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INTERIOR ESTIMATES IN CAMPANATO SPACES RELATED TO QUADRATIC FUNCTIONALS

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Abstract
In this paper we obtain interior estimates in Campanato spaces for the derivatives of the minimizers of quadratic functionals.


Key words and phrases: Variational integrals, local minima, regularity of first derivatives.

1 Introduction and Preliminary Tools

The papers concerned with the regularity problem almost always have as a common starting point the Euler's equation related to a generic functional $I$. In the paper by Giaquinta and Giusti [11], the authors investigate the Hölder continuity of the minima working directly with the functional $I$ instead of Euler's equation.

In the present paper, following the method in [11], we have studied regularity properties of the minima of variational integrals of the type:

$$A(u,\Omega) = \int f(x,u,Du)dx,$$

$\Omega \subset \mathbb{R}^n, n \geq 3$ is a bounded opens set $u : \Omega \rightarrow \mathbb{R}^N, u(x) = (u^1(x), \ldots, u^N(x))$, $Du = D_\alpha u^i$, $D_\alpha = \frac{\partial}{\partial x^\alpha}$, $i = 1, \ldots, N$, $\alpha = 1, \ldots, n$, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \rightarrow \mathbb{R}$.
defined by
\[ f(x, u, Du) = A^{\alpha\beta}_{ij}(x, u)D_{\alpha}u^{i}D_{\beta}u^{j} + g(x, u, Du). \]

Let us now give the following definitions useful in the sequel.

**Definition 1.1.** (see \[16\], \[20\]). Let \( 1 \leq p < \infty, 0 \leq \lambda < n \).

By \( L^{p,\lambda}(\Omega) \) we denote the linear space of functions \( f \in L^{p}(\Omega) \) such that
\[ \|f\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{0<\rho<\text{diam}\Omega} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y)|^{p}dy \right\}^{1/p} < +\infty \]
where \( B(x,\rho) \) ranges in the class of the balls of \( \mathbb{R}^{n} \) of radius \( \rho \) around \( x \).

We have that \( \|f\|_{L^{p,\lambda}(\Omega)} \) is a norm respect to which \( L^{p,\lambda}(\Omega) \) is a Banach space, and also that
\[ \|f\|_{L^{p}(\Omega)} \leq (\text{diam}\Omega)^{\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}(\Omega)}. \]

Before the definition of the Campanato spaces let us define \( f_{B(x,\rho)} \) as the integral average
\[ f_{B(x,\rho)} = \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} f(y)dy \]
of the function \( f(x) \) over the balls \( B(x,\rho) \) of \( \mathbb{R}^{n} \). When no confusion may arise, we will write \( f_{B_{\rho}} \) or \( f_{\rho} \) instead of \( f_{B(x,\rho)} \).

**Definition 1.2.** (see e. g. \[3\], \[9\]). Let \( 1 \leq p < \infty \) and \( \lambda \geq 0 \).

By Campanato spaces \( \mathcal{L}^{p,\lambda}(\Omega) \) we denote the linear space of functions \( u \in L^{p}(\Omega) \) such that
\[ [f]_{p,\lambda} = \left\{ \sup_{x \in \Omega} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y) - f_{\rho}|^{p}dy \right\}^{1/p} < +\infty. \]

\( \mathcal{L}^{p,\lambda}(\Omega) \) are Banach spaces with the following norm
\[ \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^{p}(\Omega)} + [f]_{p,\lambda} \]
which simply demonstrate that $u \in \mathcal{L}^{p,\lambda}(\Omega)$ if and only if

$$\sup_{x \in \Omega, 0 < \rho < \text{diam}\Omega} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega \cap B(x,\rho)} |f - c|^p \, dy < \infty.$$ 

Using Hölder inequality we have that

$$\mathcal{L}^{p_1,\lambda_1}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$$

where

$$p \leq p_1, \quad \frac{n - \lambda}{p} \geq \frac{n - \lambda_1}{p_1}.$$ 

Let us observe that

$$\int_{\Omega \cap B(x,\rho)} |f - f_\rho|^p \, dy \leq C \cdot \int_{\Omega \cap B(x,\rho)} \left( |f|^p + |\Omega \cap B(x,\rho)| \cdot |f_\rho|^p \right) \, dy$$

and also

$$|f_\rho|^p \leq \frac{1}{|\Omega \cap B(x,\rho)|} \cdot \int_{\Omega} |f|^p \, dy.$$ 

Below, we consider $0 \leq \lambda < n$.

We use

$$[f]_{p,\lambda} \leq C \|f\|_{\mathcal{L}^{p,\lambda}(\Omega)}.$$ 

to obtain the following relation between Morrey and Campanato spaces

$$L^{p,\lambda}(\Omega) \subset L^{p_1,\lambda_1}(\Omega). \quad (1.1)$$

Let us now recall the definitions of the $BMO$ and $VMO$ classes.

