Blowup analysis for $SU(3)$ Toda system

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September 23, 2004

Abstract

We study non-compact solution sequence to the $SU(3)$ Toda system in non-abelian relativistic self-dual gauge theory, i.e., the quantization of the total mass and classification of the singular limit.

Keywords: self-dual gauge theory; mean field equation; Toda system; blow-up analysis; symmetrization.

1 Introduction

The $SU(3)$ Toda system arises in non-abelian relativistic self-dual gauge theory [11, 16]. In the simplest form without the vortex term, it is given by

$$\begin{aligned} -\Delta_g u_1 &= 2\lambda_1\left(\frac{e^{u_1}}{\int_M e^{u_1}}-\frac{1}{|M|}\right)-\lambda_2\left(\frac{e^{u_2}}{\int_M e^{u_2}}-\frac{1}{|M|}\right) \\ -\Delta_g u_2 &= -\lambda_1\left(\frac{e^{u_1}}{\int_M e^{u_1}}-\frac{1}{|M|}\right)+2\lambda_2\left(\frac{e^{u_2}}{\int_M e^{u_2}}-\frac{1}{|M|}\right) \end{aligned} \qquad (1)$$

on M with

$$\int_M u_1 = \int_M u_2 = 0,$$

where (M, g) is a compact Riemannian surface with the volume $|M|$, and λ_1, λ_2 are positive constants. If $\lambda_2 = 0$, we have

$$-\Delta_g u = \lambda \left(\frac{e^u}{\int_\Omega e^u} - \frac{1}{|M|} \right) \quad \text{on} \quad M, \qquad \int_M u = 0 \tag{2}$$

for $u = 2u_1$ and $\lambda = 2\lambda_1$. This is the simplest form of the mean field equation studied in the contexts of the prescribing Gaussian curvature [14], statistical mechanics of many vortex points in the perfect fluid [3], [4], [15], and self-dual gauge theories [26]. See also the monographs [20], [25] for mean field equation, and [27] for Toda systems.

Equation (2) has a variational structure, and $u = u(x)$ is a solution if and only if it is a critical point of

$$J_\lambda(v) = \frac{1}{2}\int_M |\nabla v|^2 - \lambda \log \int_M e^v \tag{3}$$

defined for $v \in H^1(M)$ with $\int_M v = 0$. If $\lambda = 8\pi$, this functional is bounded from below by the Trudinger-Moser inequality, and it has a global minimizer for $\lambda \in [0, 8\pi)$. This functional is not bounded from below in case $\lambda > 8\pi$, but Ding-Jost-Li-Wang [10] showed that there is a saddle type critical point if M has genus $g \geq 1$ and $8\pi < \lambda < 16\pi$. This critical point may be a trivial solution $u = 0$ to (2), but we have $u \neq 0$ in the Struwe-Tarantello [24] case, that is, M is a flat torus with the fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ and $\lambda \in (8\pi, 4\pi^2)$. Discussing the general setting of the Riemannian surface, (2) has a non-trivial mountain pass solution (Struwe-Tarantello solution) if $\lambda \in (8\pi, \mu_1 |M|)$, where μ_1 denotes the principal eigenvalue of $-\Delta_g$. Then, Ding-Jost-Li-Wang solution is non-tirival if $\lambda \in (8\pi, \min\{\mu_1 |M|, 16\pi\})$. This solution is different even from the mountain pass solution and we will have at least two non-trivial solutions in this range.

In more detail, we have Chen-Lin's formula [7] to (2) concerning the total degree denoted by d_λ. If g denotes the genus of M, then we have $d_\lambda = 2g - 1$ for $\lambda \in (8\pi, 16\pi)$. This formula suggests that the Ding-Jost-Li-Wang solution has Morse index 2 and is different from the Struwe-Tarantello solution of Morse index 1, and furthermore, that the former's non-triviality survives until the second bifurcation from the trivial solution. For example, if $g = 1$, we expect five and three solutions including the trivial solution for $\lambda \in (8\pi, \min\{\mu_1 |M|, 16\pi\})$ and $\lambda \in (\mu_1 |M|, \min\{\mu_2 |M|, 16\pi\})$, respectively, where μ_2 denotes the second eigenvalue of $-\Delta_g$. Furthermore, such a multiplicity result will be valid even for the equation with vortex terms.

Problem (1) has an analogous variational structure and (u_1, u_2) is a solution if and only if it is a critical point of

$$\begin{aligned} J_{\lambda_1,\lambda_2}(v_1, v_2) &= \frac{1}{3}\int_M |\nabla v_1|^2 + \nabla v_1 \cdot \nabla v_2 + |\nabla v_2|^2 \\ &-\lambda_1 \log \int_M e^{v_1} - \lambda_2 \log \int_M e^{v_2} \end{aligned} \tag{4}$$

defined on $E \times E$, where E denotes the Hilbert space

$$E = \left\{ v \in H^1(M) \mid \int_M v = 0 \right\}$$

provided with the inner product $\langle u, v \rangle = \int_M \nabla u \cdot \nabla v$. Jost-Wang [12] showed that this new functional is bounded from below in the case of $\lambda_1 = \lambda_2 = 4\pi$, and has a global minimizer if $(\lambda_1, \lambda_2) \in [0, 4\pi) \times [0, 4\pi)$. On the other hand, Lucia-Nolasco [19] obtained a mountain pass solution if (M, g) is a flat torus with the fundamental cell domain $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, and if λ_1, λ_2 are in

$$4\pi < \max(\lambda_1, \lambda_2) < 8\pi, \qquad \min(\lambda_1, \lambda_2) \neq 4\pi, \tag{5}$$

and

$$\left(\lambda_1 - \frac{8\pi^2}{3}\right)\left(\lambda_2 - \frac{8\pi^2}{3}\right) > \left(\frac{4\pi^2}{3}\right)^2. \tag{6}$$

Concerning the Ding-Jost-Li-Wang type solution we have the following.

Theorem 1. *If M has genus ≥ 1, the functional J_{λ_1,λ_2} of (4) defined on $E \times E$ has a saddle type critical point for any (λ_1, λ_2) in (5) and*

$$\left(\lambda_1 - \frac{32\pi}{3}\right)\left(\lambda_2 - \frac{32\pi}{3}\right) > \left(\frac{16\pi}{3}\right)^2. \tag{7}$$

We refer to [5] for the precise definition of this mini-max value. The important question of its non-triviality will be studied in a forthcoming paper. Note that conditions (7) and (6) are equivalent to

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} - \frac{1}{16\pi}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} > 0 \tag{8}$$

and

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} - \frac{1}{4\pi^2}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} > 0, \tag{9}$$

respectively, and therefore, (6) implies (7). In [5], we did not eliminate the residual set of (λ_1, λ_2) completely. This is the problem of blowup analysis in which the present paper is concerned. We employ the methods of symmetrization [22], [23] and rescaling [19] and settle down the problem. A more detailed analysis will guarantee that the mass of non-compact solution sequence is in $(4\pi\mathbf{N} \times \mathbf{R}_+) \cup (\mathbf{R}_+ \times 4\pi\mathbf{N})$. Our results obtained so far are complicated, and we state them in the following section.

2 Summary

We are concerned with the solution sequence $\{(u_{1,n}, u_{2,n}, \lambda_{1,n}, \lambda_{2,n})\}$ of (1), that is;

$$-\Delta_g u_{1,n} = 2\lambda_{1,n}\left(\frac{e^{u_{1,n}}}{\int_M e^{u_{1,n}}} - \frac{1}{|M|}\right) - \lambda_{2,n}\left(\frac{e^{u_{2,n}}}{\int_M e^{u_{2,n}}} - \frac{1}{|M|}\right)$$
$$-\Delta_g u_{2,n} = -\lambda_{1,n}\left(\frac{e^{u_{1,n}}}{\int_M e^{u_{1,n}}} - \frac{1}{|M|}\right) + 2\lambda_{2,n}\left(\frac{e^{u_{2,n}}}{\int_M e^{u_{2,n}}} - \frac{1}{|M|}\right)$$

in M with

$$\int_M u_{1,n} = \int_M u_{2,n} = 0.$$

In terms of $(v_{1,n}, v_{2,n})$ defined by

$$\begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} v_{1,n} \\ v_{2,n} \end{pmatrix},$$

it holds that

$$-\Delta_g v_{1,n} = \lambda_{1,n}\left(\frac{e^{2v_{1,n}-v_{2,n}}}{\int_M e^{2v_{1,n}-v_{2,n}}} - \frac{1}{|M|}\right)$$
$$-\Delta_g v_{2,n} = \lambda_{2,n}\left(\frac{e^{-v_{1,n}+2v_{2,n}}}{\int_M e^{-v_{1,n}+2v_{2,n}}} - \frac{1}{|M|}\right)$$

in M with

$$\int_M v_{1,n} = \int_M v_{2,n} = 0,$$

namely, $\{(v_{1,n}, v_{2,n}, \lambda_{1,n}, \lambda_{2,n})\}$ is a solution sequence to

$$-\Delta_g v_1 = \lambda_1\left(\frac{e^{2v_1-v_2}}{\int_M e^{2v_1-v_2}} - \frac{1}{|M|}\right)$$
$$-\Delta_g v_2 = \lambda_2\left(\frac{e^{-v_1+2v_2}}{\int_M e^{-v_1+2v_2}} - \frac{1}{|M|}\right) \tag{10}$$

in M with

$$\int_M v_1 = \int_M v_2 = 0.$$

Henceforth, $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$ indicate the exponents. Letting

$$\mu_{i,n} = \lambda_{i,n}\frac{e^{2v_{i,n}-v_{j,n}}}{\int_M e^{2v_{i,n}-v_{j,n}}} = \lambda_{i,n}\frac{e^{u_{i,n}}}{\int_M e^{u_{i,n}}},$$

we can assume the following relations without loss of generality, where

$$\mathcal{M}(M) = C(M)'$$

denotes the set of measures on M:

$$\mu_{i,n} \rightharpoonup \mu_i \quad * \text{ weakly in } \mathcal{M}(M) \qquad \text{and} \qquad \lambda_{i,n}(>0) \to \lambda_i \geq 0.$$

Given $x_0 \in M$, we take the iso-thermal chart (Ψ, U) satisfying

$$\Psi(x_0) = 0, \quad \Psi(x) = X, \quad g = e^{\xi}(dX_1^2 + dX_2^2),$$

and each function $f(x)$ defined on M induces $f \circ \Psi^{-1}$ denoted by

$$f(X) = f\left(\Psi^{-1}(X)\right).$$

Furthermore, $G = G(x, y)$ indicates the Green's function:

$$-\Delta_g G(\cdot, y) = \delta_y - \frac{1}{|M|} \quad \text{in} \quad M, \qquad \int_M G(\cdot, y) = 0.$$

Then, we can show the following.

