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**ZETA FUNCTIONS FOR THE RENEWAL SHIFT**

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**ABSTRACT.** We exhibit a topological Markov shift on a countable alphabet with the property that for every sequence of complex numbers \( c_n \) such that \( \limsup_{n \to \infty} \sqrt{|c_n|} < \infty \) there exists a weight function \( A : X \to \mathbb{C} \) which depends only on the first two coordinates such that the corresponding weighted dynamical zeta function satisfies \( \zeta_A(z) = 1 + \sum_{i \geq 1} a_i z^i \).

1. INTRODUCTION

Let \( S \) be a countable set and \( A = (a_{ij})_{S \times S} \) a matrix of zeroes and ones. \( S \) is called the set of states. \( A \) is called a topological transition matrix if \( \forall a \in S \exists i, j \ (a_{ai} = a_{ija} = 1) \). If this is the case then one defines the (one sided) countable Markov shift generated by \( A \) to be

\[
X = \Sigma^+_A = \{ x \in S^{\mathbb{N} \cup \{0\}} : \forall i \ t_{x_i, x_{i+1}} = 1 \}.
\]

We endow this set with the metric \( d(x, y) := \left( \frac{1}{2} \right)^{\min\{n : x_n \neq y_n\}} \), and equip it with the action of the left shift map:

\[
T : \Sigma^+_A \to \Sigma^+_A, \quad (Tx)_i = x_{i+1}.
\]

Let \( \text{Fix}(T^n) := \{ x \in \Sigma^+_A : T^n x = x \} \).

Let \( A : X \to \mathbb{C} \) be some function, called a weight function. The generalized dynamical zeta function, for the weight function \( A \) is

\[
\zeta_A(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \text{Fix}(T^n)} \prod_{k=0}^{n-1} A(T^k x).
\]

These functions were introduced (in a more general context) by Ruelle [8],[9], as a generalization of certain generating functions which were considered by Artin and Mazur [1].

If \( |S| < \infty \) and \( A \) is regular enough (e.g., when \( \log A \) is Hölder continuous), then \( \zeta_A \) is holomorphic in a neighborhood of zero and its first pole is in \( e^{-P} \), where \( P \) is the topological pressure of \( \log A \) (see [9]). A series of studies have focused on meromorphic extensions of \( \zeta_A \) to larger domains (see for example [8], [5], [7], [4]).

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We show here that if \( |S| = \infty \) then no such results are possible, even if one restricts one attention to locally constant potentials. We do this by exhibiting a specific topological Markov shift with the following property: Every function \( f \) such that \( f(0) = 1 \), which is holomorphic in a neighborhood of zero, can be represented a dynamical zeta function for a suitable weight function \( A : X \rightarrow \mathbb{C} \) which depends only on the first two coordinates.

This topological Markov shift is the shift with set of states \( \mathbb{N} \) and transition matrix

\[
R = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We call this shift the renewal shift because of its obvious connection to renewal theory (see [2]). We prove:

**Theorem 1.** Let \( X \) be the renewal shift and \( \{c_n\}_{n=1}^{\infty} \) a sequence of complex numbers such that \( \lim_{n \to \infty} \sqrt[n]{|c_n|} < \infty \). There exists a function \( A : X \rightarrow \mathbb{C} \) which depends only on the first two coordinates, for which in the neighborhood of zero

\[
\frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i.
\]

In particular, any type of singular behavior can occur away from zero. This should be contrasted with the case \( |S| < \infty \), for which every zeta function with a weight function of the form \( A(x) = A(x_0, x_1) \) is rational [6]. We remark that the dynamical zeta functions without meromorphic extensions have been constructed before [3].

2. **Proof of Theorem 1**

Set

\[
c_i^* = \begin{cases} 
  c_i & c_i \neq 0 \\
  1 & c_i = 0
\end{cases}
\]

and

\[
\alpha_1 = c_1^* ; \quad \alpha_i = c_i^* / c_{i-1}^* \\
\beta_1 = -c_1 ; \quad \beta_i = -c_i / c_{i-1}^*.
\]
Let $A = (a_{ij})_{\mathbb{N} \times \mathbb{N}}$ be the matrix given by

$$A = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\
\alpha_1 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & \cdots
\end{pmatrix}$$

Let $A_n$ be the upper left $n \times n$ block. Set $r = \left( \lim_{n \to \infty} \sqrt[|c_n|] \right)^{-1}$. This number is positive or infinite, by the assumptions of the theorem.

