore, we have that

$$d_Y(u, v) \leq \rho d_Y(T_x(u), T_x(v)).$$

The system $(X \times Y, T)$ is called topologically exact along fibers if, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that, for any $(x, y) \in X \times Y$ and $n \geq N$, we have that

$$T_x^n(B(y, \varepsilon)) = Y,$$

where $B(y, \varepsilon) \subset Y$ denotes the ball of radius $\varepsilon$ centered at the point $y$ and where $T_x^n = T_{\tau^{-1}(x)} \circ T_x^{-1}$ for $n \geq 1$. Under these conditions Gibbs families do exists (see [1]).

The (weak*) continuity of the Gibbs family depends on properties of the map

$$i : X \times Y \to \{ (x, (z, y)) \in X^2 \times Y : \varphi = \tau(x) \}$$

defined by $i((x, y)) = (x, T((x, y)))$. In order that a Gibbs family is weak* continuous it is sufficient that $i$ is a local homeomorphism.

## 3 Applications

Let $S : [0, 1] \to [0, 1]$ be a piecewise monotone and continuous map. By this we mean that there are finitely many points $0 = p_0 < p_1 < \ldots < p_s = 1$ partitioning the unit interval, so that for each $k \in \{0, 1 \ldots s-1\}$, $S|_{(p_k, p_{k+1})}$ can be extended to a monotone and continuous map on $J_k = [p_k, p_{k+1}]$. We first recall the Hofbauer-Keller construction in [2]. Dividing each point $p_k$ and all its forward and backward
iterates $p$ into two points $p^+ = \lim_{x \to p} x$ and $p^- = \lim_{x \to p} x$, one constructs a compact extension $(X, \tilde{S})$ of $([0,1], S)$, such that $\tilde{S}$ is an open map and the natural projection $\pi : X \to [0,1]$ is one-to-one except in countably many points. Hence for every continuous potential $\varphi : [0,1] \to \mathbb{R}$ there is a Gibbs measure $\tilde{m}$ on $X$ so that
\[
\int_X \tilde{V}f(\pi(z))\tilde{m}(dz) = \lambda \int_X f(\pi(z))\tilde{m}(dz).
\]
If $\tilde{m}$ has no atoms, then $m = \tilde{m} \circ \pi$ defines a Gibbs measure on $[0,1]$ for the potential $\varphi$.

**Proposition 3.1.** Let $S : [0,1] \to [0,1]$ be a continuous and piecewise monotone map with positive topological entropy $h(S)$. Then there exists a non-atomic Gibbs measure for the potential $\varphi = 0$ and $\lambda = e^{h(S)}$.

**Proof.** Let $(X, \tilde{S})$ denote the extension of $([0,1], S)$ as above. Let $\tilde{m}$ denote the Gibbs measure for $\varphi \circ \pi$ on $X$. It is well known that for piecewise continuous maps of the interval topological entropy equals the asymptotic growth rate of the number of inverse branches of $S^n$. By inspecting the construction in \cite{2} one can easily show that $\log \lambda$ is also equal to this asymptotic growth rate with respect to $\tilde{S}^n$, which implies that $\lambda > 1$ by assumption. Let $x \in X$. Then $\tilde{m}(\tilde{S}^n(x)) = \lambda^n \tilde{m}(\{x\})$. In case $x$ is non-periodic we have $\tilde{m}(\tilde{S}^n(x)) \to \infty$ unless $\tilde{m}(\{x\}) = 0$, and in case $\tilde{S}^n(x) = x$ for some $n \geq 1$ we get $\lambda = 1$ unless $\tilde{m}(\{x\}) = 0$. It follows that $\tilde{m}$ has no atoms, whence $\pi$ is a measure theoretic isomorphism and $m = \tilde{m} \circ \pi$ is a non-atomic Gibbs measure with $\lambda = \exp[h(S)]$. \hfill $\square$

Applying Propositions 2.1 and 2.2 in this situation immediately gives the following result which is the advertised generalization of the result in \cite{3}.

