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Parabolic-elliptic systems with general density–pressure relations

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Abstract

This survey paper contains generalizations of results on the behavior of solutions of particular systems describing the interaction of gravitationally attracting particles that obey either the Maxwell–Boltzmann or the Fermi–Dirac statistics.

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1 Introduction and derivation of the system

We study in this paper parabolic-elliptic systems of the form

$$n_t = \nabla\cdot\left(D_*\left(\nabla p + n\nabla\varphi\right)\right), \tag{1}$$

$$\Delta\varphi = n, \tag{2}$$

which are considered in statistical mechanics as hydrodynamical models for self-interacting particles. Here $n = n(x,t) \geq 0$ is the density function defined for $(x,t) \in \Omega\times\mathbb{R}^+$, $\Omega \subset \mathbb{R}^d$, $\varphi = \varphi(x,t)$ is the Newtonian potential generated by the particles of density n, and the pressure $p \geq 0$ is determined by the density–pressure relation with a sufficiently regular (say, $C^2(\mathbb{R}^+\times\mathbb{R}^+)$) function p

$$p = p(n,\vartheta). \tag{3}$$

The parameter $\vartheta > 0$ plays the role of the temperature, and $D_* > 0$ is a coefficient which may depend on n, ϑ, φ, x, $\ldots$. Such systems can be studied either in the *canonical ensemble* (i.e. the *isothermal* setting), when $\vartheta =$ const is fixed, or in the *microcanonical ensemble* with a variable temperature: either $\vartheta = \vartheta(t)$ or $\vartheta = \vartheta(x,t)$. Then, the energy balance is described either by a relation like

$$E = c_0\int_\Omega p\,dx + \frac{1}{2}\int_\Omega n\varphi\,dx = \text{const} \tag{4}$$

(which, for a given n, defines $\vartheta = \vartheta(t)$ in an implicit way) in the former case, or by an evolution partial differential equation for $\vartheta = \vartheta(x,t)$ as was in [26, 27, 6] in the latter case. Thus, (1)–(2) can be viewed as evolution equations for self-consistent gravitational field φ. In fact, the starting point of the derivation of related *mean field* models in astrophysics in [17] consists of an analysis of kinetic equations whose evolution in time is governed by the Maximum Entropy Production Principle. More precisely, if $0 \le f = f(x,v,t)$ is the density of particles at the point $(x,t) \in \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^d$, moving at the velocity v, then f satisfies a kinetic equation

$$f_t + v \cdot \nabla_x f - \nabla\varphi \cdot \nabla_v f = -\nabla_v \mathbf{j}$$

with a general dissipation flux term $-\nabla_v \mathbf{j}$. Moreover, *a priori* an integral functional $S = \int_{\mathbb{R}^d} s(f(x,v,t))\,dv$, called the *local entropy*, is given. Distribution functions f that maximize this functional (under local density and pressure constraints, see (5), (6) below) have a particular form depending on some parameters $\lambda = \lambda(x,t)$, $\vartheta = \vartheta(t)$, etc., see the examples presented in Section 2. Then, the term $\nabla_v \mathbf{j}$ is determined (up to a positive diffusion coefficient) by the requirement that the system evolves with maximal entropy production rate at each moment t. Averaging f over the velocities $v \in \mathbb{R}^d$, and then passing to the limit of large friction (or large times) lead to "hydrodynamic" equations in the (x,t) space. For the details of that construction we refer to [17, 16] and [7].
Given the distribution f on the kinetic level, the spatio-temporal density is

$$n(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv, \tag{5}$$

and the pressure is defined by

$$p(x,t) = \frac{1}{d}\int_{\mathbb{R}^d} |v|^2 f(x,v,t)\,dv. \tag{6}$$

Thus, the first term in the energy (4) corresponds to the kinetic energy of particles (with the natural choice $c_0 = d/2$ in (4)), and the second is the potential energy of the system of self-gravitating particles. As we will see further, mild assumptions (26) on the form of f lead to natural density–pressure relations between n and p, see (28).

We consider the system (1)–(2) with the natural no-flux boundary condition on $\partial\Omega$

$$(\nabla p + n\nabla\varphi) \cdot \overline{\nu} = 0 \tag{7}$$

($\overline{\nu}$ is the unit exterior normal vector to $\partial\Omega$), and an initial condition

$$n(x,0) = n_0(x) \ge 0. \tag{8}$$

We impose on the potential φ either a physically acceptable "free" condition

$$\varphi = E_d * n, \tag{9}$$

$E_d(z) = -((d-2)\sigma_d)^{-1}|z|^{2-d}$ being the fundamental solution of the Laplacian in $\mathbb{R}^d$, $d \geq 3$, or the homogeneous Dirichlet boundary condition

$$\varphi_{|\partial\Omega} = 0, \tag{10}$$

which is mathematically somewhat simpler. In the case of radially symmetric solutions (10) is, however, equivalent to (9) which can be obtained by adding a constant to the potential φ, cf. the discussion of this issue in [8, 9, 3]. As a consequence of (7), total mass

$$M = \int_\Omega n(x,t)\, dx \tag{11}$$

is conserved during the evolution. Moreover, sufficiently regular solutions of the evolution problem with $n_0 \geq 0$ remain positive.

The main mathematical questions concerning the system are the following:

- existence, nonexistence and multiplicity of steady states, either for given M, ϑ or for M, E fixed,
- local in time existence of solutions of the evolution problem,
- asymptotics of global in time solutions,
- possibility of finite time blow up of solutions (corresponding to either a gravitational collapse or an explosion).

Two systems with particular density–pressure relations have been recently studied: for Brownian (or Maxwell–Boltzmann) and for Fermi–Dirac particles.

The model of self-gravitating Brownian particles, which consists of (1)–(2), (13) below, supplemented by (4), has been considered in [17, 15, 19] for radially symmetric solutions (n, φ) (with $c_0 = d/2$), and in [18, 10] without this symmetry assumption. Studies of the corresponding isothermal problem with $\vartheta \equiv 1$ had been conducted earlier, see e.g. [8, 1]. However, the motivations there had been a bit different — stemming from statistical mechanics of interacting charged particles in semiconductors, electrolytes, plasmas (see also [4] and [21] for different density–pressure relations) with (2) replaced by $\Delta\varphi = -n$, and afterwards for gravitating particles. Besides the statistical mechanics, systems of the form (1)–(2) appear also in modelling of chemotaxis phenomena in biology, generalizing the classical approach of E. F. Keller and L. A. Segel which involved the linear diffusion with $p = \text{const}\, n$. These biological models, supplemented by the homogeneous Neumann conditions for n and φ are applicable in description of concentration of either cells or microorganisms due to chemical agents. We refer the reader to [20] for a comprehensive review of these aspects of parabolic-elliptic systems like (1)–(2). The papers [15, 18, 10, 3, 9] deal with the Brownian particles models. The main issues are:

- gravitational collapse is possible for $d \geq 2$ in the isothermal model and for $d \geq 3$ in the nonisothermal model,
- the existence of steady states with prescribed mass and energy in $d \geq 3$ dimensions is controlled by the parameter E/M^2 which should be large enough.

Since the Fermi–Dirac model, see (17) below, involves nonlinear diffusion, even local in time existence of solutions is much harder to establish than in the Brownian (linear diffusion) case, see [7] where in the isothermal case a specific choice of the coefficient D_* has been considered. There are also many results on radially symmetric stationary solutions in [12, 13, 14]. In particular:

- structure of the set of steady states with given M and ϑ is different (and less complicated) than in the Brownian case ([13]),
- the existence of steady states with given mass and energy is controlled by the parameter $\min\left\{\frac{E}{M^2}, \frac{E}{M^{1+2/d}}\right\}$ ([11, 25]),
- gravitational collapse cannot occur in $d \leq 3$ dimensions in the isothermal case ([7]),
- the gravitational collapse is possible for $d \geq 4$ for suitable initial data in the nonisothermal case ([11]).

We will study in this paper:

- examples of density–pressure relations more general than Maxwell–Boltzmann and Fermi–Dirac (in Section 2),
- existence of entropy functionals and entropy production rates (in Section 3),
- existence of steady states with prescribed mass and sufficiently large energy (in Section 4),
- nonexistence of global in time solutions of (1)–(2) with general density–pressure relations (3), $D_* = 1$, one of the conditions (9), (10), and either negative initial entropy or low energy (in Section 5), and thus, *a fortiori*, we obtain nonexistence of steady states for arbitrary D_*,
- continuation of local in time solutions of (1)–(2) with polytropic density–pressure relations (in Section 6).