Theorem 2. *Up to a subsequence, we have the following alternatives.*

1. *(compactness) We have $(v_{1,n}, v_{2,n}) \to (v_1, v_2)$ in $E \times E$ and this*

$$(v_1, v_2, \lambda_1, \lambda_2)$$

is a solution to (10).

2. *(half compactness) There is $i \in \{1, 2\}$ such that $v_{i,n} \to v_i$ in E and the blowup set of $\{v_{j,n}\}$ defined by*

$$S_j = \{x_0 \in M \mid \text{there exists } x_n \to x_0 \text{ such that } v_{j,n}(x_n) \to +\infty\}$$

is finite and non-empty. This v_i satisfies

$$-\Delta_g v_i = \lambda_i \left(\frac{K_j(x) e^{2v_i}}{\int_M K_j(x) e^{2v_i}} - \frac{1}{|M|} \right), \qquad \int_M v_i = 0 \tag{11}$$

for $K_j(x) = e^{-4\pi \sum_{x_0 \in S_j} G(x, x_0)}$. It holds that $\mu_j = 4\pi \sum_{x_0 \in S_j} \delta_{x_0}$ and $\mu_{j,n} \to 0$ locally uniformly in $M \setminus S_j$. Each $x_0 \in S_j$ is governed by

$$\nabla_X \left\{ 8\pi H_\Psi(X, x_0) + \sum_{x_0' \in S_j \setminus \{x_0\}} 8\pi G(X, x_0') - v_i(X) + \xi(X) \right\} \Bigg|_{X=0} = 0, \tag{12}$$

where (Ψ, U) is the iso-thermal chart and

$$H_\Psi(X, Y) = G(X, Y) + \frac{1}{2\pi} \log|X - Y|.$$

3. *(concentration) It holds that $\mathcal{S}_1, \mathcal{S}_2 \neq \emptyset$ and $\sharp\mathcal{S}_1, \sharp\mathcal{S}_2 < +\infty$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the blowup sets of $\{v_{1,n}\}$ and $\{v_{2,n}\}$, respectively. For each $i = 1, 2$, we have*

$$\mu_i = r_i + \sum_{x_0 \in \mathcal{S}_i} m_i(x_0)\delta_{x_0}$$

with $m_i(x_0) \geq 2\pi$ and $r_i \in L^1(M) \cap L^\infty_{loc}(M \setminus \mathcal{S}_i)$, and $\mu_{i,n} \to r_i$ in $L^t_{loc}(M \setminus \mathcal{S}_i)$ for any $t \in [1, \infty)$. Here, the limit measure μ_i is specified more as follows.

   (a) *(mass quantization)*
   If $x_0 \in \mathcal{S}_i \setminus (\mathcal{S}_1 \cap \mathcal{S}_2)$, then we have $m_i(x_0) = 4\pi$. In the case of $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, it holds that

$$m_1(x_0)^2 - m_1(x_0)m_2(x_0) + m_2(x_0)^2 = 4\pi \{m_1(x_0) + m_2(x_0)\} \tag{13}$$

   and $\max\{m_1(x_0), m_2(x_0)\} \geq 8\pi$. Consequently, we have $m_i(x_0) \geq 4\pi$ for any $x_0 \in \mathcal{S}_i$.

   (b) *(residual vanishing)*
   If $\mathcal{S}_i \setminus \mathcal{S}_j \neq \emptyset$, then $r_i = 0$. In the case of $\mathcal{S}_i \subset \mathcal{S}_j$, on the contrary, $r_i = 0$ follows if there is $x_0 \in \mathcal{S}_i$ such that $2m_i(x_0) - m_j(x_0) > 4\pi$. This condition is relaxed as $2m_i(x_0) - m_j(x_0) \geq 4\pi$ if $r_j = 0$ is known.

   (c) *(blowup set control) If $\mathcal{S}_i \setminus \mathcal{S}_j \neq \emptyset$, in which case $r_i = 0$ holds as is described above, we have (12) at each $x_0 \in \mathcal{S}_i \setminus \mathcal{S}_j$. If $r_1 = r_2 = 0$, then for each $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ we have*

$$\begin{aligned}
&m_1(x_0)\, \nabla_X \left\{ 8\pi H_\Psi(X, x_0) + \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} 2m_1(x_0)G(X, x_0') \right. \\
&\left. \left. - \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0')G(X, x_0') + \xi(X) \right\} \right|_{X=0} \\
&+ m_2(x_0)\, \nabla_X \left\{ 8\pi H_\Psi(X, x_0) - \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0')G(X, x_0') \right. \\
&\left. \left. + \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} 2m_2(x_0')G(X, x_0') + \xi(X) \right\} \right|_{X=0} = 0.
\end{aligned} \tag{14}$$

Now, we shall give a few remarks on the above theorem. First, the blowup sets introduced in the above theorem coincide with those for $\{(u_{1,n}, u_{2,n})\}$. Therefore, we have

$$\mathcal{S}_j = \{x_0 \in M \mid \text{there exists } x_n \to x_0 \text{ such that } u_{j,n}(x_n) \to +\infty\}$$

in each case. Next, possible limits of (λ_1, λ_2) for the non-compact solution sequence $\{(u_{1,n}, u_{2,n})\}$ are restricted as follows by the above theorem. To begin with, in the half compactness case these values are contained in $L = (4\pi\mathbf{N} \times \mathbf{R}_+) \cup (\mathbf{R}_+ \times 4\pi\mathbf{N})$. Next, in the non-compact case without collision, that is, $\mathcal{S}_1, \mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, the residual vanishing is achieved and hence they are contained in $V = 4\pi\mathbf{N} \times 4\pi\mathbf{N}$. The non-compact case with collision, on the other hand, is complicated, and we put

$$\begin{aligned}
&\mathcal{E} = \{(m_1, m_2) \mid \max\{m_1, m_2\} \geq 8\pi,\ m_1^2 + m_2^2 - m_1 m_2 = 4\pi(m_1 + m_2)\} \\
&\mathcal{E}_j = \{(m_1, m_2) \in \mathcal{E} \mid 2m_i - m_j < 4\pi\ (i \neq j)\} \\
&\mathcal{E}_0 = \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)
\end{aligned}$$

as illustrated in Figure 4 of [5]. In more detail, $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ is a division of $\mathcal{E}$, and if $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, then it holds that $(m_1(x_0), m_2(x_0)) \in \mathcal{E}$. According to $(m_1(x_0), m_2(x_0))$ is in $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$, we have $r_1 = r_2 = 0$, $r_1 = 0$, and $r_2 = 0$, respectively. In any case, either r_1 or r_2 vanishes. If $\sharp(\mathcal{S}_1 \cap \mathcal{S}_2) = n$, then

$$\left(\sum_{x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2} m_1(x_0),\ \sum_{x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2} m_2(x_0) \right) \in \mathcal{E}^n,$$

where $\mathcal{E}^n$ is defined inductively by $\mathcal{E}^1 = \mathcal{E}$ and $\mathcal{E}^n = \mathcal{E}^{n-1} + \mathcal{E}$ $(n = 2, \cdots)$. In this case, if r_i does not vanish, then

$$\left(\sum_{x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2} m_1(x_0),\ \sum_{x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2} m_2(x_0) \right) \in \mathcal{E}_j^n$$

for $j \neq i$, where $\mathcal{E}_j^1 = \mathcal{E}_j$ and $\mathcal{E}_j^n = \mathcal{E}_j^{n-1} + \mathcal{E}_j$ $(n = 2, \cdots)$.

In other words, the collision case $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ is classified in accordance with (a) $\mathcal{S}_1 = \mathcal{S}_2$, (b) $\mathcal{S}_2 \subset \mathcal{S}_1$ and $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \emptyset$, (c) $\mathcal{S}_1 \subset \mathcal{S}_2$ and $\mathcal{S}_2 \setminus \mathcal{S}_1 \neq \emptyset$, and (d) $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}_2 \setminus \mathcal{S}_1 \neq \emptyset$. To state them in more detail, we put $\mathcal{E}^\infty = \cup_{n=1}^\infty \mathcal{E}^n$, $\mathcal{E}_i^\infty = \cup_{n=1}^\infty \mathcal{E}_i^n$, and $M_{c,i} = \sum_{x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2} m_i(x_0)$ for $i = 1, 2$.