**Definition 1.3.** (see [15]). We say that a function $f$ belongs to the John-Nirenberg space $BMO$, or that $f$ has "bounded mean oscillation", if

$$\|f\|_* \equiv \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_{B} |f(y) - f_\rho| \, dy < \infty$$

where $f_\rho$ is the integral average of the function $f$ over the balls $B_\rho$.

Let us define, for a function $f \in BMO$,

$$\eta(r) = \sup_{x \in \mathbb{R}^n, r \leq \rho} \frac{1}{|B_\rho|} \int_{B_\rho} |f(x) - f_\rho| \, dx.$$
Definition 1.4. (see [22]). A function $f \in BMO$ belongs to the class $VMO$, or $f$ has "vanishing mean oscillation" if
\[
\lim_{r \to 0^+} \eta(r) = 0.
\]

We are now ready to formulate the hypothesis on the terms $A_{ij}^{\alpha\beta}(x,u)$ and $g(\cdot,u,Du)$.

We suppose that $A_{ij}^{\alpha\beta}(x,u)$ are bounded functions in $\Omega \times \mathbb{R}^N$, such that:

(A1) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.

(A2) For every $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot,u) \in VMO(\Omega)$.

(A3) For every $x \in \Omega$ and $u, v \in \mathbb{R}^N$,
\[
|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)| \leq \omega(|u-v|^2)
\]
for some monotone increasing concave function $\omega$ with $\omega(0) = 0$.

(A4) There exists a positive constant $\nu$ such that
\[
\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x,u)\xi_i^j \xi_j^i
\]
for a.e. $x \in \Omega$, all $u \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$.

We suppose that the function $g$ is a Characéodory function, that is:

(g1) $g(\cdot,u,Du)$ is measurable in $x \forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^{nN}$;

(g2) $g(x,\cdot,\cdot)$ is continuous in $(u,z)$ a.e. $x \in \Omega$;

moreover we consider $g$ satisfying the condition:

(g3)
\[
g(x,u,z) \leq g_1(x) + H|z|^\gamma,
\]
$g_1 \geq 0$, a.e. in $\Omega$, $g_1 \in L^p(\Omega), 2 < p \leq \infty$, $H \geq 0$, $0 \leq \gamma < 2$.

We point out that since $C^0$ is a proper subset of $VMO$, the continuity of $A_{ij}^{\alpha\beta}(x,u)$ with respect to $x$ is not assumed.

Let us make some remarks on $VMO$ class. It was at first defined by Sarason in 1975 and later it was considered by many others. For instance we recall the papers by Chiarenza, Frasca and Longo [4] where the authors
answer a question raised thirty years before by C. Miranda in [18]. In his paper he considers a linear elliptic equation where the coefficients $a_{ij}$ of the higher order derivatives are in the class $W^{1,n} (\Omega)$ and asks whether the gradient of the solution is bounded, if $p > n$. Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that $Du$ is H"older continuous for all $p \in ]1, +\infty[$. We point out that $W^{1,n} \subset VMO$ because, using Poincare's inequality

$$\int_B |f(x) - f_B| \leq c(n) \left( \int_B |\nabla u| dx \right)^{\frac{1}{k}}$$

and the term on the right-hand side tending to zero as $|B| \to 0$. Later the interior estimates obtained by Chiarenza, Frasca and Longo were extended to boundary estimates in [5]. From these papers on, many authors have used this space $VMO$ to obtain regularity results for partial differential equations and systems with discontinuous coefficients. We recall for example Bramanti and Cerutti [2] for parabolic equations and many others.

With this useful assumption we investigate the regularity of the minimizers for the quadratic functional. Its existence is guaranteed, being the functional $A$ sequentially lower semicontinuous with respect to the $H^{1,2}$--weak topology (see [10]).

## 2 Main Results

**Theorem 2.5.** Let $u \in W^{1,2} (\Omega, \mathbb{R}^N)$ be a minimum of the functional $A(u, \Omega)$ defined above. Suppose that assumptions $(A-1)$, $(A-2)$, $(A-3)$, $(A-4)$, $1 < q \leq 2$ and $(g-1)$, $(g-2)$ and $(g-3)$ are satisfied. Then for $\lambda = n(1 - \frac{q}{p})$ we have

$$Du \in L^{q, \lambda}_{\text{loc}} (\Omega_0, \mathbb{R}^{nN})$$

where

$$\Omega_0 = \{ x \in \Omega : \liminf_{R \to 0} \frac{1}{R^{n-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \}.$$