We do not consider here the question of the local in time existence of solutions of the evolution problem which seems to be a rather difficult question (mainly because of nonlinear boundary conditions for n and possible degeneracies of diffusion) in such a general setting with (3), cf. the isothermal Fermi–Dirac case in [7]. Note that results for systems of (several species of) *repulsing* particles with *linear* boundary conditions for n are in the monograph [21].

It is worth noting that while "local" results on the existence of steady states (i.e. for a small range of control parameters M, ϑ, E) are quite similar for general $p = p(n, \vartheta)$, see Th. 4.2 below, the global structure of the set of steady states is rather sensitive to variations of the form of p in (3), cf. results for the Boltzmann and Fermi–Dirac models.

Notation. In the sequel $|\,.\,|_p$ will denote the $L^p(\Omega)$ norm. The letter C will denote inessential constants which may vary from line to line.

2 Examples of density–pressure relations

In this section we recall some important distributions and density–pressure relations in statistical mechanics, generalize these examples, and formulate our assumptions on the structure of the pressure function p.

2.1. Maxwell–Boltzmann distributions

They are characterized by

$$f = \lambda^{-1} e^{-|v|^2/2\vartheta}, \tag{12}$$

where $\lambda = \lambda(x,t) > 0$ is a physical parameter of *fugacity*, and $\vartheta = \vartheta(t) > 0$ is an instantaneous *temperature* uniform in x. With some abuse of notation we will write $f = f(x,v,t) = f(\lambda, \vartheta, v)$. These are maximizers of the Boltzmann entropy

$$S = \frac{-1}{\int_{\mathbb{R}^d} f\, dv} \int_{\mathbb{R}^d} f \log f \, dv.$$

The corresponding macroscopic density and the pressure are

$$n = \lambda^{-1} \sigma_d 2^{d/2-1} \vartheta^{d/2} \Gamma\left(\frac{d}{2}\right), \quad p = \lambda^{-1} \sigma_d 2^{d/2-1} \vartheta^{d/2+1} \Gamma\left(\frac{d}{2}\right),$$

so that

$$p_{\mathrm{MB}}(n, \vartheta) = \vartheta n. \tag{13}$$

This is the classical Boltzmann relation leading to *linear* (Brownian) diffusion term in (1). Indeed, $n(x,t) = \int_{\mathbb{R}^d} \lambda^{-1} e^{-|v|^2/2\vartheta}\, dv = \lambda^{-1}\sigma_d (2\vartheta)^{d/2} \int_0^\infty e^{-r^2} r^{d-1}\, dr$, and p is calculated in a similar manner.

2.2. Fermi–Dirac distributions

They have the form

$$f = \frac{\eta_0}{\lambda e^{|v|^2/2\vartheta} + 1} \tag{14}$$

with a fixed $\eta_0 > 0$, and again $\lambda = \lambda(x,t) > 0$, $\vartheta = \vartheta(t) > 0$. They maximize the entropy

$$S = \frac{-1}{\int_{\mathbb{R}^d} f\, dv} \int_{\mathbb{R}^d} \left(\frac{f}{\eta_0} \log \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0}\right) \log\left(1 - \frac{f}{\eta_0}\right)\right) dv$$

whose form *a priori* prevents from the overcrowding of particles at (x,v,t): $0 \le f \le \eta_0$. Then we have (see e.g. [15], [7, (1.1)–(1.3)])

$$n = \eta_0 2^{d/2-1} \sigma_d \vartheta^{d/2} I_{d/2-1}(\lambda), \quad p = \eta_0 2^{d/2} \frac{\sigma_d}{d} \vartheta^{d/2+1} I_{d/2}(\lambda), \tag{15}$$

where I_α denotes the Fermi integral of order $\alpha > -1$ defined for all $\lambda > 0$

$$I_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha \, dy}{\lambda e^y + 1}. \tag{16}$$

Hence

$$p_{\mathrm{FD}}(n,\vartheta) = \frac{\mu}{d} \vartheta^{d/2+1} \left(I_{d/2} \circ I_{d/2-1}^{-1} \right) \left(\frac{2}{\mu} \frac{n}{\vartheta^{d/2}} \right) \tag{17}$$

for a constant $\mu > 0$, which leads to a *nonlinear* diffusion in (1). Properties of Fermi integrals (16) (convexity, asymptotics, etc.) relevant to study of the system (1)–(2) are collected in [7, Sec. 2] and [11, Sec. 5].

2.3. Bose–Einstein distributions

They are of the form

$$f = \frac{\eta_0}{\lambda e^{|v|^2/2\vartheta} - 1} \tag{18}$$

with $\eta_0 > 0$, $\lambda = \lambda(x,t) > 1$, $\vartheta = \vartheta(t) > 0$. Similarly as before, we obtain

$$p_{\mathrm{BE}}(n,\vartheta) = \frac{\mu}{d} \vartheta^{d/2+1} \left(J_{d/2} \circ J_{d/2-1}^{-1} \right) \left(\frac{2}{\mu} \frac{n}{\vartheta^{d/2}} \right), \tag{19}$$

where J_α denotes the integral

$$J_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha \, dy}{\lambda e^y - 1} \tag{20}$$

defined either for $\alpha > -1$ and $\lambda > 1$, or $\alpha > 0$ and $\lambda \ge 1$. Note that $\sup_{\lambda > 1} J_\alpha(\lambda)$ is finite if and only if $\alpha > 0$.

Examples 2.2 and 2.3 are well known in quantum statistical mechanics of particles with spin: fermions and bosons, respectively.

2.4. Polytropic equations of state

The distributions corresponding to polytropic equations of state of a gas on the kinetic level have the form

$$f = A \left(\alpha - \frac{|v|^2}{2} \right)_+^{\frac{1}{q-1}} \tag{21}$$

(or, more generally $f = A \left(\alpha - \varphi - \frac{|v|^2}{2} \right)_+^{\frac{1}{q-1}}$), where $r_+ = \max(r,0)$, $q > 1$, $A = \left(\frac{q-1}{q\vartheta} \right)^{\frac{1}{q-1}}$, $\alpha = \frac{1-(q-1)\lambda}{q-1} \vartheta$ with $\lambda = \lambda(x,t) \in (-\infty, 1/(q-1))$, $\vartheta = \vartheta(t) > 0$. They are obtained by extremizing the Rényi–Tsallis entropy

$$S = \frac{-1}{q-1} \int_{\Omega \times \mathbb{R}^d} (f^q - f) \, dx \, dv$$

at fixed mass and energy, cf. [16]. After integrating with respect to $r = |v|$ we obtain $n = A2^{d/2-1}\sigma_d\alpha^{\frac{1}{q-1}+\frac{d}{2}}B\left(\frac{d}{2}, 1+\frac{1}{q-1}\right)$, $p = A2^{d/2-1}\sigma_d\alpha^{\frac{1}{q-1}+\frac{d}{2}+1}B\left(\frac{d}{2}, 1+\frac{1}{q-1}\right)\frac{1}{\frac{d}{2}+\frac{1}{q-1}+1}$, where B denotes the Euler Beta function $B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$, see [16]. Thus we arrive at the relation

$$p_{1+\gamma}(n, \vartheta) = \kappa_\gamma \vartheta^{1-\gamma\frac{d}{2}} n^{1+\gamma} \tag{22}$$

with $1/\gamma = 1/(q-1) + d/2 > d/2$, so that $q = 1 + \frac{1}{1/\gamma - d/2}$, and the polytropic constant $\kappa_\gamma = \frac{\gamma}{\gamma+1}\left(2^{d/2-1}\sigma_d B\left(\frac{d}{2}, 1+\frac{1}{\gamma}-\frac{d}{2}\right)\right)^{-\gamma}\left(1+\frac{1}{\gamma}-\frac{d}{2}\right)^{\gamma\frac{d}{2}-1}$, since $\left(\frac{q-1}{q}\right)^{\frac{1}{q-1}} = \left(1+\frac{1}{\gamma}-\frac{d}{2}\right)^{\frac{1}{\gamma}-\frac{d}{2}}$. Passing to the limit $q \nearrow \infty$, we have $\gamma = 2/d$, and p is independent of ϑ

$$p_{1+2/d}(n, \vartheta) = \kappa_{2/d} n^{1+2/d}. \tag{23}$$

The limit case $q \searrow 1$ corresponds to the Boltzmann density–pressure relation (13), cf. also the limit Boltzmann entropy S.

