Interior gradient estimate for curvature flow

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Abstract

Our purpose is to understand the anisotropic curvature flow. Especially we like to prove the interior gradient estimate. We establish the interior gradient estimate for general 1-D anisotropic curvature flow. The estimate depends only on the height of the graph and not on the gradient at initial time.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. A surface given as a graph $u : \Omega \to \mathbb{R}$ is a minimal surface when $u$ satisfies

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. \tag{1.1}$$

For this equation, the following interior gradient estimates are well-known ([5, 6, 7]): Given a constant $M$ and $\bar{\Omega} \subset \subset \Omega$, there exists a constant $C$ depending only on $M$ and $\bar{\Omega}$ such that if $\sup_{\bar{\Omega}} |u| \leq M$, then $\sup_{\bar{\Omega}} |\nabla u| \leq C$. The similar estimates are also known for the mean curvature flow equation ([3]). That is, if $u : \Omega \times (0, T) \to \mathbb{R}$ satisfies

$$\frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \tag{1.2}$$
and $\sup_{\Omega \times [0,T]} |u| \leq M$, $\tilde{\Omega} \subset \subset \Omega$, $0 < T_0 < T$, then there exists $C$ such that $\sup_{\tilde{\Omega} \times [T_0, T]} |\nabla u| \leq C$. Again, $C$ is a constant depending only on $M$, $\tilde{\Omega}$ and $T_0$. Note that $C$ is independent of the gradient at $t = 0$.

One direction to extend those results are to consider general anisotropic curvature problem, namely, to consider the variational problem corresponding to the energy functional

$$F(u) = \int_{\Omega} a(\nu) \sqrt{1 + |\nabla u|^2},$$

where $\nu = (\nabla u, -1)/\sqrt{1 + |\nabla u|^2}$ is the unit normal vector to the graph of $u$ and the function $a : \mathbb{R}^{n+1} \to \mathbb{R}^+$ is the surface energy density and should satisfy certain convexity property. The Euler-Lagrange equation is

(1.3) $\text{div}_x a_p(\nu) = 0$,

and the curvature flow equation is

(1.4) $\frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div}_x a_p(\nu)$.

The left-hand side of the equation (1.4) corresponds to the normal velocity of the curve $(x, u(x, \cdot))$ while the right-hand side is the weighted anisotropic curvature. This is a gradient flow of the anisotropic surface energy functional

$$\int_{\Omega} a(\nu) ds,$$

where $ds = \sqrt{1 + |\nabla u|^2} dx$ and $\nu = (-\nabla u, 1)/\sqrt{1 + |\nabla u|^2}$ with homogeneous Dirichlet ($u = 0$) or Neumann ($a_p(-\nabla u, 1) = 0$) boundary conditions, since

$$\frac{d}{dt} \int_{\Omega} a(\nu) ds = \int_{\Omega} a_p(-\nabla u, 1) \cdot \nabla u_t \, dx = -\int_{\Omega} |\text{div}_x a_p(-\nabla u, 1)|^2 \, ds.$$

We show the interior gradient estimates for general anisotropic curvature flow for one-dimensional case which is independent of the initial time gradient.
2 Main Theorem

Let \( r > 0 \) be given. The graph \( u : [-r, r] \times [0, T] \to \mathbb{R} \) is said to be an anisotropic curvature flow if smooth function \( u \) satisfies

\[
\frac{u_t}{\sqrt{1 + u_x^2}} = (a_p(u_x, -1))_x.
\]

where \( a : \mathbb{R}^2 \to [0, \infty) \) is an anisotropic surface energy density function satisfying the following assumptions:

(a) \( a(tp, tq) = t a(p, q) \) for all \( t > 0 \),

(b) \( a \) is a convex function,

(c) there exists \( \delta_0 > 0 \) such that \( a(p, q) - \delta_0|p, q| \) is a convex function,

(d) \( a \) is smooth except at \((0, 0)\).