**Lemma 1.** The following limit holds and is uniform on compacts in $D_r := \{ z : |z| < r \}$:

$$\lim_{n \to \infty} \det(1-zA_n) = 1 + \sum_{i=1}^{\infty} c_i z^i$$

**Proof.**

\[
\det(1-zA_n) = \begin{vmatrix}
1 - \beta_1 z & -\beta_2 z & \cdots & -\beta_{n-1} z & -\beta_n z \\
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & -\alpha_{n-1} z & 1
\end{vmatrix}
\]

\[
= (1 - \beta_1 z) \begin{vmatrix}
0 & -\alpha_3 z & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
-\alpha_1 z & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
-\beta_2 z & -\alpha_2 z & \cdots & 0 & +\cdots
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
a_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & 0 & 0 \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
(-1)^{n+1}(-\beta_n z) & 0 & -\alpha_3 z & \cdots & 1 \\
0 & 0 & 0 & \cdots & -\alpha_{n-1} z
\end{vmatrix}
\]
\[ \begin{align*}
&= 1 - \beta_1 z - \beta_2 \alpha_1 z^2 - \ldots - \beta_n \alpha_1 \cdot \ldots \cdot \alpha_{n-1} z^n \\
&= 1 + c_1 z + \frac{c_2}{c_1^*} \cdot c_1^* \cdot z^2 + \ldots + \frac{c_n}{c_{n-1}^*} \cdot c_{n-1}^* \cdot \ldots \cdot c_1^* \cdot z^n \\
&= 1 + c_1 z + \ldots + c_n z^n \xrightarrow{n \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i.
\end{align*} \]

This convergence is uniform on compacts in \( D_r \), because \( r \) is the radius of convergence of this power series.

\[ \square \]

**Lemma 2.** \( E := \{ \lambda \in \mathbb{C} : \exists n \det (\lambda I - A_n) = 0 \} \) is a bounded subset of \( \mathbb{C} \).

**Proof.** Else, \( \exists n_k \to \infty \) and \( |\lambda_{n_k}| \to \infty \), such that \( \det (\lambda_{n_k} I - A_{n_k}) = 0 \). Without loss of generality, assume that \( \forall k \ |\lambda_{n_k}| \geq \frac{2}{r} \) (if \( r = \infty \) assume that \( |\lambda_{n_k}| \geq 1 \).

According to the previous lemma, the following limit exists and is uniform on compacts in \( D_r = \{ z : |z| < r \} \):

\[ (3) \quad f(z) = \lim_{n \to \infty} \det (1 - zA_n) \]

Note that \( f(0) = 1 \), and that \( f \) is continuous in 0. In particular, since \( \lambda_{n_k}^{-1} \to 0 \) and \( \lambda_{n_k} \in D_r \)

\[ |f(0) - f(\lambda_{n_k}^{-1})| \xrightarrow{k \to \infty} 0. \]

By the uniform convergence of (3) in \( D_{r/2} \) (or in \( D_1 \) if \( r = \infty \)) we have that

\[ |f(\lambda_{n_k}^{-1}) - \det (1 - \lambda_{n_k}^{-1} A_{n_k})| \xrightarrow{k \to \infty} 0 \]

Hence, since \( \forall k \ \det (1 - \lambda_{n_k}^{-1} A_{n_k}) = 0 \),

\[ |f(0) - 0| \leq |f(0) - f(\lambda_{n_k}^{-1})| + |f(\lambda_{n_k}^{-1}) - \det (1 - \lambda_{n_k}^{-1} A_{n_k})| \xrightarrow{k \to \infty} 0 \]

which implies that \( 1 = f(0) = 0 \), a contradiction. \[ \square \]

We are now ready to prove the theorem. Let \( A : X \to \mathbb{C} \) be given by

\[ A(x_0, x_1, \ldots) = A_{x_0 x_1} \]

where \( A \) is given by (1).