**Theorem 1.** Let $S : [0,1] \to [0,1]$ be a piecewise monotone and continuous transformation of the unit interval. Assume that
\[
\limsup_{n \to \infty} \frac{1}{n} \log c_n = h(S) = M > 0,
\]
where $c_n$ denotes the number of monotone branches of $S^n$. Then there exists a Gibbs measure $m$ for the constant potential with no atoms, and
\[
h(x) = m([0, x]) \quad 0 \leq x \leq 1
\]
defines a semiconjugacy between $S$ and a piecewise linear and continuous map $T$ of the interval with slope $e^M$.

**Remark 3.2.** The map $T : [0,1] \to [0,1]$ in Theorem 1 is defined as follows:
Let $p_0 = 0 < p_1 < \ldots < p_r = 1$ denote the coarsest partition so that $S$ is monotone on each of the intervals $J_k = [p_k, p_{k+1}]$. Let $a_k = h(p_k)$. In case that $S$ is non-decreasing on $[p_0, p_1]$, for $a_k \leq y \leq a_{k+1}$

$$T(y) = h(S(p_0)) + e^M \left( 2 \sum_{j=1}^{k} (-1)^{j+1} a_j + (-1)^k y \right).$$

(2)

Similarly, if $S$ is non-increasing on $[p_0, p_1]$, for $a_k \leq y \leq a_{k+1}$

$$T(y) = h(S(p_0)) - e^M \left( 2 \sum_{j=1}^{k} (-1)^{j+1} a_j - (-1)^k y \right).$$

(3)

If $S$ is unimodal with turning point $p_1 = c$ and $T(0) = T(1) = 0$, then

$$T(y) = \begin{cases} 
  e^M y & \text{if } y \leq 1/2 \\
  e^M (1-y) & \text{if } y \geq 1/2.
\end{cases}$$

It is also immediately clear that $h$ is a conjugacy if the Gibbs measure $m$ is positive on non-empty open intervals. This occurs for example, if the map $T$ is piecewise expanding.

We give a short proof of (2) and (3). For $x \in [p_k, p_{k+1})$ and $S(x) \geq S(p_k)$ one has

$$h(S(x)) = m([0, S(x)]) = m([0, S(p_k)]) + m(S(p_k, x]) = h(S(p_k)) + e^M (h(x) - h(p_k)).$$

Similarly, for $x \in [p_k, p_{k+1})$ and $S(x) \leq S(p_k)$ one has

$$h(S(x)) = m([0, S(x)]) = m([0, S(p_k)]) - m(S(p_k, x]) = h(S(p_k)) - e^M (h(x) - h(p_k)).$$

By induction one shows in case that $S$ is non-decreasing on the first interval

$$h(S(p_k)) = h(S(p_0)) + 2e^M \sum_{j=1}^{k-1} (-1)^{j+1} a_j + e^M (-1)^{k+1} a_k,$$

and similarly if $S$ is non-increasing on the first interval. If $T$ is defined as in Remark 3.2, we get $h \circ S = T \circ h$.

Suppose $T$ is semiconjugate to the piecewise linear map $S$ with slope $\lambda$ and with semiconjugacy $h$. Clearly, $h$ defines a probability measure $m$ on $[0,1]$ and satisfies

$$h(T(x)) = m([0, T(x)]) = h(T(p_k)) \pm \lambda m([p_k, x])$$
for $x \in [p_k, p_{k+1}]$. This implies that $m$ is a Gibbs measure. If this Gibbs measure is unique, there is only one semiconjugacy to a piecewise linear map $S$ with constant slope.

In case of skew products, the existence of a Gibbs family is equivalent to the existence of an eigenspace for some relative version of the transfer operator. Namely, for a skew product $(X \times Y, T)$ and a Borel measurable function $\varphi : X \times Y \to \mathbb{R}$ the family $\{\mu_x | x \in X\}$ is a Gibbs family (cf. section 1) for $\varphi$ if and only if there exists a Borel measurable function $A_\varphi : X \to \mathbb{R}$ such that for $x \in X$ and $f \in L^1(\mu_x)$ we have that

$$\int V_x f(y) \mu_{\tau(x)}(dy) = A_\varphi(x) \int f(y) \mu_x(dy),$$

where $V_x f(y) := \sum_{T_n(y')=y} f(y') e^{\varphi(y')}$ denotes the relative transfer operator.