Remark. The Fermi–Dirac model results, in certain sense, as an interpolation between mean field models involving the Brownian diffusion and diffusion in gases. In the *classical limit* $\lambda \nearrow \infty$ (i.e. $n/\vartheta^{d/2} \searrow 0$, e.g. for fixed $\vartheta > 0$ and the small density $n \to 0$) the relation

$$\frac{p_{\mathrm{FD}}}{n} = \frac{2}{d}\vartheta\frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)} \sim \vartheta \tag{24}$$

holds. This limit corresponds to the linear Brownian diffusion as was in [15, 10, 18]. The completely degenerate case (*white dwarf* in astrophysics), $\lambda \searrow 0$ (i.e. $n/\vartheta^{d/2} \nearrow \infty$, e.g. for $\vartheta > 0$ fixed, and the large density $n \to \infty$), corresponds to the relation

$$\frac{p_{\mathrm{FD}}}{n^{1+2/d}} \sim \frac{2}{d+2}\left(\frac{d}{\mu}\right)^{2/d} = \kappa = \mathrm{const}, \tag{25}$$

i.e., to a polytropic equation of state of a gas.

Note that the distribution function f in (21) can be written using n, ϑ and v as

$$f = C_{d,\gamma}\left(\left(\frac{n}{\vartheta^{d/2}}\right)^\gamma - \frac{\gamma}{(\gamma+1)\kappa_\gamma}\frac{|v|^2}{2\vartheta}\right)_+^{1/\gamma - d/2}$$

These polytropic relations define evolution equations with nonlinear diffusions as, e.g., in the porous media equation.

There are also other relevant density–pressure relations including nonlocal ones considered in [5] and references therein, and leading to linear diffusions defined by pseudodifferential operators of Lévy–Khinchin type so that the associated stochastic processes have discontinuous trajectories.

In the examples 2.1–2.4 the scaling relation

$$f(\lambda, \vartheta, \vartheta^{1/2}v) \equiv f(\lambda, 1, v) \tag{26}$$

holds for each $\vartheta > 0$. Its immediate consequence (see (5) and (6)) is

$$p(\vartheta^{d/2}n, \vartheta) = \vartheta^{d/2+1}p(n, 1), \tag{27}$$

for each $\vartheta > 0$, and thus the self-similar relation

$$p(n, \vartheta) = \vartheta^{d/2+1}P\left(\frac{n}{\vartheta^{d/2}}\right), \tag{28}$$

is valid with a function P defined on $\mathbb{R}^+$ — as was for the pressure in each of the examples (13), (17), (19), (22) and (23).

Remark. For the general energy relation (4) with the constant c_0 measuring the importance of the kinetic energy compared to the potential one, the proper scaling is, of course,

$$p(\vartheta^{c_0}n, \vartheta) = \vartheta^{c_0+1}p(n, 1),$$

and the self-similar form of the pressure becomes $p(n, \vartheta) = \vartheta^{c_0+1}P\left(\frac{n}{\vartheta^{c_0}}\right)$, see Lemma 3.1.

Given E and the instantaneous value of $n = n(x, t)$, the property $\frac{\partial p}{\partial \vartheta} > 0$ permits us to define the temperature $\vartheta = \vartheta(t)$ (and thus $p = p(n, \vartheta)$) in a unique way. If p has the self-similar form (28) and $\frac{\partial p}{\partial \vartheta} > 0$, then for large s $P(s) \le Cs^{1+2/d}$. Indeed,

$$0 < \frac{\partial p}{\partial \vartheta} = \left(\frac{d}{2} + 1\right)\vartheta^{d/2}P\left(\frac{n}{\vartheta^{d/2}}\right) - \frac{d}{2}nP'\left(\frac{n}{\vartheta^{d/2}}\right)$$

implies $\left(\frac{d}{2} + 1\right)P(s) \ge \frac{d}{2}sP'(s)$, and an integration leads to $P(s) \le Cs^{1+2/d}$ for large s.

3 Entropies and energies

In the isothermal setting (i.e. *canonical ensemble*) the function

$$W_{\text{iso}} = \frac{1}{\vartheta}\int_\Omega \left(\vartheta nh - p + \frac{1}{2}n\varphi\right) dx \tag{29}$$

is a (neg)entropy for the problem (1)–(2), (7)–(8). Here the function h is defined (up to a constant depending on the temperature) for an arbitrary increasing function p of $n > 0$ and $\vartheta > 0$ by the relation

$$\frac{\partial h}{\partial n} = \frac{1}{\vartheta n}\frac{\partial p}{\partial n}. \tag{30}$$

Indeed, the following differential inequality holds for sufficiently regular solutions of the isothermal problem

$$\frac{d}{dt}W_{\text{iso}} = -\int_\Omega \vartheta n D_* \left|\nabla\left(h + \frac{\varphi}{\vartheta}\right)\right|^2 dx \le 0. \tag{31}$$

This is obtained multiplying (1) by $(h+\varphi/\vartheta)$, integrating by parts using the no-flux condition (7), and noting that $\frac{\partial}{\partial n}\left(nh-\frac{p}{\vartheta}\right)=h$, $\frac{d}{dt}\left(\frac{1}{2}\int_\Omega n\varphi\,dx\right)=\int_\Omega \frac{dn}{dt}\varphi\,dx$.

In the nonisothermal setting (i.e. *microcanonical ensemble*) the existence of a nontrivial entropy needs an assumption on the structure of p in (3), i.e. on the dependence on ϑ. A simple sufficient condition is (28).

Lemma 3.1 *If for the density-pressure relation (3) the condition (28) is satisfied with a function $P\in C^1$, then*

$$W=\int_\Omega\left(nh-\left(\frac{d}{2}+1\right)\frac{p}{\vartheta}\right)dx \tag{32}$$

is an entropy for the problem (1)-(2), (7)-(8), one of the boundary condition (9) or (10), with the energy relation (4), $c_0=d/2$. Moreover, the following production of entropy formula holds

$$\frac{d}{dt}W=-\int_\Omega \vartheta n D_*\left|\nabla\left(h+\frac{\varphi}{\vartheta}\right)\right|^2dx\le 0. \tag{33}$$

Proof. Note that $p\in C^2$ is required in order to have (1) defined classically. However, we only need $P\in C^1$ in the following calculations.

Observe that $W=W_{\rm iso}-E/\vartheta$. Let us compute $\frac{d}{dt}\left(\frac{1}{\vartheta}\int_\Omega p\,dx\right)$ in two ways. First, using the energy relation (4) and denoting $\frac{d}{dt}\vartheta$ by $\dot\vartheta$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{d}{2}\frac{1}{\vartheta}\int_\Omega p\,dx\right) &= \frac{1}{\vartheta}\frac{d}{dt}\left(\frac{d}{2}\int_\Omega p\,dx\right)-\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx\\
&= \frac{1}{\vartheta}\frac{d}{dt}\left(E-\frac{1}{2}\int_\Omega n\varphi\,dx\right)-\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx\\
&= -\frac{1}{\vartheta}\int_\Omega n_t\varphi\,dx-\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Second, since by (28) the relation $\frac{\partial p}{\partial\vartheta}=\left(\frac{d}{2}+1\right)\frac{p}{\vartheta}-\frac{d}{2}\frac{n}{\vartheta}\frac{\partial p}{\partial n}$ holds, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{\vartheta}\int_\Omega p\,dx\right) &= \frac{1}{\vartheta}\int_\Omega\frac{\partial p}{\partial n}n_t\,dx+\frac{\dot\vartheta}{\vartheta}\int_\Omega\frac{\partial p}{\partial\vartheta}\,dx-\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx\\
&= \frac{1}{\vartheta}\int_\Omega\frac{\partial p}{\partial n}n_t\,dx-\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega n\frac{\partial p}{\partial n}\,dx+\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Now we get

$$\begin{aligned}
\frac{d}{dt}W &= \int_\Omega n_t h\,dx+\int_\Omega nh_t\,dx+\frac{1}{\vartheta}\int_\Omega n_t\varphi\,dx+\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx\\
&\quad -\frac{1}{\vartheta}\int_\Omega\frac{\partial p}{\partial n}n_t\,dx+\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega n\frac{\partial p}{\partial n}\,dx-\frac{d}{2}\frac{\dot\vartheta}{\vartheta^2}\int_\Omega p\,dx.
\end{aligned}$$

Let us observe that by (28) h has the self-similar form

$$h(n,\vartheta) = H\left(\frac{n}{\vartheta^{d/2}}\right) \tag{34}$$

with a function H satisfying $H'(s) = P'(s)/s$. In other words

$$P'(s) = \frac{d}{ds}G(H(s)), \quad \text{where} \quad G' = g = H^{-1} \tag{35}$$

is the inverse function of H. So we obtain $\frac{\partial h}{\partial \vartheta} = -\frac{d}{2}\vartheta^{-2}\frac{\partial p}{\partial n}$ and $\frac{d}{dt}h = \frac{\partial h}{\partial n}\frac{d}{dt}n + \dot{\vartheta}\frac{\partial h}{\partial \vartheta}$. Therefore, we arrive at the entropy production formula (33).