The set $\Omega_0$ is obligatory, in fact when we pass from the regularity theory for scalar minimizers of solutions of elliptic equations to the regularity theory for vector-valued minimizers of solutions of elliptic systems, the situation changes completely: regularity is an exceptional occurrence everywhere, excluding the two dimensional case. In 1968 De Giorgi in [7] showed that his
regularity result for solutions of second order elliptic equations with measurable bounded coefficients cannot be extended to solutions of elliptic systems. He presented the quadratic functional

\[ S = \int_{\Omega} A^{\alpha\beta}_{ij}(x) D_{\alpha}u^{i} D_{\beta}u^{j} dx \]

with \( A^{\alpha\beta}_{ij} \in L^{\infty}(\Omega) \), such that

\[ \exists \nu > 0 : A^{\alpha\beta}_{ij} \chi_{\alpha\beta} \varphi^{i} \geq \nu |\varphi|^{2} \text{a.e.} x \in \Omega, \chi \in \mathbb{R}^{nN} \]

De Giorgi proves that \( S \) has a minimizer that is a function having a point of discontinuity in the origin. Later, Souček in [23] showed that minimizers of functionals of the type \( S \) can be discontinuous, not only in a point, but also on a dense subset of \( \Omega \). Modifying De Giorgi's example, Giusti and Miranda in [13] showed that solutions of elliptic quasilinear systems of the type

\[ \int_{\Omega} A^{\alpha\beta}_{ij}(u) D_{\alpha}u^{i} D_{\beta}^{j} \varphi dx = 0, \forall \varphi \in C_{0}^{\infty}(\Omega, \mathbb{R}^{N}) \]

with analytic elliptic coefficients \( A^{\alpha\beta}_{ij} \) have singularities in dimension \( n \geq 3 \). We observe that we can get global regularity for some special cases, see for example [24].

Similar examples were presented in the meantime independently by Maz’ya in [17]. Even Giaquinta in 1993 in [10], Morrey in [19] and others were interested in these problems of solutions of elliptic systems, solutions in general non regular. Then we can prove regularity except on a set, hopefully not too large.

For linear systems, regularity results assuming \( A^{\alpha\beta}_{ij} \) constant or in \( C^{0}(\Omega) \), have been obtained by Campanato (see [3]). Without assuming continuity of coefficients, we mention the study by Acquistapace [1] where Campanato’s results are refined considering that \( A^{\alpha\beta}_{ij} \) belongs to a class that neither contains nor is being contained by \( C^{0}(\Omega) \), hence in general discontinuous. Moreover, we recall the study made by Huang in [14] where he shows the regularity of weak solutions of linear elliptic systems with coefficients \( A^{\alpha\beta}_{ij}(x) \in VMO \). So, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms \( A^{\alpha\beta}_{ij} \in VMO \), even for nonlinear cases. Daneček and Viszus in [6] consider the regularity of minimizer for the functional

\[ \int_{\Omega} \left\{ A^{\alpha\beta}_{ij}(x) D_{\alpha}u^{i} D_{\beta}u^{j} + g(x, u, Du) \right\} dx \]
where the term $g(x, u, Du)$ is such that
\[ |g(x, u, z)| \leq f(x) + |z|^{\gamma} \]
where $f \in L^{p}(\Omega), 2 < p \leq \infty, f \geq 0$ a.e. on $\Omega$, $L$ is a non-negative constant and $0 \leq \gamma < 2$.

They obtained Hölder regularity of minimizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.

We also recall the paper [8] where the authors obtain regularity results for minimizers of the quasilinear functionals
\[ \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha}u^{i} D_{\beta}u^{j} dx, \]
where the coefficients $A_{ij}^{\alpha\beta}(x, u)$ have VMO dependence on the variable $x$ and continuous dependence on $u$.

In the paper [21] we improve the last mentioned result in Morrey spaces because we have considered inside the integral the term $g(x, u, Du)$ and the result by Daneček and Viszus because we consider $A_{ij}^{\alpha\beta}$ dependent not only on $x$ but also on $u$. In the present note we extend our above cited regularity results because we consider the more general class Campanato spaces.

Before an outline of the proof we state a preliminary Lemma by Campanato.

**Lemma 2.6.** Let $B(x_{0}, R)$ be a fixed ball and $u \in W^{1,2}(B(x_{0}, R), \mathbb{R}^{N})$ be a weak solution of the system
\[ D_{\alpha}(A_{ij}^{\alpha\beta} D_{\beta}u^{j}) = 0, \quad i = 1, \ldots, N \]
where $A_{ij}^{\alpha\beta}$ are constant and satisfy the ellipticity condition. Then $\forall t \in (0, 1]$
\[ \int_{B(x_{0}, tR)} |Du|^{2} dx \leq c \cdot t^{n} \int_{B(x_{0}, R)} |Du|^{2} dx. \]

**PROOF OF THE THEOREM**
For simplicity we'll consider the case $g = 0$. Let $R > 0$ and $x_{0} \in \Omega$ such that $B(x_{0}, R) \subset \subset \Omega$.