1. $(\mathcal{S}_1 = \mathcal{S}_2)$. It holds that $(M_{c,1}, M_{c,2}) \in \mathcal{E}^\infty$. There is a possibility that one of r_j does not vanish, so that $(\lambda_1, \lambda_2) \in (\{M_{c,1}\} \times [M_{c,2}, \infty)) \cup ([M_{c,1}, \infty) \times \{M_{c,2}\})$, or equivalently, $(\lambda_1, \lambda_2) \in \mathcal{E}^\infty \cup \Lambda_c$, where

$$\begin{aligned}
\Lambda_c = &\{(\lambda_1, \lambda_2) \mid \text{there exists } \lambda_{1,0} \leq \lambda_1 \text{ such that } (\lambda_{1,0}, \lambda_2) \in \mathcal{E}_2^\infty\} \\
&\cup \{(\lambda_1, \lambda_2) \mid \text{there exists } \lambda_{2,0} \leq \lambda_2 \text{ such that } (\lambda_1, \lambda_{2,0}) \in \mathcal{E}_1^\infty\}.
\end{aligned}$$

2. $(\mathcal{S}_2 \subset \mathcal{S}_1$ and $\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \emptyset)$. This case gives $r_1 = 0$ and hence $\lambda_1 \in \{M_{c,1}\} + 4\pi\mathbf{N}$. Therefore, it holds that $(\lambda_1, \lambda_2) \in \Lambda_c^1\ (\subset \Lambda_c + 4\pi\mathbf{N} \times \{0\})$, where

$$\begin{aligned}
\Lambda_c^1 = &\{(\lambda_1, \lambda_2) \mid \text{there exists } \lambda_{2,0} \leq \lambda_2 \text{ and } n \in \mathbf{N} \\
&\text{such that } (\lambda_1 - 4\pi n, \lambda_{2,0}) \in \mathcal{E}_1^\infty\}.
\end{aligned}$$

3. ($\mathcal{S}_1 \subset \mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}_2 \setminus \mathcal{S}_1$). Similarly, we have $(\lambda_1, \lambda_2) \in \Lambda_c^2 (\subset \Lambda_c + \{0\} \times 4\pi\mathbf{N})$, where

$$\Lambda_c^2 = \{(\lambda_1, \lambda_2) \mid \text{there exists } \lambda_{1,0} \leq \lambda_1 \text{ and } n \in \mathbf{N} \text{ such that } (\lambda_{1,0}, \lambda_2 - 4\pi n) \in \mathcal{E}_1^\infty\}.$$

4. ($\mathcal{S}_1 \setminus \mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}_2 \setminus \mathcal{S}_1 \neq \emptyset$). In this case, we have $r_1 = r_2 = 0$, and hence $(\lambda_1, \lambda_2) \in \mathcal{E}^\infty + V (= \mathcal{E}^\infty + (4\pi\mathbf{N} \times 4\pi\mathbf{N}))$.

Consequently, the residual set of the collision case $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ is contained in

$$\mathcal{E}^\infty \cup \Lambda_c + (4\pi\mathbf{N}_0 \times 4\pi\mathbf{N}_0)$$

for $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$, and we obtain the following.

Theorem 3. *A solution sequence $\{(u_{1,n}, u_{2,n}, \lambda_{1,n}, \lambda_{2,n})\}$ of (1) is compact in $E \times E$ if (λ_1, λ_2) is not in the residual set $L \cup (\mathcal{E}^\infty \cup \Lambda_c + (4\pi\mathbf{N}_0 \times 4\pi\mathbf{N}_0))$, where $\lambda_{i,n} \to \lambda_i$ for $i = 1, 2$.*

Some estimates necessary for the proof of the above theorem are obtained just by regarding (1) as a mean field equation. This is done in the following section, and then we apply the method of symmetrization [22, 23] in §4, which makes the blowup mechanism clearer. The proof of Theorem 2 is completed in §5 by the rescaling argument [17], whereby Lemma 5.8 of [19] is justified, namely, $\max\{m_1(x_0), m_2(x_0)\} \geq 8\pi$ holds for each $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$. This enables us to eliminate all the redidual points in Theorem 1.

Recently, C.-S. Lin [18] informed us that

$$(m_1(x_0), m_2(x_0)) \in \{(4\pi, 8\pi), (8\pi, 4\pi), (8\pi, 8\pi)\}$$

holds for any $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$. In this case, each solution sequence to (1) is compact in $E \times E$ except for $(\lambda_1, \lambda_2) \in (4\pi\mathbf{N} \times \mathbf{R}_+) \cup (\mathbf{R}_+ \times 4\pi\mathbf{N})$, althogh the residual vanishing may not occur for $(m_1(x_0), m_2(x_0)) = (4\pi, 8\pi), (8\pi, 4\pi)$.

3 Preliminaries

Writing $v_n = 2v_{i,n}$, $K_n(x) = e^{-v_{j,n}}$, and $\lambda_n = 2\lambda_{i,n}$, we get

$$-\Delta_g v_n = \lambda_n \left(\frac{K_n(x) e^{v_n}}{\int_M K_n(x) e^{v_n}} - \frac{1}{|M|} \right), \qquad \int_M v_n = 0 \tag{15}$$

from (10), where $i = 1, 2$ and $j \in \{1, 2\} \setminus \{i\}$. This is the mean field equation with the inhomogeneous coefficient and we can apply [23] to control the solution sequence.

In fact, from the elliptic L^1 estimate we have $\limsup \|v_{i,n}\|_{W^{1,q}(M)} < +\infty$ for $q \in [1, 2)$ and hence, passing to a subsequence, $v_{i,n} \to v_i$ follows in $L^t(M)$

for $t \in [1,\infty)$ and for a.e. $x \in M$. On the other hand, by [1] there is $A \in \mathbf{R}$ satisfying $G(x,y) \geq -A$, and hence we have

$$v_{i,n} = \lambda_{i,n} \int_M G(\cdot, y) \frac{e^{u_{i,n}(y)}}{\int_M e^{u_{i,n}}} dg_y \geq -\lambda_{i,n} A,$$

namely, there is $C > 0$ independent of n such that

$$v_{i,n} \geq -C. \tag{16}$$

This implies $\limsup \|e^{-v_{j,n}}\|_\infty < +\infty$, and hence

$$e^{-v_{j,n}} \to e^{-v_j} \quad \text{in} \quad L^t(M)$$

for any $t \in [1,\infty)$ and a.e. $x \in M$. Therefore, Theorem 2.1 of [23] is applicable and we obtain the following.

Lemma 1. *Under the assumptions and notations of Theorem 2, we have the following alternatives up to a subsequence.*

1. *(compactness) It holds that $(v_{1,n}, v_{2,n}) \to (v_1, v_2)$ in $E \times E$ and this $(v_1, v_2, \lambda_1, \lambda_2)$ is a solution to (10).*
2. *(half compactness) It holds that $v_{i,n} \to v_i$ in E and the blowup set S_j of $\{v_{j,n}\}$ is finite and non-empty, where $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$. This v_i satisfies (11) for $K_j = e^{-v_i} = e^{-\sum_{x_0 \in S_j} m(x_0) G(\cdot, x_0)}$, while μ_j takes the form $\mu_j = \sum_{x_0 \in S_j} m_j(x_0)\delta_{x_0}$ with $m_j(x_0) \geq 2\pi$.*
3. *(concentration) For each $i = 1,2$, the blowup set S_i of $\{v_{i,n}\}$ is finite and non-empty. We have*

$$\mu_i = r_i + \sum_{x_0 \in S_i} m_i(x_0)\delta_{x_0}$$

with $m_i(x_0) \geq 2\pi$ and $r_i \in L^1(M) \cap L^\infty_{loc}(M \setminus S_i)$ and $\mu_{i,n} \to r_i$ in $L^t(M \setminus S_i)$ for any $t \in [1,\infty)$. Furthermore, $r_i = 0$ if $S_i \setminus S_j \neq \emptyset$.

Let us recall that S_i denotes the blowup set of $\{v_{i,n}\}$. Now, we show that it coincides with the blowup set of $\{u_{i,n}\}$, denoted by S_{u_i}.

Lemma 2. *It holds that $S_{u_i} = S_i$.*

Proof: We have $u_{i,n} = 2v_{i,n} - v_{j,n}$ and the half compactness case is obvious. In the concentration case, we have $u_{i,n} \leq 2v_{i,n} - C$ by (16), and it holds that $S_{u_i} \subset S_i$. Therefore, we have only to show $S_i \subset S_{u_i}$ in the concentration case.

In fact, the blowup set S_i coincides with the singular support of μ_i, and

$$\mu_{i,n} = \lambda_{i,n} \frac{e^{u_{i,n}}}{\int_M e^{u_{i,n}}} \left(= \lambda_{i,n} \frac{e^{2v_{i,n} - v_{j,n}}}{\int_M e^{2v_{i,n} - v_{j,n}}} \right)$$

is L^∞ un-bounded around $x_0 \in \mathcal{S}_i$. Therefore, we may suppose

$$\lim_{n\to\infty} \sup_{B(x_0,r_0)} \left(u_{i,n} - \log \int_M e^{u_{i,n}} \right) = +\infty$$

for any $r_0 > 0$. Then, we obtain $r_0 > 0$ and $x_n \in \overline{B(x_0, r_0)}$ satsifying $\overline{B(x_0, r_0)} \cap \mathcal{S}_i = \{x_0\}$ and

$$u_{i,n}(x_n) - \log \int_M e^{u_{i,n}} = \max_{x \in \overline{B(x_0,r_0)}} \left(u_{i,n}(x) - \log \int_M e^{u_{i,n}} \right) (\to +\infty),$$

respectively. On the other hand, we have

$$\log \left(\frac{1}{|M|} \int_M e^{u_{i,n}} \right) \geq \frac{1}{|M|} \int_M u_{i,n} = 0$$

by Jensen's inequality, and hence $u_{i,n}(x_n) \to +\infty$ follows from

$$u_{i,n}(x_n) - \log \int_M e^{u_{i,n}} \leq u_{i,n}(x_n) - \log |M|. \tag{17}$$

Therefore, if $x_n \to x_0$ is proven, then we have $x_0 \in \mathcal{S}_{u_i}$.

Suppose the contrary, $x_n \to \overline{x} \neq x_0$. This means $\overline{x} \notin \mathcal{S}_i$, and hence $\limsup v_{i,n}(x_n) < +\infty$. Then, it holds that

$$\begin{aligned} \limsup \left(u_{i,n}(x_n) - \log \int_M e^{u_{i,n}} \right) &\leq \limsup u_{i,n}(x_n) - \log |M| \\ &\leq \limsup 2v_{i,n}(x_n) - \log |M| + C < +\infty, \end{aligned}$$

a contradiction.

Lemma 12 of [5] concerning the residual vanishing is stated as follows.

Lemma 3. *In the concentration case of Lemma 1, $r_i = 0$ is obtained if $\mathcal{S}_i \subset \mathcal{S}_j$ and there exists $x_0 \in \mathcal{S}_i \cap \mathcal{S}_j$ such that $2m_i(x_0) - m_j(x_0) > 4\pi$. The last condition is relaxed as $2m_i(x_0) - m_j(x_0) \geq 4\pi$ if $r_j = 0$ is known.*

The last statement of the above lemma is a direct consequence of Theorem 2.1 of [23], while the lack of summability of $r_j \neq 0$ around x_0 is compensated by the strict inequality, $2m_i(x_0) - m_j(x_0) > 4\pi$.