Set

\[ Z_n = \sum_{x \in F(x)(T^n)} \prod_{k=0}^{n-1} A(T^k x) \]

Then

\[ \log \zeta_A = \sum_{n=1}^{\infty} \frac{Z_n}{n} \cdot Z_n. \]
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By the definition of $A$,

$$Z_n = \sum_{x \in Fix(T^n)} A_{x_0 x_1} A_{x_1 x_2} \cdots A_{x_{n-1} x_0}.$$  

$\forall x_0, \ldots, x_{n-1} \in \mathbb{N}$ if $A_{x_0 x_1} A_{x_1 x_2} \cdots A_{x_{n-1} x_0} > 0$ then

$$(x_0, x_1, \ldots, x_{n-1}; x_0, x_1, \ldots, x_{n-1}, \ldots)$$

belongs to $\Sigma^+_A$ and constitutes a periodic point of order $n$. Thus

$$Z_n = \sum_{x \in Fix(T^n)} \prod_{i=0}^{n-1} A(T^i x) = \sum_{x_1 \cdots x_n} A_{x_0 x_1} \cdots A_{x_{n-1} x_0}.$$  

By the definition of the renewal shift, if $(x_0, x_1, \ldots, x_{n-1}, x_0)$ is admissible then $\forall i x_i \leq n$ (if $m$ appears, so must $m-1, m-2, \ldots, 1$. Since there are at most $n$ different symbols $x_i$, $m$ must be smaller than $n$). Thus,

$$\forall n \leq N : Z_n = \sum_{x_0 \cdots x_{n-1} = 1}^{n} A_{x_0 x_1} \cdots A_{x_{n-1} x_0} =$$

$$= \sum_{x_0 \cdots x_{n-1} = 1}^{N} A_{x_0 x_1} \cdots A_{x_{n-1} x_0} = tr(A^n_N).$$

This shows that

$$\left| \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^{\infty} \frac{z^n}{n} \cdot tr(A^n_k) \right| \leq \sum_{n>N} \frac{z^n}{n} \cdot Z_n \left| + \sum_{n>N} \frac{z^n}{n} \cdot tr(A^n_N) \right|.$$  

We estimate these tails. According to the previous lemma, $E = \{ \lambda \in \mathbb{C} : \exists n \det(\lambda I - A_n) = 0 \}$ is bounded. Let $\lambda = \sup \{ |z| : z \in E \}$. Let $\lambda_1(k), \ldots, \lambda_k(k)$ the eigenvalues of $A_k$, written with multiplicities. Then $|\lambda_i(k)| \leq \lambda$. Using the fact that every matrix can be triangulated, it is easy to verify that

$$|tr(A^n_k)| = |\lambda_1(k)^n + \ldots + \lambda_k(k)^n| \leq k \lambda^n$$

Thus, for every $|z| < \lambda^{-1}$,

$$\left| \sum_{n>N} \frac{z^n}{n} \cdot Z_n \right| = \left| \sum_{n>N} \frac{z^n}{n} \cdot tr(A^n_n) \right|$$

$$\leq \sum_{n>N} \frac{|z^n|}{n} \cdot n \lambda^n = \sum_{n>N} |z \cdot \lambda|^n \xrightarrow{N \to \infty} 0.$$
and
\[
\left| \sum_{n>N} \frac{z^n}{n} \cdot \text{tr}(A_N^n) \right| \leq \sum_{n>N} |z\cdot \lambda|^n \xrightarrow{N \to \infty} 0.
\]
Thus, \( \forall |z| < \lambda^{-1} \)
\[
\left| \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^{\infty} \frac{z^n}{n} \cdot \text{tr}(A_N^n) \right|
\leq \left| \sum_{n>N} \frac{z^n}{n} \cdot Z_n \right| + \left| \sum_{n>N} \frac{z^n}{n} \cdot \text{tr}(A_N^n) \right| \xrightarrow{N \to \infty} 0.
\]
Using the Taylor expansion of \( z \mapsto \log(1-z) \) and the identities
\[
\text{tr}(A_N^n) = \lambda_1 (N)^n + \ldots + \lambda_N (N)^n
\]
and
\[
\det(1-zA_N) = (1-z\lambda_1 (N)) \cdot \ldots \cdot (1-z\lambda_N (N))
\]
it is not difficult to show that if \( |z| < \lambda^{-1} \) then
\[
- \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot \text{tr}(A_N^n) = \ln \det(1-zA_N)
\]
Thus, the following limit holds in \( D_{\lambda^{-1}} \)
\[
\ln \det(1-zA_N) \xrightarrow{N \to \infty} - \log \zeta_A (z).
\]
But by (2) if \( |z| < r \) then
\[
\det(1-zA_N) \xrightarrow{N \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i
\]
Hence, for \( |z| < \min\{r, \lambda^{-1}\} \) we have
\[
\frac{1}{\zeta_A (z)} = 1 + \sum_{i=1}^{\infty} c_i z^i
\]
as required. \( \square \)

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