Under these assumptions, we show

**Theorem 1**

Suppose \( u \) is a smooth solution of (2.1) on \([-r, r] \times [0, T]\) satisfying

\[
\sup_{[-r, r] \times [0, T]} |u| \leq M.
\]

Given \( 0 < s < r \) and \( 0 < t_0 < T \), there exists a constant \( C > 0 \) depending only on \( \delta_0, M, t_0, s, r \) such that

\[
\sup_{[-(r-s), r-s] \times [t_0, T]} |u_x| \leq C.
\]

Note that the estimate is independent the gradient of the initial data. Also we point out that the dependence of \( C \) on \( a \) is only through the lower bound of the uniform convexity \( \delta_0 \), but not on the upper bound (such as \( C^1 \) bound). Thus, the result in this paper can be extended equally to the non-smooth anisotropic curvature flow problem [4] by approximations.

**Remark 1** For example, \( a(p, q) = (p^2 + q^2)^{\frac{1}{2}} \) is isotropic curvature flow (mean curvature flow) and satisfies above assumptions. \( a(p, q) = (|p|^r + |q|^r)^{\frac{1}{r}} \) \((1 < r < \infty)\) is anisotropic curvature flow and also satisfies assumptions.

**Remark 2** In general dimension, if we assume the axis symmetry of the graph of \( u \), we expect to prove the same interior gradient estimate.
3 Proof

We cite the following theorem due to Angenent [2] which says that the number of zeros of the solution of parabolic equations is nonincreasing as time increases.

**Lemma 1** (Angenent [2])
Suppose \( u \in C^\infty([x_1, x_2] \times [0, T]) \) satisfies the equation

\[
    u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u
\]
on \([x_1, x_2] \times [0, T]\) and

\[
    u(x_j, t) = 0 \quad \text{for} \quad t \in [0, T] \quad j = 1, 2.
\]

Here, \( a, b, c \) are smooth functions of \((x, t)\) and \( a > 0 \). Then for all \( t \in (0, T] \), the zero set of \( x \to u(x, t) \) will be finite, even when counted with multiplicity.

**The number of zeros of** \( x \to u(x, t) \) **counted with multiplicity is nonincreasing function of** \( t \).

**Proof of Theorem.** Given \( 0 < s < r \) and \( 0 < t_0 < T \), we construct a solution \( v \) for \((2.1)\) on \([-s, s] \times (0, T]\) with the following properties:

(a) \( v(-s, t) = -M - 1 \) and \( v(s, t) = M + 1 \) for \( 0 < t \leq T \),

(b) \( v_x > 0 \) on \([-s, s] \times (0, T]\),

(c) for any \( -s < x \leq s \), \( \lim_{t \to 0} v(x, t) > M \).

The property (c) means that \( v \) has an initial data which is vertical at \( x = -s \). We show that the function \( v \) has a gradient bound \( 0 < v_x \leq C \) on \([-s, s] \times [t_0, T]\), where \( C \) depends only on \( M, \delta_0, s, t_0 \). We show the existence of such \( v \) later in the proof. Assuming such \( v \) exists for now, we then prove that any solution with \( \sup_{[-r, r] \times [0, T]} |u| \leq M \) satisfies \( \sup_{[-(r-s), r-s] \times [t_0, T]} u_x \leq C \).

The same argument using \(-u\) will show \( \sup_{[-(r-s), r-s] \times [t_0, T]} |u_x| \leq C \). For a contradiction, assume that there exists a point \((\bar{x}, \bar{t}) \in [-\r, \r] \times [t_0, T]\) with \( u_x(\bar{x}, \bar{t}) > C \). Since \( \sup |u| \leq M \) and by (a), we may choose \( \lambda \) so that \( |\bar{x} - \lambda| < s \) and \( v(\bar{x} - \lambda, \bar{t}) = u(\bar{x}, \bar{t}) \). With this \( \lambda \), define \( v_\lambda(x, t) = v(x - \lambda, t) \). Since \( u_x(\bar{x}, \bar{t}) > C \geq (v_\lambda)_x(\bar{x}, \bar{t}) \) and \( v_\lambda(\lambda + s, \bar{t}) = v(s, \bar{t}) = M + 1 > u(\lambda + s, \bar{t}) \), there has to be at least another point \( \bar{x} < \tilde{x} < \lambda + s \) such that \( u(\tilde{x}, \bar{t}) = v_\lambda(\tilde{x}, \bar{t}) \). Thus \( u - v_\lambda \) has at least two zeros at \( t = \bar{t} \) on
\( \lambda - s < x < \lambda + s \). Function \( u - v_\lambda \) satisfies the equation of the type (3.1) on \([\lambda - s, \lambda + s] \times (0, T] \), with non-zero boundary values for all \( t > 0 \) due to \( \sup |u| \leq M \) and (a). Thus we may use Lemma 1 and conclude that \( u - v_\lambda \) has at least two zeros in \( x \) variable for all \( \tilde{t} > t > 0 \). Since \( v_\lambda > M \) for \( x \) away from \( \lambda - s \) and all small \( t \), and since we assume that \( u \) is a smooth function up to \( t = 0 \), this is impossible to satisfy for all small enough \( t \). (See fig. 3 and 4.)