We can also apply Theorem 2.2 of [23], and obtain the following.

Lemma 4. *In the half compactness case of Lemma 1, we have $m_j(x_0) = 4\pi$ and (12) for each $x_0 \in \mathcal{S}_j$. This is also true in the concentration case of $x_0 \in \mathcal{S}_j \setminus \mathcal{S}_i$.*

4 Symmetrization

In this section we apply the method of symmetrization [22, 23] to (1) regarded as a system of equations. In fact, letting

$$f_{i,n} = \lambda_{i,n} \frac{e^{2v_{i,n}-v_{j,n}}}{\int_M e^{2v_{i,n}-v_{j,n}}}$$

for $i, j = 1, 2$ with $i \neq j$, we have

$$\begin{aligned} \nabla f_{i,n} &= f_{i,n} \nabla \left(2v_{i,n} - v_{j,n}\right) \\ \Delta f_{i,n} &= \nabla \cdot \left(f_{i,n} \nabla \left(2v_{i,n} - v_{j,n}\right)\right), \end{aligned}$$

and hence it holds that

$$\begin{aligned} -\int_M f_{i,n} \Delta\psi = 2 \int_M \int_M \nabla_x G(x,y) \cdot \nabla\psi(x) f_{i,n}(x) f_{i,n}(y) \\ - \int_M \int_M \nabla_x G(x,y) \cdot \nabla\psi(x) f_{j,n}(x) f_{i,n}(y) \end{aligned}$$

for any $\psi \in C^2(M)$. Adding those equalities for $(i, j) = (1, 2), (2, 1)$, we have

$$\begin{aligned} &-\int_M \left(f_{1,n} + f_{2,n}\right) \Delta\psi \\ &= 2 \int_M \int_M \nabla_x G(x,y) \cdot \nabla\psi(x) \left\{f_{1,n}(x) f_{1,n}(y) + f_{2,n}(x) f_{2,n}(y)\right\} \\ &- \int_M \int_M \nabla_x G(x,y) \cdot \nabla\psi(x) f_{1,n}(x) f_{2,n}(y) \\ &- \int_M \int_M \nabla_x G(x,y) \cdot \nabla\psi(x) f_{2,n}(x) f_{1,n}(y), \end{aligned}$$

where the last term is equal to

$$\int_M \int_M \nabla_y G(x,y) \cdot \nabla\psi(y) f_{1,n}(x) f_{2,n}(y)$$

by $G(x, y) = G(y, x)$. The first term is also symmetrized, and we have

$$\begin{aligned} &-\int_M (f_{1,n} + f_{2,n}) \Delta\psi \\ &= 2 \int_M \int_M \rho_\psi(x,y) \left\{f_{1,n}(x) f_{1,n}(y) - f_{1,n}(x) f_{2,n}(y) + f_{2,n}(x) f_{2,n}(y)\right\}, \end{aligned}$$

where

$$\rho_\psi(x,y) = \frac{1}{2} \left(\nabla_x G(x,y) \cdot \nabla\psi(x) + \nabla_y G(x,y) \cdot \nabla\psi(y)\right).$$

All the results in this section are obtained by this relation. First, we note the following.

Lemma 5. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain containing the origin with smooth boundary $\partial\Omega$, and $\{g_{1,n}\}, \{g_{2,n}\}$ be sequences in $W^{1,\infty}(\Omega)$ satisfying*

$$\nabla g_{i,n} \to G_i \qquad \text{in } L^\infty(\Omega)^2$$

with $G_1, G_2 \in C(\overline{\Omega})^2$. Let $\{v_{1,n}\}$ and $\{v_{2,n}\}$ be sequences in $H_0^1(\Omega)$ satisfying

$$\begin{aligned} -\Delta v_{i,n} &= e^{2v_{i,n}-v_{j,n}+g_{i,n}} \qquad \text{in } \Omega \\ v_{i,n} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for $i,j = 1,2$ with $i \neq j$, and suppose that

$$\begin{aligned} e^{2v_{i,n}-v_{j,n}+g_{i,n}} &\to m_i\delta_0 + r_i(x) \qquad * \text{ weakly in } \mathcal{M}(\overline{\Omega}) \\ e^{2v_{i,n}-v_{j,n}+g_{i,n}} &\to r_i \qquad \text{in } L^1_{loc}(\overline{\Omega}\setminus\{0\}) \end{aligned}$$

for $i = 1,2$, where $r_i \in L^1(\Omega)$ and $m_i > 0$. Then, we have

$$m_1^2 + m_2^2 - m_1 m_2 = 4\pi(m_1 + m_2). \tag{18}$$

If $r_1 = r_2 = 0$, furthermore, it holds that

$$\frac{m_1 G_1(0) + m_2 G_2(0)}{m_1 + m_2} = -8\pi \nabla_x H_\Omega(x,0)\big|_{x=0}, \tag{19}$$

where

$$H_\Omega(x,y) = G_\Omega(x,y) + \frac{1}{2\pi}\log|x-y|$$

with $G_\Omega = G_\Omega(x,y)$ standing for the Green's function of $-\Delta$ in Ω under the Dirichlet boundary condition.

Proof: Letting $f_{i,n} = e^{2v_{i,n}-v_{j,n}+g_{i,n}}$, we have

$$\Delta f_{i,n} = \nabla \cdot f_{i,n} \nabla\left(2v_{i,n} - v_{j,n} + g_{i,n}\right),$$

similarly. Therefore, it holds that

$$\begin{aligned} &-\int_\Omega (f_{1,n} + f_{2,n})\Delta\psi - \int_\Omega\int_\Omega ((\nabla g_{1,n}\cdot\nabla\psi) f_{1,n} + (\nabla g_{2,n}\cdot\nabla\psi) f_{2,n}) \\ &= 2\int_\Omega\int_\Omega \rho_\psi(x,y)\left\{f_{1,n}(x)f_{1,n}(y) - f_{1,n}(x)f_{2,n}(y) + f_{2,n}(x)f_{2,n}(y)\right\}, \end{aligned}$$

where $\psi \in C_0^2(\Omega)$. We take $\psi(x) = |x-a|^2\varphi(x)$ for $\varphi \in C_0^2(\Omega)$ with $\varphi(x) \equiv 1$ near 0 and $a \in \mathbf{R}^2$. In this case we have

$$\nabla\psi(x) = 2(x-a), \qquad \Delta\psi = 4 \qquad \text{near } 0,$$

and hence

$$\begin{aligned} \int_\Omega (f_{1,n} + f_{2,n})\,\Delta\psi &\to 4(m_1+m_2) + \int_\Omega (r_1+r_2)\Delta\psi \\ \int_\Omega (\nabla g_{i,n}\cdot\nabla\psi) f_{i,n} &\to -2m_i a\cdot G_i(0) - 2\int_\Omega ((x-a)\cdot\nabla\psi) r_i \end{aligned}$$

from the assumption. Furthermore,

$$\begin{aligned}
\rho_\psi(x,y) &= \frac{1}{2}\left\{\nabla_x G_\Omega(x,y)\cdot\nabla\psi(x) + \nabla_y G_\Omega(x,y)\cdot\nabla\psi(y)\right\} \\
&= -\frac{1}{4\pi}\frac{(x-y)\cdot\{\nabla\psi(x)-\nabla\psi(y)\}}{|x-y|^2} \\
&+\frac{1}{2}\left\{\nabla_x H_\Omega(x,y)\cdot\nabla\psi(x) + \nabla_y H_\Omega(x,y)\cdot\nabla\psi(y)\right\} \\
&= -\frac{1}{2\pi} + \{(x-a)\cdot\nabla_x H_\Omega(x,y) + (y-a)\cdot\nabla_y H_\Omega(x,y)\}
\end{aligned}$$

holds near $(x,y)=(0,0)$, and therefore, we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \rho_\psi(x,y) f_{i,n}(x) f_{i,n}(y) \to -\frac{m_i^2}{2\pi} + m_i^2(-a)\cdot\nabla_x H_\Omega(0,0) \\
&+m_i^2(-a)\cdot\nabla_y H_\Omega(0,0) + m_i\int_\Omega \rho_\psi(0,y) r_i(y) + m_i\int_\Omega \rho_\psi(x,0) r_i(x) \\
&+\int_\Omega\int_\Omega \rho_\psi(x,y) r_i(x) r_i(y) = -\frac{m_i^2}{2\pi} - 2m_i^2 a\cdot\nabla_x H_\Omega(0,0) \\
&+2m_i\int_\Omega \rho_\psi(x,0) r_i(x) + \int_\Omega\int_\Omega \rho_\psi(x,y) r_i(x) r_i(y)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega\int_\Omega \rho_\psi(x,y) f_{1,n}(x) f_{2,n}(y) \to -\frac{m_1 m_2}{2\pi} - m_1 m_2 a\cdot\nabla_x H_\Omega(0,0) \\
&-m_1 m_2 a\cdot\nabla_y H_\Omega(0,0) + m_1\int_\Omega \rho_\psi(0,y) r_2(y) + m_2\int_\Omega \rho_\psi(x,0) r_1(x) \\
&+\int_\Omega\int_\Omega \rho_\psi(x,y) r_1(x) r_2(y) = -\frac{m_1 m_2}{2\pi} - 2m_1 m_2 a\cdot\nabla_x H_\Omega(0,0) \\
&+m_1\int_\Omega \rho_\psi(x,0) r_2(x) + m_2\int_\Omega \rho_\psi(x,0) r_1(x) + \int_\Omega\int_\Omega \rho_\psi(x,y) r_1(x) r_2(y).
\end{aligned}$$

In this way, we obtain

$$\begin{aligned}
&-4(m_1+m_2) - \int_\Omega (r_1+r_2)\Delta\psi + 2a\cdot[m_1 G_1(0) + m_2 G_2(0)] \\
&+2\int_\Omega [(x-a)\cdot\nabla\psi](r_1+r_2) = -\frac{1}{\pi}(m_1^2+m_2^2-m_1m_2) \\
&-4(m_1^2+m_2^2-m_1m_2)a\cdot\nabla_x H_\Omega(0,0) \\
&+2\left((2m_1-m_2)\int_\Omega \rho_\psi(x,0) r_1(x) + (2m_2-m_1)\int_\Omega \rho_\psi(x,0) r_2(x)\right) \\
&+2\int_\Omega\int_\Omega \rho_\psi(x,y)\{r_1(x)r_1(y) - r_1(x)r_2(y) + r_2(x)r_2(y)\}.
\end{aligned}$$

and therefore, can apply the argument in the proof of Lemma 4.1 of [23]. Namely, first, we put $a=0$ and shrink the diameter of the support of ψ. This implies

$$-4(m_1+m_2) = -\frac{1}{\pi}(m_1^2+m_2^2-m_1m_2),$$

or equivalently, (18). Next, from the arbitrariness of a we get

$$m_1G_1(0)+m_2G_2(0) = -2(m_1^2+m_2^2-m_1m_2)\nabla_x H_\Omega(0,0)$$

in the case of $r_1=r_2=0$, which is equivalent to (19).

