ZETA FUNCTIONS FOR THE RENEWAL SHIFT

OMRI SARIG

ABSTRACT. We exhibit a topological Markov shift on a countable alphabet with the property that for every sequence of complex numbers $c_n$ such that $\limsup_{n \to \infty} \sqrt{|c_n|} < \infty$ there exists a weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates such that the corresponding weighted dynamical zeta function satisfies $\frac{1}{\zeta_A(z)} = 1 + \sum_{i \geq 1} c_i z^i$.

1. INTRODUCTION

Let $S$ be a countable set and $A = (t_{ij})_{S \times S}$ a matrix of zeroes and ones. $S$ is called the set of states. $A$ is called a topological transition matrix if $\forall a \in S \exists i, j$ $(t_{ai} = t_{ja} = 1)$. If this is the case then one defines the (one sided) countable Markov shift generated by $A$ to be

$$X = \Sigma_A^+ = \{ x \in S^{N \cup \{0\}} : \forall i t_{x_i x_{i+1}} = 1 \}.$$ 

We endow this set with the metric $d(x, y) := (\frac{1}{2})^{\min\{n : x_n \neq y_n\}}$, and equip it with the action of the left shift map:

$$T : \Sigma_A^+ \to \Sigma_A^+, \quad (Tx)_i = x_{i+1}.$$ 

Let $Fix(T^n) := \{ x \in \Sigma_A^+ : T^n x = x \}$.

Let $A : X \to \mathbb{C}$ be some function, called a weight function. The generalized dynamical zeta function, for the weight function $A$ is

$$\zeta_A(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in Fix(T^n)} \prod_{k=0}^{n-1} A(T^k x).$$

These functions were introduced (in a more general context) by Ruelle [8],[9], as a generalization of certain generating functions which were considered by Artin and Mazur [1].

If $|S| < \infty$ and $A$ is regular enough (e.g., when $\log A$ is Hölder continuous), then $\zeta_A$ is holomorphic in a neighborhood of zero and its first pole is in $e^{-P}$, where $P$ is the topological pressure of $\log A$ (see [9]). A series of studies have focused on meromorphic extensions of $\zeta_A$ to larger domains (see for example [8], [5], [7], [4]).

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OMRI SARIG

We show here that if $|S| = \infty$ then no such results are possible, even if one restricts one attention to locally constant potentials. We do this by exhibiting a specific topological Markov shift with the following property: Every function $f$ such that $f(0) = 1$, which is holomorphic in a neighborhood of zero, can be represented a dynamical zeta function for a suitable weight function $A : X \to \mathbb{C}$ which depends only on the first two coordinates.

This topological Markov shift is the shift with set of states $\mathbb{N}$ and transition matrix

$$R = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

We call this shift the renewal shift because of its obvious connection to renewal theory (see [2]). We prove:

**Theorem 1.** Let $X$ be the renewal shift and $\{c_n\}_{n=1}^\infty$ a sequence of complex numbers such that $\lim_{n \to \infty} \sqrt[n]{|c_n|} < \infty$. There exists a function $A : X \to \mathbb{C}$ which depends only on the first two coordinates, for which in the neighborhood of zero

$$\frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^\infty c_i z^i.$$

In particular, any type of singular behavior can occur away from zero. This should be contrasted with the case $|S| < \infty$, for which every zeta function with a weight function of the form $A(x) = A(x_0, x_1)$ is rational [6]. We remark that the dynamical zeta functions without meromorphic extensions have been constructed before [3].

2. PROOF OF THEOREM 1

Set

$$c_i^* = \begin{cases}
c_i & c_i \neq 0 \\
1 & c_i = 0
\end{cases}$$

and

$$\alpha_1 = c_1^* ; \quad \alpha_i = c_i^*/c_{i-1}^*; \quad \beta_1 = -c_1 ; \quad \beta_i = -c_i/c_{i-1}^*.$$
Let $A = (a_{ij})_{N \times N}$ be the matrix given by

$$A = \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \cdots \\
\alpha_1 & 0 & 0 & 0 & \cdots \\
0 & \alpha_2 & 0 & 0 & \cdots \\
0 & 0 & \alpha_3 & 0 & \cdots \\
0 & 0 & 0 & \alpha_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$  

(1)

Let $A_n$ be the upper left $n \times n$ block. Set $r = \left( \lim_{n \to \infty} \sqrt[n]{|c_n|} \right)^{-1}$. This number is positive or infinite, by the assumptions of the theorem.