Note that for an arbitrary $p = p(n,\vartheta)$ the expression

$$\frac{\dot{\vartheta}}{\vartheta^2}\left\{\left(\frac{d}{2}+1\right)\int_\Omega p\,dx - \vartheta\int_\Omega \frac{\partial p}{\partial \vartheta}\,dx + \vartheta^2\int_\Omega n\frac{\partial h}{\partial \vartheta}\,dx\right\} \equiv \frac{\dot{\vartheta}}{\vartheta^2}K$$

(obtained from $\int_\Omega n_t\left(h + \frac{\varphi}{\vartheta}\right)dx$ by subtracting time derivative of a quantity) may not be a perfect time derivative of a function of n, ϑ satisfying the original problem. This holds, however, if the expression K in braces is a function of ϑ only (but not of n). The condition $\frac{\partial}{\partial n}K = 0$ is equivalent to the first order differential equation $\frac{d}{2}n\frac{\partial h}{\partial n} + \vartheta\frac{\partial h}{\partial \vartheta} = 0$, which characterizes C^1 functions h of the self-similar form (34). □

Remark. The entropies corresponding to the nonisothermal problems in Examples 2.1 and 2.4 read

$$W_{\mathrm{MB}} = \int_\Omega n\left(\log n - \frac{d}{2}\log\vartheta\right)dx, \tag{36}$$

$$W_{1+\gamma} = \kappa_\gamma\left(\frac{1}{\gamma} - \frac{d}{2}\right)\vartheta^{-\gamma d/2}\int_\Omega n^{1+\gamma}\,dx, \tag{37}$$

with the limit case $\lim_{\gamma\nearrow 2/d} W_{1+\gamma} = 0$ when $p_{1+2/d}$ is defined in (23) and the dependence on ϑ also disappears in the energy (4).

The energy relation (4) sometimes leads to interesting *a priori* estimates.

Lemma 3.2 *If $P(s) \ge \varepsilon s^{1+\gamma}$ for some $\varepsilon > 0$, $\gamma > 1 - 2/d$ and all large $s \gg 0$, then the total energy (4) controls the thermal energy $c_0\int_\Omega p\,dx$ and the absolute value of the potential energy $\frac{1}{2}\left|\int_\Omega n\varphi\,dx\right|$ from above. More precisely, for each $0 < c_1 < c_0$ there exists $C = C(c_1,\Omega)$ such that*

$$E \ge c_1\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - CM^{1+\nu}, \tag{38}$$

where $\nu = 2\gamma/(\gamma d + 2 - d)$.

If $\gamma = 1 - 2/d$ such an estimate is meaningful for small values of mass M only, namely for $0 < M < M_0$

$$E \ge \left(c_0 - c_2 M^{2/d}\right)\int_\Omega p\,dx + \left|\int_\Omega n\varphi\,dx\right| - C \tag{39}$$

holds for some c_2, $C > 0$.

Proof. Clearly, the integral $\int_\Omega n\varphi\,dx$ is bounded from above. In fact, for either $d \geq 3$ or $d \geq 1$ and (10), we have even $\int_\Omega n\varphi\,dx \leq 0$.

Using the Hölder inequality and the Sobolev imbedding theorem (and taking into account boundary conditions for φ), we obtain for each $\varepsilon > 0$

$$\begin{aligned}\left|\int_\Omega n\varphi\,dx\right| &\leq |n|_q|\varphi|_{q'} \leq C|n|_q^2 \\ &\leq C|n|_{1+\gamma}^{2r}|n|_1^{2-2r} \leq \varepsilon|n|_{1+\gamma}^{1+\gamma} + CM^{1+2\gamma/(\gamma d+2-d)}\end{aligned}$$

with $1/q = r/(1+\gamma) + (1-r) = (d+2)/(2d) < 1$ if $d \geq 3$. For $d \leq 2$ we take q close enough to 1, cf. [7, Lemma 3.5]. Since $\int_\Omega p\,dx$ dominates $|n|_{1+\gamma}^{1+\gamma}$, this implies (38).

In the case $\gamma = 1 - 2/d$ a similar argument leads to the inequality

$$\left|\int_\Omega n\varphi\,dx\right| \leq C|n|_{1+\gamma}^{1+\gamma}M^{1-\gamma},$$

and (39) follows immediately. □

Corollary 3.3 *Under the assumptions in Lemma 3.2, either the boundedness of $W_{\rm iso}$ or the energy relation (4) gives a uniform apriori estimate from above for each of the quantities $\int_\Omega p(x,t)\,dx$, $\left|\int_\Omega n(x,t)\varphi(x,t)\,dx\right|$ and (if, moreover, $\gamma < 2/d$) also for $\vartheta(t)$ in the evolution problem, whenever solutions $n(x,t)$ with any fixed M, E exist.* □

Remark. The results in Lemma 3.2 apply to the case of polytropic density–pressure relations and the Fermi–Dirac model ($\gamma = 2/d$, $d \leq 3$, arbitrary $M > 0$, and $d = 4$, small $M > 0$, see [11, (29)]). They are also valid (however, with different proofs) for the Maxwell–Boltzmann case ($d = 2$ and small M, see [3] and [10]).

Note that, since the boundary conditions (7) give no immediate information on n, φ (unlike the situation in [21]), estimates following from energy and entropy are, in general, too weak to obtain substantial bounds on n and φ.

4 Steady state problem with fixed mass and energy

For general density–pressure relations (3) the steady states (N, Φ) of (1)–(2) are determined from the equation

$$h(N) + \frac{\Phi}{\vartheta} = c, \tag{40}$$

where the constant c is chosen from the mass constraint $\int_\Omega N\,dx = M$, $\Delta\Phi = N$, and the function $h = h(n,\vartheta)$ is defined in (30). Indeed, multiplying the stationary version $\nabla\cdot(D_*(\nabla p + N\nabla\Phi)) = 0$ of (1) by $h(N) + \Phi/\vartheta$ and integrating over Ω, we obtain

$$\int_\Omega \vartheta N D_* \left|\nabla\left(h(N) + \frac{\Phi}{\vartheta}\right)\right|^2 dx = 0, \tag{41}$$

which implies (40).
Whenever h is of self-similar form (34), the equation (40) can be written as

$$-\vartheta\Delta\Psi = \vartheta^{d/2}g(\Psi + c), \tag{42}$$

where $\Psi = -\Phi/\vartheta$ and $g = H^{-1}$ as in (35). Then, mass normalization becomes

$$\vartheta^{d/2-1}\int_\Omega g(\Psi + c)\,dx = \frac{M}{\vartheta}. \tag{43}$$

The energy constraint (4) with $c_0 = d/2$ is now

$$E = \frac{d}{2}\int_\Omega p\,dx - \frac{\vartheta}{2}\int_\Omega N\Psi\,dx. \tag{44}$$

Entropies give an alternative way to derive and solve the equation (40) for steady states. Namely, the production of entropy formula leads to (40) as was in the Brownian case studied in [3, 9] and the Fermi–Dirac one in [7, 11]. Moreover, we can use the entropy $W_{\rm iso}$ for the isothermal problem to obtain steady states as global minimizers of $W_{\rm iso}$. This is possible for $P(s) \geq Cs^{1+\gamma}$ with $\gamma \geq 1 - 2/d$. Using Lemma 3.2 we get relative weak compactness of minimizing sequences for $W_{\rm iso}$ considered on the set $\{W_{\rm iso} \leq W_0 < \infty\}$, and $W_{\rm iso}$ satisfies usual weak lower semicontinuity properties.