Now, we show the following.

Lemma 6. *In the concentration case of Lemma 1, we have (13) for each $x_0 \in S_1\cap S_2$. Furthermore, if $r_1=r_2=0$, then (14) holds true.*

Proof: Given $x_0\in S_1\cap S_2$, we take the iso-thermal chart (Ψ, U) satisfying $\Psi(x_0)=0$, $\overline{U}\cap(S_1\cup S_2)=\{x_0\}$, $g=e^\xi\left(dX_1^2+dX_2^2\right)$ for $X=\Psi(x)$, and $\partial\Omega$ smooth for $\Omega=\Psi(U)$. Then, $v_{i,n}(X)=v_{i,n}\circ\Psi^{-1}(X)$ is a solution to

$$-\Delta v_{i,n} = \lambda_{i,n}\left(\frac{e^{2v_{i,n}-v_{j,n}}}{\int_M e^{2v_{i,n}-v_{j,n}}} - \frac{1}{|M|}\right)e^\xi.$$

Taking $h_{i,n}, h_\xi$ by

$$\begin{aligned} &\Delta h_{i,n}=0 \quad \text{in } \Omega \qquad h_{i,n}=v_{i,n} \quad \text{on } \partial\Omega \\ &\Delta h_\xi = e^\xi \quad \text{in } \Omega \qquad h_\xi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{20}$$

we put $\tilde{v}_{i,n}=v_{i,n}-h_{i,n}-\frac{\lambda_{i,n}}{|M|}h_\xi$. Then, it holds that

$$\begin{aligned} -\Delta\tilde{v}_{i,n} &= e^{2\tilde{v}_{i,n}-\tilde{v}_{j,n}+g_{i,n}} \quad \text{in } \Omega \\ \tilde{v}_{i,n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$g_{i,n} = 2h_{i,n}-h_{j,n}+\frac{2\lambda_{i,n}-\lambda_{j,n}}{|M|}h_\xi+\xi+\log\lambda_{i,n}-\log\int_M e^{2v_{i,n}-v_{j,n}}$$

belongs to $W^{1,\infty}(\Omega)$. Furthermore, the elliptic regularity guarantees

$$\begin{aligned} \nabla g_{i,n} &= \nabla\left(2h_{i,n}-h_{j,n}+\frac{2\lambda_{i,n}-\lambda_{j,n}}{|M|}h_\xi+\xi\right) \\ &\to \quad \nabla\left(2h_i-h_j-\frac{2\lambda_i-\lambda_j}{|M|}h_\xi+\xi\right) \qquad \text{in } L^\infty(\Omega) \end{aligned}$$

by $\overline{U}\cap(S_1\cup S_2)=\{x_0\}$, where h_i is a solution to

$$\Delta h_i = 0 \quad \text{in } \Omega, \qquad h_i = v_i \quad \text{on } \partial\Omega.$$

It is obvious that

$$\nabla\left(2h_i - h_j - \frac{2\lambda_i - \lambda_j}{|M|}h_\xi + \xi\right) \in C(\overline{\Omega})^2,$$

and Lemma 5 is applicable. Therefore, (13) holds true.

If $r_1 = r_2 = 0$, then we get (19). In this case we have

$$v_i = \sum_{x_0' \in S_i} m_i(x_0')G(\cdot, x_0')$$

from the assumption, and therefore, the relation

$$-\Delta\left(2h_i - h_j + \frac{2\lambda_{i,n} - \lambda_{j,n}}{|M|}h_\xi\right) = -\frac{2\lambda_{i,n} - \lambda_{j,n}}{|M|}e^\xi \quad \text{in } \Omega$$

$$2h_i - h_j + \frac{2\lambda_{i,n} - \lambda_{j,n}}{|M|}h_\xi = 2v_i - v_j \qquad \text{on } \partial\Omega$$

implies

$$2h_i - h_j + \frac{2\lambda_{i,n} - \lambda_{j,n}}{|M|}h_\xi = 2\sum_{x_0' \in S_i} m_i(x_0')G(\cdot, x_0')$$
$$- \sum_{x_0' \in S_j} m(x_0')G(\cdot, x_0') - \{2m_i(x_0) - m_j(x_0)\}\, G_\Omega(X, 0).$$

The right-hand side is equal to

$$\{2m_i(x_0) - m_j(x_0)\}\, H_\Psi(X, x_0) + 2\sum_{x_0' \in S_i \setminus \{x_0\}} m_i(x_0')G(\cdot, x_0')$$
$$- \sum_{x_0' \in S_j \setminus \{x_0\}} m_j(x_0')G(\cdot, x_0') - (2m_i(x_0) - m_j(x_0))\, H_\Omega(X, 0),$$

and hence it holds that

$$\begin{aligned}
&m_1(x_0)G_1(0) + m_2(x_0)G_2(0) \\
&= \nabla_X \{[m_1(x_0)(2m_1(x_0) - m_2(x_0)) + m_2(x_0)(-m_1(x_0) + 2m_2(x_0))] \\
&\qquad \{H_\Psi(X, x_0) - H_\Omega(X, 0)\} \\
&+ (2m_1(x_0) - m_2(x_0)) \sum_{x_0' \in S_1 \setminus \{x_0\}} m_1(x_0')G(\cdot, x_0') \\
&+ (-m_1(x_0) + 2m_2(x_0)) \sum_{x_0' \in S_2 \setminus \{x_0\}} m_2(x_0')G(\cdot, x_0') \\
&+ (m_1(x_0) + m_2(x_0))\xi(X)\}|_{X=0}\,.
\end{aligned}$$

Since we have

$$\begin{aligned}
&m_1(x_0)\,(2m_1(x_0) - m_2(x_0)) + m_2(x_0)\,(-m_1(x_0) + 2m_2(x_0)) \\
&= 8\pi\,(m_1(x_0) + m_2(x_0))\,,
\end{aligned}$$

relation (19) is equivalent to

$$\nabla_X \left[8\pi H_\Psi(X, x_0) + \frac{2m_1(x_0) - m_2(x_0)}{m_1(x_0) + m_2(x_0)} \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0') G(X, x_0') \right.$$
$$\left. \left. + \frac{-m_1(x_0) + 2m_2(x_0)}{m_1(x_0) + m_2(x_0)} \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0') G(X, x_0') + \xi(X) \right] \right|_{X=0} = 0.$$

This means (14) and the proof is complete.

5 Rescaling

Given $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$, we have (13) and

$$\min\{m_1(x_0), m_2(x_0)\} \geq 2\pi \tag{21}$$

by the results obtained so far. In this section, we refine (21) to

$$\min\{m_1(x_0), m_2(x_0)\} \geq 4\pi. \tag{22}$$

This implies $\max\{m_1(x_0), m_2(x_0)\} \geq 8\pi$ by (18), i.e., the inequality asserted in Lemma 5.8 of [19], and then Theorem 2 follows.

For this purpose, we take the local chart (U, ψ) as in the proof of Lemma 6 and the function h_ξ defined by (20). Then, putting

$$w_{1,n}(X) = u_{1,n}\left(\Phi^{-1}(X)\right) - \log \int_M e^{u_{1,n}} - (2\lambda_{1,n} - \lambda_{2,n}) h_\xi$$
$$w_{2,n}(X) = u_{2,n}\left(\Phi^{-1}(X)\right) - \log \int_M e^{u_{2,n}} - (-\lambda_{1,n} + 2\lambda_{2,n}) h_\xi,$$

we obtain

$$\begin{aligned} -\Delta w_{1,n} &= 2V_{1,n}(x) e^{w_{1,n}} - V_{2,n} e^{w_{2,n}} \\ -\Delta w_{2,n} &= -V_{1,n}(x) e^{w_{1,n}} + 2V_{2,n}(x) e^{w_{2,n}} \end{aligned} \tag{23}$$

in Ω for

$$V_{1,n} = \lambda_{1,n} e^{\xi + (2\lambda_{1,n} - \lambda_{2,n})h}$$
$$V_{2,n} = \lambda_{2,n} e^{\xi + (-\lambda_{1,n} + 2\lambda_{2,n})h}$$

satisfying

$$\begin{aligned} &0 \leq V_{1,n}(X) \leq b, \quad 0 \leq V_{2,n}(X) \leq b \qquad (X \in \Omega) \\ &\int_\Omega e^{w_{1,n}} \leq c, \quad \int_\Omega e^{w_{2,n}} \leq c \end{aligned} \tag{24}$$

with some constants $b, c > 0$ independent of n, and

$$\begin{array}{lll} V_{1,n} & \to & V_1 = \lambda_1 e^{\xi + (2\lambda_1 - \lambda_2) h_\xi} \\ V_{2,n} & \to & V_2 = \lambda_2 e^{\xi + (-\lambda_1 + 2\lambda_2) h_\xi} \end{array} \tag{25}$$

uniformly on $\overline{\Omega}$. By (21) we have only to consider the case $\min(\lambda_1, \lambda_2) > 0$, that is, $V_1, V_2 > 0$. We have $x_{i,n} \to x_0$ such that $u_{i,n}(x_{i,n}) \to +\infty$ for $i = 1, 2$. This implies $X_{i,n} = \Phi(x_{i,n}) \to 0$ and also

$$u_{i,n}(x_{i,n}) - \log \int_M e^{u_{i,n}} \to +\infty$$

from the proof of Lemma 2, or equivalently, $w_{i,n} \to +\infty$. This means $0 \in \mathcal{S}_i^0$, where

$$\mathcal{S}_i^0 = \{X_0 \in \Omega \mid \text{there exists } X_n \to X_0 \text{ such that } w_{i,n}(X_n) \to +\infty\}.$$

We also obtain $\mathcal{S}_i^0 \subset \Psi(U \cap \mathcal{S}_i)$ similarly from the proof of Lemma 2.