**Lemma 1.** The following limit holds and is uniform on compacts in $D_r := \{ z : |z| < r \}$:

$$\lim_{n \to \infty} \det(1-zA_n) = 1 + \sum_{i=1}^{\infty} c_i z^i$$

(2)

Proof.

\[
\begin{vmatrix}
1 - \beta_1 z & -\beta_2 z & \cdots & -\beta_{n-1} z & -\beta_n z \\
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & -\alpha_{n-1} z & 1 \\
\end{vmatrix}
\]

\[
= (1 - \beta_1 z) \begin{vmatrix}
0 & -\alpha_3 z & \cdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
\end{vmatrix} \\
\]

\[
-(-\beta_2 z) \begin{vmatrix}
-\alpha_2 z & \cdots & 0 & + \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\alpha_{n-1} z & 1 \\
\end{vmatrix} \\
\]

\[
+(-1)^{n+1}(-\beta_n z) \begin{vmatrix}
-\alpha_1 z & 1 & 0 & \cdots & 0 \\
0 & -\alpha_2 z & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\alpha_{n-1} z \\
\end{vmatrix} \\
\]
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\[
1 - \beta_1 z - \beta_2 \alpha_1 z^2 - \ldots - \beta_n \alpha_1 \cdots \alpha_{n-1} z^n = 1 + c_1 z + c_2 \cdot c_1^{-1} \cdot z^2 + \ldots + \frac{c_n}{c_{n-1}^*} \cdot c_1^{-1} \cdot \cdots \cdot \frac{c_{n-1}^*}{c_{n-2}^*} \cdot z^n = 1 + c_1 z + \ldots + c_n z^n \xrightarrow{n \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i.
\]

This convergence is uniform on compacts in \(D_r\), because \(r\) is the radius of convergence of this power series.

\[\blacksquare\]

Lemma 2. \(E := \{ \lambda \in \mathbb{C} : \exists n \ \det(\lambda 1 - A_n) = 0 \}\) is a bounded subset of \(\mathbb{C}\).

\[\textbf{Proof.} \ \exists n \ k \neq \infty \text{ and } |\lambda_{n_k}| \to \infty, \text{ such that } \det(\lambda_{n_k} 1 - A_{n_k}) = 0. \text{ Without loss of generality, assume that } \forall k \ |\lambda_{n_k}| \geq \frac{2}{r} \ (\text{if } r = \infty \text{ assume that } |\lambda_{n_k}| \geq 1).

According to the previous lemma, the following limit exists and is uniform on compacts in \(D_r = \{ z : |z| < r \} : \)

\[(3) \ f(z) = \lim_{n \to \infty} \det(1 - zA_n)\]

Note that \(f(0) = 1\), and that \(f\) is continuous in \(0\). In particular, since \(\lambda_{n_k}^{-1} \to 0\) and \(\lambda_{n_k} \in D_r\)

\[|f(0) - f(\lambda_{n_k}^{-1})| \xrightarrow{k \to \infty} 0.\]

By the uniform convergence of (3) in \(\overline{D}_{r/2}\) (or in \(\overline{D}_1\) if \(r = \infty\)) we have that

\[|f(\lambda_{n_k}^{-1}) - \det(1 - \lambda_{n_k}^{-1}A_{n_k})| \xrightarrow{k \to \infty} 0\]

Hence, since \(\forall k \ \det(1 - \lambda_{n_k}^{-1}A_{n_k}) = 0,\)

\[|f(0) - 0| \leq |f(0) - f(\lambda_{n_k}^{-1})| + |f(\lambda_{n_k}^{-1}) - \det(1 - \lambda_{n_k}^{-1}A_{n_k})| \xrightarrow{k \to \infty} 0\]

which implies that \(1 = f(0) = 0\), a contradiction. \(\blacksquare\)

We are now ready to prove the theorem. Let \(A : X \to \mathbb{C}\) be given by

\[A(x_0, x_1, \ldots) = A_{x_0 x_1}\]

where \(A\) is given by (1).