The second variational approach is possible via an analysis of the dual functional as was in [28]. Without entering into the details (cf. [7, Sec. 4.1]), we recall that this gives for each $M > 0$ and $\vartheta > 0$, solvability of (42) supplemented with the Dirichlet condition (10) and mass constraint expressed as $\int_{\partial\Omega}\frac{\partial\Psi}{\partial\nu}\,d\sigma = -M/\vartheta$, whenever $P(s) \sim Cs^{1+\gamma}$ for large $s > 0$ with $\gamma > 1 - 2/d$. Indeed, then $H(s) \sim Cs^\gamma$, so $g(s) = H^{-1}(s) \sim Cs^{1/\gamma}$, and the sufficient condition in [28] reads $\lim_{s\to\infty} g(s)/s^{p^*} = 0$ for $p^* = d/(d-2)$ if $d \geq 3$ or $p^* < \infty$ if $d = 2$. This is, of course, $1/\gamma < d/(d-2)$, i.e. the same condition as before for minimization of $W_{\rm iso}$.

Summarizing, we get for ϑ = const and either of the boundary conditions (9), (10), the following existence results similar to those in Propositions 4.1, 4.2 in [11].

Proposition 4.1 *Under the assumption $P(s) \geq \varepsilon s^{1+\gamma}$ with $\gamma > 1 - 2/d$ and large $s \gg 0$ (as in Lemma 3.2), given $M > 0$ there exists at least one solution Ψ of the equation (42) satisfying the condition (9) and (43). A similar result holds true for the case of boundary condition (10). If $\gamma = 1 - 2/d$ those solutions exist for sufficiently small $M > 0$.* □

The question of the existence of multiple solutions of the equation (41), and their stability as solutions of the evolution problem (1)–(8), is rather delicate. The problem is relatively well understood in the case of the Boltzmann model, cf. [3] and references therein. There are some numerical results in the case of radially symmetric solutions of the Fermi–Dirac model in the ball of $\mathbb{R}^3$ in [12, 13, 14].

For another approach to the problem of the existence of stationary solutions of the Fermi–Dirac model with Ψ satisfying the Dirichlet condition (10) for small $M > 0$ and each $d \geq 3$, we refer the reader to [23, 24, 25, 11]. These results are proved in the spirit of fixed point theorems based on the compactness properties of the operator $N \mapsto \Psi$ in [23], and using contraction arguments in [24]. For the free condition (9) the proofs of analogous results are practically the same. Here, we will generalize these results to an abstract setting with the density–pressure relation (28). Also, by an application of the Pohozaev identity, it is shown in [23] that for $d \geq 5$, the equation (42) with the Dirichlet boundary condition in star-shaped domains has no solution for sufficiently large M, $M \geq M(\vartheta)$.

Now we study stationary solutions satisfying the energy and mass constraints, i.e. in the *microcanonical ensemble*. In the problem of finding steady states for fixed E and M, (40) is supplemented by (4) and (11), so that the temperature ϑ is to be determined from these constraints. The approach via fixed point arguments is like that in [23, 24]. However, it should be noted that in the aforementioned papers $p = p_{\rm FD}$ and the dependence on ϑ was not explicitly stated (as irrelevant in the *canonical ensemble*), while it is of crucial importance in [11] and [25] (still with $p = p_{\rm FD}$) and herein — with rather general p's and either of the conditions (9) or (10).

Theorem 4.2 *Suppose that $d \geq 2$, $0 < P'(s) < \infty$ for $s \geq 0$ and $P'(s)/s$ is a decreasing function. Then for $0 < M/\vartheta \ll 1$ there exists a solution of (42) with (43). These solutions form a branch in $L^\infty(\Omega)$ depending continuously on M and ϑ.*
Moreover, for each $M > 0$ there exists $E_0 = E_0(M) > -\infty$ such that given $E > E_0$ there exists at least one solution of (42), (43), (4).

Remark. Note that under assumptions of Lemma 3.2 steady states exist for each $M > 0$ and $\vartheta > 0$. However, the corresponding values of the energy E are not arbitrary: by (38) the quantity $E/M^{1+\nu}$ is bounded from below.

For steady states in bounded d-dimensional star-shaped domains, $d \geq 4$ (which exist for $\vartheta > 0$ and each $0 \leq M < M(\vartheta)$), the ratio E/M^2 is bounded from below, see results in Theorems 5.3 and 5.4.

Yet another approach is possible whenever an entropy exists for the nonisothermal problem, so, e.g., if (28) holds. The direct minimization of W in (32) under mass and energy constraints leads to steady states. Note that $|W|$ can be rather small compared to $|W_{\rm iso}|$. For instance, in the Fermi–Dirac case $W_{\rm FD} \geq -C\vartheta \int_\Omega n^{1-2/d}\,dx$, see [11, Lemma 3.5].

Proof. If $P'(0) \in (0, \infty)$, then the strictly increasing function H defined in (30), (34) satisfies the relation

$$\lim_{s\to 0} \frac{H(s)}{\log s} = \ell \in (0, \infty). \tag{45}$$

Moreover, if $P'(s)/s$ decreases (it suffices to assume that for small $s > 0$ only), then H is a concave function. This is, of course, satisfied in each of the examples 2.1–2.3, see [7, 11] for the Fermi–Dirac case. Since $g = H^{-1}$ is a convex function, the function g defined in (35), as well as its derivative g', are strictly increasing on $(-\infty, 0)$ with $g(z) \sim \ell g'(z) \sim \text{const}\, e^{z/\ell}, \quad z \to -\infty$.

Using this together with the positivity of Ψ, we get from (43)

$$c \leq g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right). \tag{46}$$

Moreover, for any $\Psi \in L^\infty(\Omega)$ there exists a unique value c satisfying (43), which we will denote $c(\Psi)$. The function c is Lipschitz continuous: $|c(\Psi_1) - c(\Psi_2)| \leq |\Psi_1 - \Psi_2|_\infty$ by (43) and the mean value theorem, cf. [23, (1.11)]. Thus, looking for a solution of the problem (42)–(43) is reduced to finding a fixed point of the integral operator

$$\mathcal{T}(\Psi) = \vartheta^{d/2-1}(-\Delta)^{-1}\left(g(\Psi + c(\Psi))\right). \tag{47}$$

Here $(-\Delta)^{-1}$ denotes the inverse of $-\Delta$ with an appropriate boundary condition, defined either by the Green function of Ω or by the convolution with the fundamental solution. Its norm, as the operator considered in $L^\infty(\Omega)$, is denoted by A. Applying the mean value theorem for the function g and monotonicity properties of g and g', we see that the Lipschitz constant $L(M, \vartheta)$ of $\mathcal{T}$ on the unit ball $B(0,1) \subset L^\infty(\Omega)$ satisfies the estimate

$$L(M, \vartheta) \leq 2A\vartheta^{d/2-1}\, g'\left(1 + g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)\right). \tag{48}$$

Moreover, $\mathcal{T}$ maps the ball $B(0,1)$ into itself provided the inequality

$$A\vartheta^{d/2-1} g\left(1 + g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)\right) \leq 1, \tag{49}$$

holds. Since $1 - d/2 \leq 0$, we conclude from (45), (48) that for sufficiently large ϑ: $0 < M/\vartheta \ll 1$, say $M/\vartheta \leq m_0$, the conditions

$$L(M, \vartheta) \leq C\frac{M}{\vartheta} < \frac{1}{2}, \tag{50}$$

and (49) are satisfied. Applying the Contraction Mapping Principle we get, for each positive M, ϑ such that (49)–(50) are satisfied, a solution $\Psi = \Psi_{M,\vartheta}$, unique in the ball $B(0,1)$. The norm of that solution $\Psi = \mathcal{T}(\Psi)$ satisfies by (50) and $|\mathcal{T}(0)|_\infty \leq C\frac{M}{\vartheta}$

$$|\Psi_{M,\vartheta}|_\infty \leq 2C\frac{M}{\vartheta} \leq 1. \tag{51}$$

Stationary solutions constructed above have positive energy. Indeed, the assumption $P'(0) > 0$ implies $p(n, \vartheta) \geq k\vartheta n$ for some $k > 0$ and small $n/\vartheta^{d/2}$. Thus for fixed $M > 0$, the energy E defined in (44) for the solution $-\vartheta\Psi = -\vartheta\Psi_{M,\vartheta}$ of (42) with mass M in (43), at the sufficiently large temperature ϑ, satisfies, by $|\Psi|_\infty \le 2C\frac{M}{\vartheta}$

$$E \ge \frac{dk}{2}\vartheta M - \frac{1}{2}\vartheta M|\Psi|_\infty \ge \left(\frac{dk}{2} - C\frac{M}{\vartheta}\right)\vartheta M > 0$$

for M/ϑ small enough, thus $\sup_{\vartheta \ge \frac{M}{m_0}} E = \infty$. Since by the contraction mapping arguments both $\Psi_{M,\cdot}$ and E are continuous functions of ϑ along the branch of solutions constructed above, we obtain the conclusion. □