By Lemma 1 we have

$$\begin{array}{lll} V_{1,n} e^{w_{1,n}} & \to & m_1 \delta_0 + r_1 \\ V_{2,n} e^{w_{2,n}} & \to & m_2 \delta_0 + r_2 \end{array}$$

in $\mathcal{M}(\overline{\Omega})$ with $\min(m_1, m_2) \geq 2\pi$, $r_1, r_2 \in L^1(\Omega) \cap L^\infty_{loc}(\overline{\Omega} \setminus \{0\})$, and

$$V_{i,n} e^{w_{i,n}} \to r_i \qquad \text{in } L^t_{loc}(\overline{\Omega} \setminus \{0\})$$

for any $1 \leq t < \infty$. These m_i coincide with $m_i(x_0)$ $(i = 1, 2)$. By Lemma 3 we have $r_1 = 0$ and $r_2 = 0$ in the cases of $2m_1 - m_2 \geq 4\pi$ and $-m_1 + 2m_2 \geq 4\pi$, respectively, and it holds that

$$m_1^2 + m_2^2 - m_1 m_2 = 4\pi(m_1 + m_2) \tag{26}$$

by Lemma 6. These relations guarantee

$$\max(m_1, m_2) \leq 4(1 + \frac{2}{\sqrt{3}})\pi = 8.6188\ldots \times \pi.$$

We study (23), (24), and (25) in a bounded domain $\Omega \subset \mathbf{R}^2$, taking $x = (x_1, x_2)$ to indicate the standard coordinate in $\mathbf{R}^2$. For this purpose, we apply Theorem 4.2 of [19], which is regarded as Brezis-Merle's theorem [2] to (1).

Lemma 7. *If $\{(w_{1,n}, w_{2,n})\}_n$ is a solution sequence to (23) and (24), then there is a subsequence (denoted by the same symbol) satisfying the following alternatives, where*

$$\mathcal{S}_i^0 = \{x_0 \in \Omega \mid \textit{there is } x_n \to x_0 \textit{ such that } w_{i,n}(x_n) \to +\infty\}$$

denotes the blowup set of $\{w_{i,n}\}_n$.

1. *Both* $\{w_{1,n}\}_n$ *and* $\{w_{2,n}\}_n$ *are locally uniformly bounded in* Ω.

2. *There is* $i \in \{1,2\}$ *such that* $\{w_{i,n}\}_n$ *is uniformly bounded in* Ω *and* $w_{j,n} \to -\infty$ *locally uniformly in* Ω *for* $j \neq i$.

3. *We have both* $w_{1,n} \to -\infty$ *and* $w_{2,n} \to -\infty$ *locally uniformly in* Ω.

4. *For the blowup sets* S_1^0, S_2^0 *defined to this subsequence, we have* $S_1^0 \cup S_2^0 \neq \emptyset$ *and* $\sharp\left(S_1^0 \cup S_2^0\right) < +\infty$. *Furthermore, for each* $i \in \{1,2\}$, *either* $\{w_{i,n}\}_n$ *is locally uniformly bounded in* $\Omega \setminus \left(S_1^0 \cup S_2^0\right)$ *or* $w_{i,n} \to -\infty$ *locally uniformly in* $\Omega \setminus \left(S_1^0 \cup S_2^0\right)$. *Finally, if* $S_i^0 \setminus \left(S_1^0 \cap S_2^0\right) \neq \emptyset$, *then* $w_{i,n} \to -\infty$ *locally uniformly in* $\Omega \setminus \left(S_1^0 \cup S_2^0\right)$, *and each* $x_0 \in S_i^0$ *takes* $m(x_0) \geq 2\pi$ *such that*
$$V_{i,n}(x)e^{w_{i,n}} \rightharpoonup \sum_{x_0 \in S_i^0} m_i(x_0)\delta_{x_0} \qquad *\text{-weakly in } \mathcal{M}(\Omega).$$

If we perform the rescaling argument using the above lemma, then we will arrive at one of the following:

1. (Toda system in $\mathbf{R}^2$)
$$-\Delta w_1 = 2e^{w_1} - e^{w_2}, \quad -\Delta w_2 = -e^{w_1} + 2e^{w_2} \qquad \text{in } \mathbf{R}^2$$
$$\int_{\mathbf{R}^2} e^{w_1} < +\infty, \quad \int_{\mathbf{R}^2} e^{w_2} < +\infty. \tag{27}$$

2. (Liouville equation in $\mathbf{R}^2$)
$$-\Delta w = e^w \qquad \text{in } \mathbf{R}^2, \qquad \int_{\mathbf{R}^2} e^w < +\infty \tag{28}$$

3. (singular Liouville equation in $\mathbf{R}^2$)
$$-\Delta w = e^w - \sum_{x_0 \in S} m(x_0)\delta_{x_0}, \qquad \int_{\mathbf{R}^2} e^w < +\infty, \tag{29}$$
where $S \subset \mathbf{R}^2$ is a finite set and $m(x_0) \geq 2\pi$ for any $x_0 \in S$.

For these problems we have [12, 8, 9];

Lemma 8. *We have the following.*

1. *For the solution* (w_1, w_2) *to (27) we have*
$$2\alpha_1 - \alpha_2 > 4\pi, \quad -\alpha_1 + 2\alpha_2 > 4\pi, \quad \alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2 = 4\pi(\alpha_1 + \alpha_2),$$
where
$$\alpha_1 = \int_{\mathbf{R}^2} e^{w_1}, \qquad \alpha_2 = \int_{\mathbf{R}^2} e^{w_2},$$
and in particular, $\min(\alpha_1, \alpha_2) > 4(1 + \frac{1}{\sqrt{3}})\pi = 6.309\ldots \times \pi$.

2. For the solution w to (28) we have $\int_{\mathbf{R}^2} e^w = 8\pi$.

3. For the solution w to (29) we have $\int_{\mathbf{R}^2} e^w > 4\pi + \sum_{x_0 \in S} m(x_0)$.

In the first case of the above lemma, [13] asserted $\alpha_1 = \alpha_2 = 8\pi$, although we have not been able to justify it. On the other hand, we expect $\int_{\mathbf{R}^2} e^w = 8\pi + 2\sum_{x_0 \in S} m(x_0)$ in the third case. Now, we show the following.

Lemma 9. *We have (21) for each $x_0 \in S_1 \cap S_2$.*

Proof: We have $S_1^0 = S_2^0 = \{0\}$, and there are $x_{1,n}^1 \to 0$ and $x_{2,n}^1 \to 0$ such that

$$w_{1,n}(x_{1,n}^1) = \sup_\Omega w_{1,n} \to +\infty \qquad \text{and} \qquad w_{2,n}(x_{2,n}^1) = \sup_\Omega w_{2,n} \to +\infty.$$

We take the rescaling of $w_{i,n}$ around $x_{k,n}^1$ by

$$w_{i,n}^{1,k}(x) = w_{i,n}(x_{k,n}^1 + \varepsilon_{k,n}^1 x) - w_{k,n}(x_{k,n}^1),$$

where $i, k = 1, 2$ and $\varepsilon_{k,n}^1 = e^{-w_{k,n}(x_{k,n}^1)/2}$. Then, it holds that

$$\begin{aligned} -\Delta w_{1,n}^{1,k} &= 2V_{1,n}(x_{k,n}^1 + \varepsilon_{k,n}^1 x)e^{w_{1,n}^{1,k}} - V_{2,n}(x_{k,n}^1 + \varepsilon_{k,n}^1 x)e^{w_{2,n}^{1,k}} \\ -\Delta w_{2,n}^{1,k} &= -V_{1,n}(x_{k,n}^1 + \varepsilon_{k,n}^1 x)e^{w_{1,n}^{1,k}} + 2V_{2,n}(x_{k,n}^1 + \varepsilon_{k,n}^1 x)e^{w_{2,n}^{1,k}} \end{aligned}$$

in $\Omega_n^{1,k} = \left\{ x \in \mathbf{R}^2 \mid \frac{x - x_{k,n}^1}{\varepsilon_{k,n}^1} \in \Omega \right\}$ with $\int_{\Omega_{i,n}^{1,k}} e^{w_{i,n}^{1,k}} = \int_\Omega e^{w_{i,n}} \leq b$. Without loss of generality, we may suppose

$$\varepsilon_{1,n}^1 \leq \varepsilon_{2,n}^1$$

for $n = 1, 2, \cdots$, i.e., $w_{1,n}(x_{1,n}^1) \geq w_{2,n}(x_{2,n}^1)$. Then, we take the rescaled solution around $x_{1,n}^1$, i.e., $(w_{1,n}^{1,1}, w_{2,n}^{1,1})$. Since

$$\begin{aligned} &w_{1,n}^{1,1}(x) \leq w_{1,n}^{1,1}(0) = 0 \\ &w_{2,n}^{1,1}(x) \leq w_{2,n}^{1,1}\left(\frac{x_{2,n}^1 - x_{1,n}^1}{\varepsilon_{1,n}^1}\right) \leq w_{2,n}(x_{2,n}^1) - w_{1,n}(x_{1,1}^1) \leq 0 \end{aligned}$$

holds on $\Omega_n^{1,1}$, Lemma 7 assures the following alternatives:

1. Both $\left\{w_{1,n}^{1,1}\right\}$ and $\left\{w_{2,n}^{1,1}\right\}$ are locally uniformly bounded in $\mathbf{R}^2$.

2. $\left\{w_{1,n}^{1,1}\right\}$ is locally uniformly bounded in $\mathbf{R}^2$, while $w_{2,n}^{1,1} \to -\infty$ locally uniformly in $\mathbf{R}^2$.