Let $v$ be the minimum of the “freezing” functional $A$, that is
\[ A_{0}(v, B(x_{0}, R/2)) = \int_{B(x_{0}, R/2)} A_{ij}^{\alpha\beta}(x_{0}, u_{R/2}) D_{\alpha}v^{i} D_{\beta}v^{j} dx \]
with \( v \equiv u \) on \( \partial B(x_0, \frac{R}{2}) \). The idea of freezing is the same used by Chiarenza, Frasca and Longo in [4].

For \( 0 \leq \lambda < n \) and \( q \leq 2 \) we have that

\[
\|Du\|_{L^{q, \lambda}(\Omega)} = \|Du\|_{L^q(\Omega)} + [Du]_{q, \lambda} \leq \sum_{\iota} \epsilon \iota
\]

\[
\leq K \|Du\|_{L^{q, \lambda}(\Omega)} \leq K \|Du\|_{L^2(\Omega)}
\]

where the constant \( K \) is independent of \( u \). We observe that \( A_{ij}^{\alpha \beta}(x_0, u_{\frac{R}{2}}) \) are constant coefficients, then from the above Lemma, \( \forall t \in (0, 1] \),

\[
\int_{B(x_0, t\frac{R}{2})} |Du|^2 dx \leq c \cdot t^n \int_{B(x_0, \frac{R}{2})} |Du|^2 dx.
\]

Let \( w = u - v \), then \( w \in W^{1, 2}_0(B(x_0, \frac{R}{2})) \)

\[
\int_{B(x_0, \frac{R}{2})} |Du|^2 dx \leq c \int_{B(x_0, \frac{R}{2})} |Du|^2 dx + \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx
\]

Let us estimate:

\[
\int_{B(x_0, \frac{R}{2})} |Dw|^2 dx.
\]

From the hypothesis and a Lemma in [12] we have

\[
\nu \int_{B(x_0, \frac{R}{2})} |Dw|^2 dx \leq \left\{ A^0(u, B(x_0, \frac{R}{2})) - A^0(v, B(x_0, \frac{R}{2})) \right\}
\]

adding and subtracting:

\[ A_{ij}^{\alpha \beta}(x, u_{\frac{R}{2}}) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha \beta}(x, u_{\frac{R}{2}}) D_\alpha v^i D_\beta v^j \]

\[ A_{ij}^{\alpha \beta}(x, u) D_\alpha u^i D_\beta u^j, \quad A_{ij}^{\alpha \beta}(x, v) D_\alpha v^i D_\beta v^j \]

we obtain some different kinds of integrals that we now examine.
Using $L^p$ estimate, we can estimate the terms with $|Du|^2$ as follows:

$$
\int_{B(x_0, \frac{R}{2})} \left| A_{ij}^{\alpha \beta}(x_0, u_{\frac{R}{2}}) - A_{ij}^{\alpha \beta}(x, u_{\frac{R}{2}}) \right| |Du|^2 \, dx \leq 
$$

$$
\leq c \{ \eta(A(\cdot, u_{\frac{R}{2}}); R) \}^{1-\frac{2}{p}} \int_{B(x_0, R)} |Du|^2 \, dx.
$$

We also observe that:

$$
\int_{B(x_0, \frac{R}{2})} \left| A_{ij}^{\alpha \beta}(x, u_{\frac{R}{2}}) - A_{ij}^{\alpha \beta}(x, u) \right| |D_\alpha u|^i |D_\beta u|^j \, dx \leq 
$$

$$
\leq c \left( \int_{B(x_0, R)} |Du|^2 \, dx \right) \left( \int_{B(x_0, \frac{R}{2})} \omega \left( \left| u_{\frac{R}{2}} - u \right|^2 \right) \, dx \right)^{1-\frac{2}{p}} 
$$

Moreover, we can estimate the terms having $|Dv|^2$ similarly using $L^p$ estimates for $Dv$.

Then if $\rho = tR$

$$
\int_{B(x_0, \rho)} |Du|^2 \, dx \leq 
$$

$$
\leq c \left\{ \left( \frac{\rho}{R} \right)^n + \left( \omega \left( R^{2-n} \int_{B(x_0, \frac{R}{2})} |Du|^2 \, dx \right) \right)^{1-\frac{2}{p}} + \left( \eta(A(\cdot, u_{\frac{R}{2}}), R) \right)^{1-\frac{2}{p}} \right\} \left( \int_{B(x_0, \frac{R}{2})} |Du|^2 \, dx \right).
$$

Using a Lemma contained in [9] and selecting $\rho$ sufficiently small, specifically $\rho < \frac{R}{2}$, we have

$$
\int_{B(x_0, \rho)} |Du|^2 \, dx \leq c \cdot \rho^\lambda.
$$
References


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