A similar reasoning, but with completely different asymptotics of functions H, g, works in the case of density–pressure relations covering the polytropic case. Namely, we have

Theorem 4.3 *Suppose that $d \ge 2$, $0 < \gamma \le 2/d$, $\lim_{s\to 0} P(s)/s^{1+\gamma} \in (0,\infty)$, and $P'(s)/s$ is a decreasing function. Then for a fixed $M > 0$ and sufficiently large ϑ, e.g. if $M \ll \min\left\{\vartheta^{d/2}, \vartheta^{\left(1-\gamma\frac{d}{2}\right)/(1-\gamma)}\right\}$ there exists a solution of (42) with (43). These solutions form a branch in $L^\infty(\Omega)$ depending continuously on M and ϑ.*

Proof. Under the assumptions of this theorem, H is a concave function satisfying the relation

$$\lim_{s\to 0} \frac{H(s)}{s^\gamma} = \ell \in (0,\infty). \tag{52}$$

The functions g and g', defined as before: $g = H^{-1}$, for $\gamma \le 1$ satisfy $g(s) \sim \text{const}\, s^{1/\gamma}$, $g'(s) \sim \text{const}\, s^{1/\gamma - 1}$ as $s \to 0$. Now, following the lines of the proof of Theorem 4.2, we see that sufficient conditions (48)–(49) of the applicability of the Contraction Mapping Principle to the equation (47) in the ball $B(0,R) \subset L^\infty(\Omega)$ become

$$L(M,\vartheta) \le 2A\vartheta^{d/2-1} g'\left(R + g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)\right), \quad A\vartheta^{d/2-1} g\left(R + g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)\right) \le R. \tag{53}$$

Asymptotically for $M \ll \vartheta^{d/2}$, these read $A\vartheta^{d/2-1}\left(R + \left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)^\gamma\right)^{1/\gamma} \le R$ and $\frac{2}{\gamma}A\vartheta^{d/2-1}\left(R + \left(\frac{M}{\vartheta^{d/2}|\Omega|}\right)^\gamma\right)^{1/\gamma - 1} \le 1$. Thus, they are satisfied for a fixed M and ϑ large enough with, e.g., $R = g^{-1}\left(\frac{M}{\vartheta^{d/2}|\Omega|}\right) \sim \frac{M^\gamma}{\vartheta^{\gamma\frac{d}{2}}}$. Indeed, if $M \ll \min\left\{\vartheta^{d/2}, \vartheta^{\left(1-\gamma\frac{d}{2}\right)/(1-\gamma)}\right\}$, then the both conditions in (53) are verified, and (47) has a unique solution in $B(0,R) \subset L^\infty(\Omega)$.

The analysis above shows that for $\gamma = 2/d$ those solutions can be constructed for sufficiently small $M > 0$ only.

For sufficiently small domains Ω, the solutions constructed above have positive energy. This can be seen from the estimates

$$\begin{aligned} E &\ge \frac{d}{2}\kappa_\gamma \vartheta^{1-\gamma\frac{d}{2}} \int_\Omega n^{1+\gamma}\,dx - \frac{1}{2}\vartheta M|\Psi|_\infty \\ &\ge \frac{d}{2}\kappa_\gamma \vartheta^{1-\gamma\frac{d}{2}} |\Omega|^{-\gamma} M^{1+\gamma} - \frac{1}{2}\vartheta M C M^\gamma \vartheta^{-\gamma\frac{d}{2}} \\ &= \frac{1}{2} M^{1+\gamma}\vartheta^{1-\gamma\frac{d}{2}} \left(d\kappa_\gamma|\Omega|^{-\gamma} - C\right), \end{aligned}$$

the last quantity being positive for $0 < |\Omega|$ small enough. □

5 Nonexistence of global in time solutions

Here we prove some results on the nonexistence of steady states and, more generally, nonexistence of solutions of the evolution problem defined for all $t \geq 0$. These are results on the isothermal problem with the pressure p asymptotically resembling polytropic relation (22), and in the microcanonical setting with quite general density–pressure relations, the latter are recalled from [11, Sec. 2].

These results show that weakly nonlinear diffusion (i.e. that in (22) with small $\gamma > 0$) is not strong enough to prevent from a blow up of solutions, at least for initial data of negative energy. We will see in the next section that strongly nonlinear diffusion (i.e. (22) with relatively large γ) guarantees the continuation of local in time solutions to the global ones. The immediate reason for those different properties of solutions can be explained by analyzing possible singularities of parabolic-elliptic systems considered here. Indeed,

$$u_{\text{sing}}(x) = c(d,\gamma)|x|^{-2/(1-\gamma)}$$

formally satisfies (1)–(2) a.e. in $\mathbb{R}^d\setminus\{0\}$. This singular unbounded solution is a counterpart of the Chandrasekhar radially symmetric stationary solution $u_c(x) = 2(d-2)|x|^{-2}$ for the classical model with Brownian diffusion, whose role in the formation of singularities for both stationary and evolutionary problem has been studied extensively, see for instance [2]. For larger values of γ such a solution is not integrable at the origin. Thus, simply imposing integrability conditions on n we may get rid of such singular objects. On the other hand, u_{sing} is locally integrable whenever γ is small. Therefore, it is not excluded that this could appear as a singularity of a (suitable) weak solution.

Star-shaped domains $0 \in \Omega \subset \mathbb{R}^d$ are defined by the condition $\beta \geq 0$, where

$$\beta = \inf_{x\in\partial\Omega} x\cdot\overline{\nu}. \tag{54}$$

Similarly, *strictly star-shaped* domains are those with $\beta > 0$. Geometric assumptions on the shape of the domain Ω expressed in terms of β permit us to prove, under some restrictions on the ratio E/M^2, nonexistence of steady states of (1)–(2), (4), (7) with fairly general density–pressure relations (3). In this section, by solutions we mean the classical ones $n \in C^2(\Omega\times(0,\infty)) \cap C^1(\overline{\Omega}\times[0,\infty))$.

Let us begin with the isothermal case of polytropic $p = p(n)$ with small exponents in (22), i.e. with relatively weak diffusion for large n.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 3$, p is such that $p(s)/s^{2-2/d}$ is a decreasing function for all sufficiently large values of s. Then there exist initial data n_0 such that sufficiently regular, positive solutions of the problem (1)–(2), (7), (9), with a constant temperature $\vartheta > 0$, cannot be defined globally in time.*

Proof. Multiplying (1) by $|x|^2$ and integrating over Ω we get the relation

$$\begin{aligned} \frac{d}{dt}\int_\Omega n|x|^2\,dx &= -2\int_\Omega \nabla p\cdot x\,dx - 2\int_\Omega n\nabla\varphi\cdot x\,dx \\ &= 2d\int_\Omega p\,dx - 2\int_{\partial\Omega} p\,x\cdot\overline{\nu}\,d\sigma \qquad (55)\\ &\quad -\frac{2}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^d}(x-y)\cdot x\,dx\,dy \end{aligned}$$

for the second moment $V(t)=\int_\Omega n(x,t)|x|^2\,dx$. After symmetrization of the double integral above, we arrive at

$$\frac{d}{dt}V \leq 2d\int_\Omega p\,dx - \frac{1}{\sigma_d}\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy \qquad (56)$$

since $(x-y)\cdot x + (y-x)\cdot y = |x-y|^2$. Next, we see that from (29) that W_{iso} can be used to estimate

$$\begin{aligned} \frac{d}{dt}V &\leq 2d\int_\Omega p\,dx + 2(d-2)\left(\vartheta W_{\text{iso}} + \int_\Omega(-\vartheta nh + p)\,dx\right) \\ &\leq 2(d-2)\vartheta W_{\text{iso}}(0) + 2\int_\Omega((2d-2)p-(d-2)\vartheta nh)\,dx. \qquad (57) \end{aligned}$$