From the elliptic estimate, we may assume $w_{i,n}^{1,1} \to w_i^{1,1}$ in $C_{loc}^{1,\alpha}(\mathbf{R}^2)$ with $w_1^{1,1}, w_2^{1,1} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$ in the first alternative, and these $w_i^{1,1}$ satisfy

$$
\begin{aligned}
-\Delta w_1^{1,1} &= 2V_1(0)e^{w_1^{1,1}} - V_2(0)e^{w_2^{1,1}} \\
-\Delta w_2^{1,1} &= -V_1(0)e^{w_1^{1,1}} + 2V_2(0)e^{w_2^{1,1}}
\end{aligned}
$$

in $\mathbf{R}^2$ with $\int_{\mathbf{R}^2} e^{w_1^{1,1}} < +\infty$ and $\int_{\mathbf{R}^2} e^{w_2^{1,1}} < +\infty$, where $0 < \alpha < 1$. Given $R > 0$, we have $r_n \to +\infty$ satisfying $\limsup r_n \varepsilon_{1,n}^1 < R$, and in this case it follows that

$$
\int_{B_R(0)} V_{i,n}e^{w_{i,n}} \geq \int_{B_{r_n\varepsilon_{1,n}^1}(x_{1,n}^1)} V_{i,n}e^{w_{i,n}} = \int_{B_{r_n}(0)} V_{i,n}(x_{1,n}^1 + \varepsilon_{1,n}^1 x)e^{w_{i,n}^{1,1}}
$$

for large n. Making $n \to +\infty$ and then $R \downarrow 0$, we have

$$
m_i = \lim_{R\downarrow 0}\lim_{n\to\infty} \int_{B_R(0)} V_{i,n}e^{w_{i,n}} \geq \int_{\mathbf{R}^2} V_i(0)e^{w_i^{1,1}}.
$$

Using $V_i(0) > 0$, we have $\min(m_1, m_2) > 4(1 + \frac{1}{\sqrt{3}})\pi$ by the first case of Lemma 8, and the proof for this alternative is done. (If we apply [13] and (26), then we obtain $(m_1, m_2) = (8\pi, 8\pi)$ in this alternative.)

Therefore, henceforth, we consider the second alternative concerning this rescaling around $x_{1,n}^1$. Even in this case, we have a subsequence (denoted by the same symbol) such that $w_{1,n}^{1,1} \to w_1^{1,1}$ in $C_{loc}^{1,\alpha}(\mathbf{R}^2)$ and this $w_1^{1,1}$ satisfies

$$
-\Delta w_1^{1,1} = 2V_1(0)e^{w_1^{1,1}}, \qquad \int_{\mathbf{R}^2} e^{w_1^{1,1}} < +\infty.
$$

Therefore, from the second case of Lemma 8 we have $m_1 \geq \int_{\mathbf{R}^2} V_1(0)e^{w_1^{1,1}} = 4\pi$. Henceforth, we put $w_2^{1,1} = -\infty$ for simplicity, and thereforfe, this alternative is referred to as $w_1^{1,1} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$ and $w_2^{1,1} = -\infty$. Furthermore, we have $(m_1, m_2) \geq (4\pi, 2\pi)$, namely, $m_1 \geq 4\pi$ and $m_2 \geq 2\pi$.

Now, we use the rescaled solution $(w_{1,n}^{1,2}, w_{2,n}^{1,2})$ around $x_{2,n}^1$. In this case, we have

$$
\begin{aligned}
w_{2,n}^{1,2}(x) &\leq w_{2,n}^{1,2}(0) = 0 \\
w_{1,n}^{1,2}(x) &\leq w_{1,n}^{1,2}\left(\frac{x_{1,n}^1 - x_{2,n}^1}{\varepsilon_{2,n}^1}\right) = w_{1,n}(x_{1,n}^1) - w_{2,n}(x_{2,n}^1) \qquad (30)
\end{aligned}
$$

in $\Omega_n^{1,2}$. In spite of $w_{1,n}(x_{1,n}^1) - w_{2,n}(x_{2,n}^1) \geq 0$, again by Lemma 7 we have the following alternatives.

1. Both $\left\{w_{1,n}^{1,2}\right\}$ and $\left\{w_{2,n}^{1,2}\right\}$ are locally uniformly bounded in $\mathbf{R}^2$.

2. $\left\{w_{2,n}^{1,2}\right\}$ is locally uniformly bounded, while $w_{1,n}^{1,2} \to -\infty$ locally uniformly in $\mathbf{R}^2$.

3. There is a finite blowup set $\mathcal{S}_1^{1,2}$ of $\left\{w_{1,n}^{1,2}\right\}$ such that $m_1^{1,2}(x_0) \geq 2\pi$ for any $x_0 \in \mathcal{S}_1^{1,2}$ and $\left\{w_{2,n}^{1,2}\right\}$ is locally uniformly bounded in $\mathbf{R}^2 \setminus \mathcal{S}_1^{1,2}$, $w_{1,n}^{1,2} \to -\infty$ locally uniformly in $\mathbf{R}^2 \setminus \mathcal{S}_1^{1,2}$, and $V_{1,n}(x_{2,n}^1 + \varepsilon_{2,n}^1 x)e^{w_{1,n}^{1,2}} \to \sum_{x_0 \in \mathcal{S}_1^{1,2}} m_1^{1,2}(x_0)\delta_{x_0}$ in $\mathcal{M}(\mathbf{R}^2)$.

4. There is a finite blowup set $\mathcal{S}_1^{1,2}$ of $\left\{w_{1,n}^{1,2}\right\}$ such that $m_1^{1,2}(x_0) \geq 2\pi$ for any $x_0 \in \mathcal{S}_1^{1,2}$ and $w_{2,n}^{1,2}, w_{1,n}^{1,2} \to -\infty$ locally uniformly in $\mathbf{R}^2 \setminus \mathcal{S}_1^{1,2}$, and $V_{1,n}(x_{2,n}^1 + \varepsilon_{2,n}^1 x)e^{w_{1,n}^{1,2}} \to \sum_{x_0 \in \mathcal{S}_1^{1,2}} m_1^{1,2}(x_0)\delta_{x_0}$ in $\mathcal{M}(\mathbf{R}^2)$.

The first alternative may be referred to as $w_1^{1,2}, w_2^{1,2} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$, with the limit $(w_1^{1,2}, w_2^{1,2})$ satisfying the Toda system on $\mathbf{R}^2$. We shall show that this is impossible in case $w_2^{1,1} = -\infty$, the second alternative of the rescaling around $x_{1,n}^1$ that we are considering. For this purpose, first we assume

$$\limsup \frac{\left|x_{1,n}^1 - x_{2,n}^1\right|}{\varepsilon_{2,n}^1} = +\infty.$$

Then, given $R > 0$, we have $r_n \to +\infty$ such that

$$r_n \leq \frac{1}{3} \cdot \frac{\left|x_{1,n}^1 - x_{2,n}^1\right|}{\varepsilon_{2,n}^1} \qquad \text{and} \qquad \limsup r_n \varepsilon_{2,n}^1 < R,$$

passing to a subsequence. Since $\varepsilon_{1,n}^1 \leq \varepsilon_{2,n}^1$, we have

$$\begin{aligned} \int_{B_R(0)} V_{i,n} e^{w_{i,n}} &\geq \int_{B_{r_n \varepsilon_{2,n}^1}(x_{1,n}^1)} V_{i,n} e^{w_{i,n}} + \int_{B_{r_n \varepsilon_{2,n}^1}(x_{2,n}^1)} V_{i,n} e^{w_{i,n}} \\ &= \int_{B_{r_n}(0)} V_{i,n}(x_{1,n}^1 + \varepsilon_{1,n}^1 x) e^{w_{i,n}^{1,1}} + \int_{B_{r_n}(0)} V_{i,n}(x_{2,n}^1 + \varepsilon_{2,n}^1 x) e^{w_{i,n}^{1,2}} \end{aligned} \tag{31}$$

and therefore,

$$\int_{B_R(0)} V_{i,n} e^{w_{i,n}} \geq \int_{\mathbf{R}^2} V_i(0) e^{w_i^{1,1}} + \int_{\mathbf{R}^2} V_i(0) e^{w_i^{1,2}}.$$

Making $R \downarrow 0$, we obtain

$$m_i \geq \int_{\mathbf{R}^2} V_i(0) e^{w_i^{1,1}} + \int_{\mathbf{R}^2} V_i(0) e^{w_i^{1,2}}$$

for $i = 1, 2$, and therefore,

$$(m_1, m_2) \geq (4\pi, 0) + \left(4(1 + \frac{1}{\sqrt{3}})\pi, 4(1 + \frac{1}{\sqrt{3}})\pi\right),$$

which is impossible by (26).

Now, we proceed to the other case,

$$\limsup \frac{|x_{1,n}^1 - x_{2,n}^1|}{\varepsilon_{2,n}^1} < +\infty.$$

Then,

$$\limsup \left\{ w_{1,n}(x_{1,n}^1) - w_{2,n}(x_{2,n}^1) \right\} = \limsup \left\{ -2\log \varepsilon_{1,n}^1 + 2\log \varepsilon_{2,n}^1 \right\} < +\infty$$

holds by (30), because $\left\{ w_{1,n}^{1,2} \right\}$ is locally uniformly bounded in $\mathbf{R}^2$. Passing to a subsequence, we have

$$\frac{\varepsilon_{2,n}^1}{\varepsilon_{1,n}^1} \to C \geq 1, \tag{32}$$

and this implies $w_i^{1,2}(x) = w_i^{1,1}(Cx) + 2\log C$, a contradiction to $w_2^{1,1} = -\infty$ and $w_2^{1,2} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$. Thus, we observe that the first alternative of the rescaling around $x_{2,n}^1$ is impossible.

The second alternative is indicated by $w_2^{1,2} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$ and $w_1^{1,2} = -\infty$. The former function satisfies the Liouville equation on $\mathbf{R}^2$, and this implies $m_2 \geq 4\pi$. On the other hand, we have already $m_1 \geq 4\pi$ from the former rescaling, that is, $w_1^{1,1} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$ and $w_2^{1,1} = -\infty$. Therefore, it holds that $(m_1, m_2) \geq (4\pi, 4\pi)$.