Set

\[Z_n = \sum_{x \in \Phi(T^n)} \prod_{k=0}^{n-1} A(T^k x)\]

Then

\[\log \zeta_A = \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot Z_n.\]
By the definition of $A$,

$$Z_n = \sum_{x \in Fix(T^n)} A_{x_0 x_1} A_{x_1 x_2} \ldots A_{x_{n-1} x_0}.$$ 

$\forall x_0, \ldots, x_{n-1} \in \mathbb{N}$ if $A_{x_0 x_1} A_{x_1 x_2} \ldots A_{x_{n-1} x_0} > 0$ then 

$$(x_0, x_1, \ldots, x_{n-1}; x_0, x_1, \ldots, x_{n-1}; \ldots)$$

belongs to $\Sigma_A^+$ and constitutes a periodic point of order $n$. Thus

$$Z_n = \sum_{x \in Fix(T^n)} \prod_{i=0}^{n-1} A(T^i x) = \sum_{x_1 \ldots x_n} A_{x_0 x_1} \ldots A_{x_{n-1} x_0}.$$ 

By the definition of the renewal shift, if $(x_0, x_1, \ldots, x_{n-1}, x_0)$ is admissible then $\forall i x_i \leq n$ (if $m$ appears, so must $m-1, m-2, \ldots, 1$. Since there are at most $n$ different symbols $x_i$, $m$ must be smaller than $n$). Thus,

$$\forall n \leq N : Z_n = \sum_{x_0 \ldots x_{n-1} = 1}^n A_{x_0 x_1} \ldots A_{x_{n-1} x_0} = \sum_{x_0 \ldots x_{n-1} = 1}^N A_{x_0 x_1} \ldots A_{x_{n-1} x_0} = tr(A^n_N).$$

This shows that

$$\left| \sum_{n=1}^\infty \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^\infty \frac{z^n}{n} \cdot tr(A^n_k) \right| \leq \sum_{n> N} \frac{z^n}{n} \cdot Z_n \leq \sum_{n> N} \frac{z^n}{n} \cdot tr(A^n_N).$$

We estimate these tails. According to the previous lemma, $E = \{ \lambda \in \mathbb{C} : \exists n \ det(\lambda I - A_n) = 0 \}$ is bounded. Let $\lambda = \sup\{|z| : z \in E\}$. Let $\lambda_1(k), \ldots, \lambda_k(k)$ the eigenvalues of $A_k$, written with multiplicities. Then $|\lambda_i(k)| \leq \lambda$. Using the fact that every matrix can be triangulated, it is easy to verify that

$$|tr(A^n_k)| = |\lambda_1(k)^n + \ldots + \lambda_k(k)^n| \leq k \lambda^n$$

Thus, for every $|z| < \lambda^{-1}$,

$$\left| \sum_{n> N} \frac{z^n}{n} \cdot Z_n \right| = \left| \sum_{n> N} \frac{z^n}{n} \cdot tr(A^n_n) \right| \leq \sum_{n> N} \frac{|z^n|}{n} \cdot n \lambda^n = \sum_{n> N} |z \cdot \lambda|^n \xrightarrow{N \rightarrow \infty} 0.$$
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and

$$\left| \sum_{n>N} \frac{z^n}{n} \cdot tr(A_N^n) \right| \leq \sum_{n>N} |z \cdot \lambda|^n \xrightarrow{N \to \infty} 0.$$  

Thus, $\forall |z| < \lambda^{-1}$

$$\left| \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot Z_n - \sum_{k=1}^{\infty} \frac{z^n}{n} \cdot tr(A_N^n) \right| \leq \left| \sum_{n>N} \frac{z^n}{n} \cdot Z_n \right| + \left| \sum_{n>N} \frac{z^n}{n} \cdot tr(A_N^n) \right| \xrightarrow{N \to \infty} 0.$$  

Using the Taylor expansion of $z \mapsto \log(1 - z)$ and the identities

$$tr(A_N^n) = \lambda_1 (N)^n + \ldots + \lambda_N (N)^n$$  

and

$$\det(1 - zA_N) = (1 - z\lambda_1 (N)) \cdot \ldots \cdot (1 - z\lambda_N (N))$$  

it is not difficult to show that if $|z| < \lambda^{-1}$ then

$$- \sum_{n=1}^{\infty} \frac{z^n}{n} \cdot tr(A_N^n) = \ln \det(1 - zA_N)$$  

Thus, the following limit holds in $D_{\lambda^{-1}}$

$$\ln \det(1 - zA_N) \xrightarrow{N \to \infty} - \log \zeta_A(z).$$  

But by (2) if $|z| < r$ then

$$\det(1 - zA_N) \xrightarrow{N \to \infty} 1 + \sum_{i=1}^{\infty} c_i z^i$$  

Hence, for $|z| < \min\{r, \lambda^{-1}\}$ we have

$$\frac{1}{\zeta_A(z)} = 1 + \sum_{i=1}^{\infty} c_i z^i$$  

as required. \hfill \Box

REFERENCES


MATHEMATICS DEPARTMENT, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA
E-mail address: sarig@math.psu.edu