Now, observe that if, more generally than in the assumption, p is a C^1 function such that $p(s)/s^{1+\gamma}$ decreases for large s, say $s > s_*$, then $\frac{p'(s)}{s^{1+\gamma}} - (1+\gamma)\frac{p(s)}{s^{2+\gamma}} \leq 0$, so that $(1+\gamma)\left(\frac{p(s)}{s}\right)' - \gamma\frac{p'(s)}{s} \leq 0$. This reads $\left(\frac{1}{\gamma}+1\right)\frac{p(s)}{s} \leq \vartheta h(s) + C_*$ for $s \geq s_*$, or

$$sC_* + \vartheta s h(s) - p(s) \geq \frac{1}{\gamma}p(s)$$

for large s and some C_*. Finally, (57) implies $\frac{d}{dt}V \leq 2(d-2)\vartheta W_{\text{iso}}(0)+C$, where $C=C(n_*)$ takes also into account the value of the integral of $(2d-2)p-(d-2)\vartheta nh$ over small values of n, $n\leq n_*$ with a fixed n_*. Evidently, for n_0 sufficiently large and well concentrated (e.g. for Gaussian n_0 with small variance, cf. [11, Lemma 3.5]), the entropy is negative: $W_{\text{iso}} \ll -1$, so that we obtain for such initial data

$$\frac{d}{dt}V = \frac{d}{dt}\int_\Omega n|x|^2\,dx \leq -\varepsilon < 0$$

for some $\varepsilon > 0$. This leads to a contradiction with the existence of a positive solution n for $t \geq T_{\max} = \int_\Omega n_0|x|^2\,dx/\varepsilon$. □

Corollary 5.2 *For $d \geq 4$ there exist initial data such that solutions of the isothermal system (1)-(2) with the Fermi-Dirac density-pressure relation (17), conditions (7), (9), cease to exist after a finite time.*

Proof. Indeed, for $d \geq 4$ $1 + 2/d \leq 2 - 2/d$ holds and Theorem 5.1 applies to the Fermi–Dirac case with $p(s) \sim s^{1+\gamma}$ for large s and $\gamma = 2/d$. Note that for $d = 4$ the initial data leading to a blow up of solutions must have a sufficiently large mass (cf. existence results in Proposition 4.1), while for $d \geq 5$ a blow up is possible for arbitrarily small mass and highly concentrated n_0. This can be inferred from the computations in Lemma 3.2. □

Similar nonexistence results can be proved also for the isothermal problem with the Dirichlet condition (10) for the potential.

We have two results on the nonexistence of solutions in the microcanonical setting.

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 3$, and $c_0 \geq d/(d-2)$, $E/M^2 < ((d-2)c_0 - d)/\left(2d\sigma_d(d-2)(\operatorname{diam}\Omega)^{d-2}\right)$. Then sufficiently regular, positive solutions of the problem (1)–(2), (7), with $D_* = 1$, and (9), satisfying the energy relation (4) cannot be defined globally in time.*

Remark. There exist initial data (n_0, ϑ_0) leading to $E/M^2 \ll 0$. It suffices to consider an arbitrary $0 \leq n_0 \not\equiv 0$ in (8), $\vartheta_0 > 0$, and take the density Mn_0 with $M \gg 1$ large enough.

Proof. Repeating the computations (55)–(56) for the moment $V(t) = \int_\Omega n(x,t)|x|^2\,dx$ in the beginning of the proof of Theorem 5.1, we arrive at the inequality

$$\begin{aligned}\frac{d}{dt}V &\leq \frac{2d}{c_0}E + \frac{1}{\sigma_d}\left(\frac{d}{(d-2)c_0} - 1\right)\int_\Omega\int_\Omega \frac{n(x,t)n(y,t)}{|x-y|^{d-2}}\,dx\,dy \\ &\leq \frac{1}{c_0}\left(2dE + \frac{1}{\sigma_d}\frac{d-(d-2)c_0}{d-2}M^2\left(\operatorname{diam}\Omega\right)^{2-d}\right),\end{aligned}$$

where we used (4). Under the assumptions of Theorem 5.3 we obtain $\frac{d}{dt}V \leq -\varepsilon < 0$ for some $\varepsilon > 0$. As before, this contradicts the existence of a positive solution n after $T_{\max} = \int_\Omega n_0|x|^2\,dx/\varepsilon$. □

A counterpart of Theorem 5.3 for the Dirichlet condition (10) is

Theorem 5.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped domain, $d \geq 3$, $c_0 \geq d/(d-2)$ and $E/M^2 < c_0\beta/(2d|\partial\Omega|)$. Then sufficiently regular, positive solutions of the problem (1)–(2), (7), with $D_* = 1$ and (10), satisfying the energy relation (4) cannot be defined globally in time.*

Proof. Proceeding as in the proof of Theorem 5.3, let us multiply (1) by $|x|^2$ and integrate over Ω. Taking into account the no-flux boundary condition (7) and the Pohozaev–Rellich identity for $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\varphi_{|\partial\Omega} = 0$

$$\int_\Omega \Delta\varphi\,\nabla\varphi\cdot x\,dx = \frac{1}{2}\int_{\partial\Omega} x\cdot\overline{\nu}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma + \frac{d-2}{2}\int_\Omega |\nabla\varphi|^2\,dx,$$

we get a differential inequality for the second moment $V(t) = \int_\Omega n(x,t)|x|^2\,dx$ of the distribution $n(x,t)$. Namely, we have

$$\begin{aligned}
\frac{d}{dt}V &= -2\int_\Omega \nabla p\cdot x\,dx - 2\int_\Omega n\nabla\varphi\cdot x\,dx \\
&= 2d\int_\Omega p\,dx - 2\int_{\partial\Omega} p\,x\cdot\overline{\nu}\,d\sigma \\
&\quad -\int_{\partial\Omega} x\cdot\overline{\nu}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma - (d-2)\int_\Omega |\nabla\varphi|^2\,dx \\
&\le \frac{2d}{c_0}\left(E+\frac{1}{2}\int_\Omega|\nabla\varphi|^2\,dx\right) - \beta\int_{\partial\Omega}\left(\frac{\partial\varphi}{\partial\overline{\nu}}\right)^2 d\sigma - (d-2)\int_\Omega|\nabla\varphi|^2\,dx \\
&= \frac{2d}{c_0}E + \left(\frac{d}{c_0}+2-d\right)\int_\Omega|\nabla\varphi|^2\,dx - \beta\frac{M^2}{|\partial\Omega|} < 0.
\end{aligned}$$

The remainder of the reasoning is as was in the proof of Theorem 5.3. □

Corollary 5.5 *Under the assumptions of either Th. 5.3 or Th. 5.4, there is no positive steady state of the system (1)–(2), (3), (7) with the energy (4), arbitrary $D_* > 0$, in bounded star-shaped domains $\Omega \subset \mathbb{R}^d$.*

Proof. Evidently, steady states do not depend on $D_* > 0$, cf. (41), so that $\frac{d}{dt}\int_\Omega n|x|^2\,dx \equiv 0$ for each steady state. Thus, the proofs of Theorems 5.3, 5.4 apply in this situation. □

Remark. Using the comparison results discussed in [1] (see also [10, Sec. 3]) one can show that $\lim_{t\nearrow T_{\max}}\int_\Omega n(x,t)|x|^2\,dx = 0$ implies by the Shannon inequality (e.g. [1, Lemma 1]) $\lim_{t\nearrow T_{\max}}\int_\Omega n(x,t)\log n(x,t)\,dx = \infty$, and thus for each $\delta > 0$ $\lim_{t\nearrow T_{\max}}\int_\Omega n^{1+\delta}(x,t)\,dx = \infty$, so by the energy relation (4) one infers that $\lim_{t\nearrow T_{\max}}\int_\Omega n(x,t)\varphi(x,t)\,dx = -\infty$. This gives an insight into the problem: How do the blowing up solutions behave near the maximal existence time (which obviously can be strictly less than $T_{\max}$)?