In the third alternative, passing to a subsequence, we have $w_{2,n}^{1,2} \to w_2^{1,2}$ in $C_{loc}^{1,\alpha}(\mathbf{R}^2 \setminus \mathcal{S}_1^{1,2})$ and weakly in $W_{loc}^{1,q}(\mathbf{R}^2)$ for every $q \in [1,2)$ with $w_2^{1,2}$ satisfying

$$-\Delta w_2^{1,2} = -\sum_{x_0 \in \mathcal{S}_1^{1,2}} m_1^{1,2}(x_0)\delta_{x_0} + 2V_2(0)e^{w_2^{1,2}} \qquad \text{in } \mathbf{R}^2$$

$$\int_{\mathbf{R}^2} e^{w_2^{1,2}} < +\infty,$$

where $m_1^{1,2}(x_0) \geq 2\pi$ for each $x_0 \in \mathcal{S}_1^{1,2}$. In particular, it holds that

$$\int_{\mathbf{R}^2} V_2(0)e^{w_2^{1,2}} > 2\pi + \frac{1}{2}\sum_{x_0 \in \mathcal{S}_1^{1,2}} m_1^{1,2}(x_0)$$

by the third case of Lemma 8, and therefore,

$$m_1 \geq 4\pi, \qquad m_2 > 2\pi + \frac{1}{2}\sum_{x_0 \in \mathcal{S}_1^{1,2}} m_1^{1,2}(x_0).$$

First, we consider the case

$$\limsup \frac{|x_{1,n}^1 - x_{2,n}^1|}{\varepsilon_{2,n}^1} = +\infty. \tag{33}$$

Since $\mathcal{S}_1^{1,2} \neq \emptyset$, we have $x_{1,n}^2 \in \Omega$ such that

$$\limsup \frac{\left|x_{1,n}^2 - x_{2,n}^1\right|}{\varepsilon_{2,n}^1} < +\infty \tag{34}$$

$$w_{1,n}^{1,2}\left(\frac{x_{1,n}^2 - x_{2,n}^1}{\varepsilon_{2,n}^1}\right) = w_{1,n}(x_{1,n}^2) - w_{2,n}(x_{2,n}^1) \to +\infty. \tag{35}$$

The second relation implies $w_{1,n}(x_{1,n}^2) \to +\infty$ by $w_{2,n}(x_{2,n}^1) \to +\infty$, and we can consider the second rescaling around $x_{1,n}^2$;

$$w_{i,n}^{2,1}(x) = w_{i,n}(x_{1,n}^2 + \varepsilon_{1,n}^2 x) - w_{1,n}(x_{1,n}^2),$$

where $\varepsilon_{1,n}^2 = e^{-w_{1,n}(x_{1,n}^2)/2} \to 0$. We have

$$w_{1,n}^{2,1}(x) \leq w_{1,n}^{2,1}(0)$$

$$w_{2,n}^{2,1}(x) \leq w_{2,n}^{2,1}\left(\frac{x_{2,n}^1 - x_{1,n}^2}{\varepsilon_{1,n}^2}\right) = w_{2,n}(x_{2,n}^1) - w_{1,n}(x_{1,n}^2) \to -\infty$$

in $\Omega_n^{2,1} = \left\{x \in \mathbf{R}^2 \mid \frac{x - x_{1,n}^2}{\varepsilon_{1,n}^2} \in \Omega\right\}$, and therefore, Lemma 7 guarantees that $\left\{w_{1,n}^{2,1}\right\}$ is locally uniformly bounded in $\mathbf{R}^2$. Of course we have $w_{2,n}^{2,1} \to -\infty$ locally uniformly in $\mathbf{R}^2$, and this case may be referred to as $w_1^{2,1} \in C_{loc}^{1,\alpha}(\mathbf{R}^2)$ and $w_2^{2,1} = -\infty$, where $w_1^{2,1}$ satisfyies the Liouville equation in $\mathbf{R}^2$. The relation (35) implies $\varepsilon_{1,n}^2 \leq \varepsilon_{2,n}^1$ for large n, and therefore, (33) and (34) imply

$$\frac{\left|x_{1,n}^1 - x_{1,n}^2\right|}{\varepsilon_{1,n}^2} \geq \frac{\left|x_{1,n}^1 - x_{2,n}^1\right| - \left|x_{2,n}^1 - x_{1,n}^2\right|}{\varepsilon_{2,n}^1} \to +\infty.$$

From this condition, we can argue similarly to the first alternative in the previous rescaling around $x_{1,n}^1$, that is, (31). The concentrations around $x_{1,n}^1$ and $x_{1,n}^2$ are separated, and we obtain

$$m_1 \geq 4\pi + 4\pi = 8\pi. \tag{36}$$

We may suppose $\lim \frac{x_{1,n}^2 - x_{2,n}^1}{\varepsilon_{2,n}^1} = X_1^2 \in \mathcal{S}_1^{1,2}$ by (34) and (35). Since (35) guarantees $\lim \frac{\varepsilon_{1,n}^2}{\varepsilon_{2,n}^1} = 0$, given $R > 0$, we have $r_n \to +\infty$ such that

$$\limsup r_n \frac{\varepsilon_{1,n}^2}{\varepsilon_{2,n}^1} < R.$$

Therefore, we have

$$\begin{aligned}
&\int_{B(X_1^2,R)} V_{1,n}(x_{2,n}^1+\varepsilon_{2,n}^1 x)e^{w_{1,n}^{1,2}(x)} \\
&\geq \int_{B\left(\frac{x_{1,n}^2-x_{2,n}^1}{\varepsilon_{2,n}^1}, r_n\frac{\varepsilon_{1,n}^2}{\varepsilon_{2,n}^1}\right)} V_{1,n}(x_{2,n}^1+\varepsilon_{2,n}^1 x)e^{w_{1,n}(x_{2,n}^1+\varepsilon_{2,n}^1 x)}(\varepsilon_{2,n}^1)^2 dx \\
&= \int_{B(0,r_n\frac{\varepsilon_{1,n}^2}{\varepsilon_{2,n}^1})} V_{1,n}(x_{1,n}^2+\varepsilon_{2,n}^1 x)e^{w_{1,n}(x_{1,n}^2+\varepsilon_{2,n}^1 x)}(\varepsilon_{2,n}^1)^2 dx \\
&= \int_{B_{r_n}(0)} V_{1,n}(x_{1,n}^2+\varepsilon_{1,n}^2 x)e^{w_{1,n}^{2,1}(x)} dx
\end{aligned}$$

for large n. Making $n \to +\infty$ and $R \downarrow 0$, we obtain

$$m_1^{1,2}(X_1^2) \geq \int_{\mathbf{R}^2} V_1(0)e^{w_1^{2,1}} = 4\pi,$$

and therefore, it follows that

$$m_2 > 2\pi + \frac{1}{2}m_1^{1,2}(X_1^2) \geq 4\pi.$$

If (33) is not the case, we have $\frac{x_{1,n}^1-x_{2,n}^1}{\varepsilon_{2,n}^1} \to X_1^1$, passing to a subsequence. In fact, we have

$$w_{1,n}^{1,2}(x) \leq w_{1,n}^{1,2}\left(\frac{x_{1,n}^1-x_{2,n}^1}{\varepsilon_{2,n}^1}\right) = w_{1,n}(x_{1,n}^1) - w_{2,n}(x_{2,n}^1)$$

in $\Omega_n^{1,2}$, and the right-hand side is not bonded by $\mathcal{S}_1^{1,2} \neq \emptyset$. Thus, we may assume

$$w_{1,n}(x_{1,n}^1) - w_{2,n}(x_{2,n}^1) \to +\infty,$$

which implies $X_1^1 \in \mathcal{S}_1^{1,2}$ and $\frac{\varepsilon_{1,n}^1}{\varepsilon_{2,n}^1} \to 0$. Then, similarly to the case of (33), we obtain

$$m_1^{1,2}(X_1^1) \geq \int_{\mathbf{R}^2} V_1(0)e^{w_1^{1,1}} \geq 4\pi,$$

which guarantees

$$m_2 > 2\pi + \frac{1}{2}m_1^{1,2}(X_1^1) \geq 4\pi.$$

In particular, we have $(m_1, m_2) \geq (4\pi, 4\pi)$ in this alternative.

Finally, the fourth alternative does not occur. In fact, we have $w_{2,n}^{1,2}(0) = 0$, and therefore, $0 \in \mathcal{S}_1^{1,2}$. We can choose $R > 0$ satisfying $\overline{B_R(0)} \cap \mathcal{S}_1^{1,2} = \{0\}$, and define $h_{i,n}$ $(i = 1, 2)$ by

$$\begin{aligned} -\Delta h_{i,n} &= V_{i,n}(x_{2,n}^1 + \varepsilon_{2,n}^1 x)e^{w_{i,n}^{1,2}} \qquad \text{in } B_R(0) \\ h_{i,n} &= 0 \qquad \text{on } \partial B_R(0). \end{aligned}$$

Then,

$$h_{0,n} = w_{2,n}^{1,2} - (2h_{2,n} - h_{1,n})$$

is a harmonic function satsifying

$$\sup_{B_R(0)} h_{0,n} \leq \sup_{\partial B_R(0)} h_{0,n} \longrightarrow -\infty.$$

On the other hand, we have $0 \leq e^{w_{2,n}^{1,2}(x)} \leq e^0 = 1$ and $e^{w_{2,n}^{1,2}(x)} \longrightarrow 0$ locally uniformly in $\mathbf{R}^2 \setminus \mathcal{S}_1^{1,2}$, and therefore, $e^{w_{2,n}^{1,2}(x)} \longrightarrow 0$ in $L^p_{loc}(\mathbf{R}^2)$ for every $p \in [1, \infty)$. This implies

$$h_{2,n} \longrightarrow 0 \qquad \text{in } C^{1,\alpha}(B_R(0)),$$

while $h_{1,n}$ is a non-negative function. Thus, we obtain

$$\begin{aligned} 0 = w_{2,n}^{1,2}(0) &= h_{0,n}(0) + 2h_{2,n}(0) - h_{1,n}(0) \leq h_{0,n}(0) + 2h_{2,n}(0) \\ &\leq \sup_{B_R(0)} h_{0,n} + 2\,\|h_{2,n}\|_{L^\infty(B_R(0))} \longrightarrow -\infty, \end{aligned}$$

a contradiction, and the proof is complete.

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