We conclude describing two setups when Proposition 2.3 can be applied.

**Example 1.** Let $(X \times [0,1], T)$ be a skew product where $\tau : X \to X$ is bounded-to-one and each fiber map $T_x$ is a piecewise continuous and monotone map of the interval $Y = [0,1]$. Like in the case of an interval map as above we split each point in the partition $p_0(x) < p_1(x) < \ldots < p_{s(a)}(x)$ for the fiber map $T_x$ over $x \in X$ into two points, as well as their grand orbits. This procedure does not give a continuous extension in general, but we assume here it does. The extended system is then a fibered system (no longer a skew product in general), denoted by $(\tilde{Y}, \tilde{T})$. Taking the order topology we may assume w.l.o.g. that for each $x \in X$ the map $T_x$ is open. If this Hofbauer-Keller extension is fiberwise expanding and exact along fibers we can proceed by taking $\varphi : X \times [0,1] \to \mathbb{R}$ to be constant, hence its lift $\tilde{\varphi} : \tilde{Y} \to \mathbb{R}$ is Hölder continuous in the order space topology. Hence by [1], if $i : \tilde{Y} \to X \times \tilde{Y}$, $i(\tilde{y}) = (\pi(\tilde{y}), \tilde{T}(\tilde{y}))$ is a local homeomorphism, where $\pi : \tilde{Y} \to X$ denotes the canonical projection, the semiconjugacy of $T$ exists according to Proposition 2.3.

**Example 2.** If $T : X \times Y \to X \times Y$ is an open map and bounded-to-one, the operator $V_x : C(\{x\} \times Y) \to C(\{\tau(x)\} \times Y)$ acts on continuous functions for each $x \in X$. Moreover, we consider the map

$$V^* : C(X, C^*(Y)) \to C(X, C^*(Y))$$

defined by

$$\int f d\mu_x = \int V_x f(\tau(x), \cdot) \mu_{\tau(x)},$$

where $\mu \in C(X, C^*(Y))$ and $f \in C(Y)$.

For $\mu \in C(X, C^*(Y))$ define

$$(L\mu)_x = V^* \mu_x / V^* \mu_x (Y),$$
and note that it is continuous since
\[ \| V^* \mu \|_{\infty} = \sup_{x \in X} \sup_{f \in C(Y),\|f\|_\infty = 1} \| \int f V_x^* d\mu_x \| \leq \| \mu \|_{\infty} \sup_{x \in X} \| V_x \|_1 \| f \|_\infty. \]

Define $\mathcal{M}$ to be the set of all $\mu = (\mu_x)_{x \in X} \in C(X, C^*(Y))$ such that for all $f \in C(Y)$ with $\| f \|_\infty \leq 1$ the map $x \mapsto \int f d\mu_x$ is Hölder continuous with Hölder exponent $s$ and Hölder constant bounded by some $M$ (independently of $f$).

**Proposition 3.3.** Let $(X \times Y, T)$ be a skew product with open map $T$ and assume that $L$ leaves $\mathcal{M}$ invariant. For every continuous potential $\varphi : X \times Y \to \mathbb{R}$ there exists a Gibbs family $\{ \mu_x : x \in X \}$. Moreover, for this family the map $x \mapsto \mu_x$ is continuous in the weak* topology.

**Proof.** As it easily can be seen the set $M$ is convex. Assume that $(\mu^n)_{n \in \mathbb{N}}$ is a sequence in $M$ converging pointwise to $\mu$. By the triangle inequality, for any $f \in C(Y)$ with $\| f \|_\infty \leq 1$ and $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for $n \geq n_0$

\[ |\int f d\mu_x - \int f d\mu_y| \leq |\int f d\mu_x - \int f d\mu_x^n| + |\int f d\mu_x^n - \int f d\mu_y^n| + |\int f d\mu_y^n - \int f d\mu_y| \leq M d(x, y)^s + 2\epsilon. \]

Clearly $\mu \in M$, whence the set $M$ is compact. The proposition follows from the Schauder-Tychonoff fixed point theorem.

**References**