6 Uniform estimates for solutions of the evolution problem for polytropes

We present in this section an argument which would permit us to continue local in time solutions of the evolution isothermal problem with strong diffusion and $D_* = 1$ to the global in time ones. This reasoning is formal because, as far as we know, there are no local in time existence results for general p as in (3), except for the Boltzmann and Fermi–Dirac cases, see [8] and [7]. For $p(s) \sim \kappa s^{1+\gamma}$ with $\gamma > 1 - 2/d$, using the entropy W_{iso}, we will show that, first, $\sup_{t>0}\int_\Omega p(x,t)\,dx < \infty$ with a bound depending on the initial data, cf. Corollary 3.3. Then, we will prove that $\sup_{\delta\le t\le T}|n(t)|_q < \infty$ for any $q < \infty$ and $0 < \delta < T < \infty$, with a bound depending also on δ and T. Since, by results in Theorem

5.1, we know that for $\gamma \le 1 - 2/d$ blow up of solutions, caused by $\int_\Omega n(x,t)|x|^2\,dx \to 0$ as $t \nearrow T_{\max}$, is accompanied by the unboundedness of $L^q(\Omega)$ norms as $t \nearrow T_{\max}$, we can conclude that such a phenomenon do not occur in the complementary range of polytropic exponents γ, $\gamma > 1 - 2/d$, i.e. for diffusion terms $\Delta p(n)$ in (1) strong enough.

Proposition 6.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in $\mathbb{R}^d$, $d \ge 2$, p is a C^1 function such that $p(s) \ge \varepsilon s^{1+\gamma}$, $\varepsilon > 0$, $\gamma > 1 - 2/d$ and large s, and $p(s)/s^{1+\tilde\gamma}$ decreases for all sufficiently large s and some $\tilde\gamma > 1 - 2/d$. Suppose that the initial data n_0 are such that $W_{\rm iso}$ in (29) is finite. Then each local in time positive solution of the system (1)-(2), (7), with $D_* = 1$, either (9) or (10) satisfies the bound $\sup_{\delta\le t\le T} |n(t)|_q < \infty$ for each $q > 1$, $\delta > 0$ and $T < \infty$.*

Proof. Recall from the beginning of the proof of Theorem 5.1 that the assumption on p implies $\vartheta sh(s) - p(s) \ge \frac{1}{\tilde\gamma} p(s)$. This, together with the estimate recorded in Lemma 3.2, shows that the entropy $W_{\rm iso}$ controls $\int_\Omega p(x,t)\,dx$: $W_{\rm iso} \ge \frac{1}{2\tilde\gamma}\int_\Omega p\,dx - C$ with a constant $C = C(M)$. Thus, $\int_\Omega p(x,t)\,dx \sim |n(t)|_{1+\gamma}^{1+\gamma}$ is a *priori* uniformly bounded for all $t \ge 0$, see also Corollary 3.3.

From now on, avoiding clumsy notation, we will write our proof for $\gamma = \tilde\gamma$ and $p(s) = \kappa s^{1+\gamma}$, $s \ge 0$, the modifications necessary to cover the case $p(s) \sim \kappa s^{1+\gamma}$ for $s \ge s_*$ (so that, e.g., $p(s) \ge \frac{\kappa}{2} s^{1+\gamma} - C$) being fairly standard.

Let us multiply (1) by n^{q-1} and integrate over Ω taking into account (7), and either $\frac{\partial\varphi}{\partial\nu} \le 0$ on $\partial\Omega$ in (9) case or $\varphi = 0$ in (10) case, to estimate the boundary integrals. This leads to the differential inequality

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega n^q\,dx &\le -(q-1)\int_\Omega n^{q+\gamma-2}|\nabla n|^2\,dx + \frac{q-1}{q}\int_\Omega n^{q+1}\,dx\\ &= -4\frac{q-1}{(q+\gamma)^2}\int_\Omega \left|\nabla\left(n^{\frac{q+\gamma}{2}}\right)\right|^2 dx + \left(1 - \frac{1}{q}\right)|n|_{q+1}^{q+1}. \end{aligned} \tag{58}$$

Our strategy is to estimate the term $|n|_{q+1}^{q+1} = |v|_{\frac{2(q+1)}{q+\gamma}}^{\frac{2(q+1)}{q+\gamma}}$ with $v = n^{\frac{q+\gamma}{2}}$ by $|\nabla v|_2^2$ and a function $c = c\left(|v|_1, |v|_{\frac{2(1+\gamma)}{q+\gamma}}\right)$. Here we will use the Sobolev and (a version of) the Poincaré inequality: $\|v\|_{H^1(\Omega)}^2 \le C\left(|\nabla v|_2^2 + |v|_1^2\right)$.

The Sobolev inequality gives us

$$|v|_{\frac{2(q+1)}{q+\gamma}} \le C\|v\|_{H^1(\Omega)}^{\nu}|v|_{\frac{2(1+\gamma)}{q+\gamma}}^{1-\nu}$$

with $\frac{-d(q+\gamma)}{2(q+1)} = \nu\left(1 - \frac{d}{2}\right) + (1-\nu)\frac{-d(q+\gamma)}{2(1+\gamma)}$, or $\nu = \frac{d}{1+q}\frac{(q+\gamma)(q-\gamma)}{2+2\gamma+d(q-1)}$. Since $\gamma > (d-2)/d > (d-2)/(d+2)$, we have $\frac{2(q+1)}{q+\gamma}\nu < 2$, so that

$$|v|_{\frac{2(q+1)}{q+\gamma}}^{\frac{2(q+1)}{q+\gamma}} \le \varepsilon\left(|\nabla v|_2^2 + |v|_1^2\right) + C_\varepsilon |v|_{\frac{2(1+\gamma)}{q+\gamma}}^{\mu}$$

with $\mu = 2\frac{1-\nu}{\frac{q+\gamma}{q+1}-\nu} = \frac{1+\gamma}{(d+2)\gamma-(d-2)}\left(\frac{2q}{q+\gamma} + \frac{d\gamma-(d-2)}{q+\gamma}\right)$, and an arbitrarily small (but fixed) $\varepsilon > 0$.

Since $|v|_{\frac{2(1+\gamma)}{q+\gamma}}^{\frac{2(1+\gamma)}{q+\gamma}} = |n|_{1+\gamma}^{1+\gamma}$ and $|v|_1 = |n|_{\frac{q+\gamma}{2}}^{\frac{q+\gamma}{2}}$, it suffices to control the last term $|n(t)|_{\frac{q+\gamma}{2}}$. To accomplish this goal, we will apply this scheme of estimates with $q_0 = 2+\gamma$, $q_{m+1} = 2q_m - \gamma\,(= 2^{m+1}+\gamma) \nearrow \infty$, $m = 0, 1, 2, 3 \ldots$. Note that $\frac{q_0+\gamma}{2} = 1+\gamma$, so the recurrent procedure can be started with the preliminary estimate of $\int_\Omega p(x,t)\,dx \sim |n(t)|_{1+\gamma}^{1+\gamma}$, and then this works with $|n(t)|_{q_m}$, $q_m = \frac{q_{m+1}+\gamma}{2}$. □

Remark. In the context of Fermi–Dirac isothermal systems, Proposition 6.1 says that solutions in either the $d \le 3$ dimensional case with arbitrary mass $M > 0$, or in the four dimensional case with sufficiently small $M > 0$, are locally bounded in each $L^q(\Omega)$, $q < \infty$. Related results for a system with a particular coefficient $D_* \not\equiv 1$ are in [7].

Remark. Concerning the nonisothermal evolution in the polytropic case, assuming $2/d > \gamma > 1 - 2/d$, the nonisothermal entropy W in (32) controls the pressure from above: $W \ge \vartheta^{-1}\left(\frac{1}{\gamma} - \frac{d}{2}\right)\int_\Omega p\,dx$. Moreover, a locally uniform in time *a priori* estimate of the norm $|n(t)|_q$ can be proved for each $q < \infty$ under the assumptions of Proposition 6.1 together with an *a priori* estimate of the temperature $0 < \vartheta_1 \le \vartheta(t) \le \vartheta_2 < \infty$. The estimate from above is provided by Corollary 3.3. However, we are able to prove an estimate for the temperature from below only for the Fermi–Dirac model in $d \le 3$ dimensions with particular initial data, see [11, Lemma 3.6].

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