Thus it remains to prove the existence of such \( v \). To do this, we invert the role of independent variable \( x \) and dependent variable \( y = v(x,t) \). Let \( y = w(x,t) \) be the inverse function of \( v \) with respect to the space variables, i.e., \( w \) satisfies \( y = v(w(y,t), t) \) identically. Since the equation is geometric, \( w \) should satisfy the similar equation to (2.1) on \([-M-1, M+1] \times (0,T] \) with the role of \( y \) and \( x \) exchanged. Now, the conditions on \( v \) in terms of \( w \) are

\[
(a') \ w(-M-1,t) = -s \quad \text{and} \quad w(M+1,t) = s \quad \text{for} \ 0 < t \leq T,
\]

\[
(b') \ w_x > 0 \quad \text{on} \ [-M-1,M+1] \times (0,T],
\]

\[
(c') \ \text{for any} \ -M-1 \leq x \leq M, \ \lim_{t \to 0} w(x,t) = -s.
\]

Furthermore, on \([-M-1,M+1] \times (0,T], \ w \) should satisfy

\[
(3.2) \quad \frac{w_t}{\sqrt{1+w_x^2}} = (a_q(1,w_x))_x.
\]

Since \( \frac{\partial w}{\partial x} = 1/\frac{\partial x}{\partial y} \), we need to show that there exists a constant \( C > 0 \) such that \( w_x > C \) on \([-M,M] \times [t_0,T] \). We solve (3.2) with the following convex initial data. Let \( \Gamma \in C^\infty([-M-1,M+1]) \) (See fig. 2 and 4.) be

- \( \Gamma(x) = -s \) for \( x \in [-M-1,M] \),
- \( \Gamma(M+1) = s, \ \Gamma''(M+1) = 0, \)
- \( \Gamma(x) \geq -s, \ \Gamma'(x) \leq 3s, \ \Gamma''(x) \geq 0 \) for \( x \in [M,M+1] \).

Let \( w \) be the unique smooth solution of (3.2) with the initial data \( \Gamma \) and the boundary data \((a')\). Since any functions \( c_1 + c_2 x \) are solutions of (3.2), one obtains the gradient estimate

\[
(3.3) \quad 0 \leq w_x \leq 3s
\]
on $[-M - 1, M + 1] \times [0, T]$, by using these functions as barriers and the standard maximum principle applied to $w_x$. Also, note that the convexity of $w$ is preserved, i.e., $w_{xx} \geq 0$. This is seen by differentiating the equation with respect to $t$ and then applying the maximum principle to $w_t$. $w_t = 0$ on the boundary and $w_t = a_{qq}w_{xx} \geq 0$ for $t = 0$ imply $w_t \geq 0$. The equation then yields $w_{xx} \geq 0$ on $[-M - 1, M + 1] \times [0, T]$.

Now, (3.3) implies that $a_{qq}(-1, w_x) \geq c(s, \delta_0)$ (call this $\delta > 0$ by assumption (c). We claim that the solution of

$$
\begin{cases}
    z_t = \delta z_{xx} & [-M - 1, M + 1] \times [0, T], \\
    z(\pm(M + 1), t) = \pm s & t \in [0, T], \\
    z(x, 0) = \Gamma(x) & x \in [-M - 1, M + 1]
\end{cases}
$$

satisfies $w \geq z$ on $[-M - 1, M + 1] \times [0, T]$. (See fig.2) This is because of the following combined with the standard maximum principle:

$$(w - z)_t = a_{qq}(-1, w_x)w_{xx} - \delta z_{xx} = a_{qq}(-1, w_x)(w - z)_{xx} + (a_{qq}(-1, w_x) - \delta)z_{xx} \geq \delta a_{qq}(-1, w_x)(w - z)_{xx}.$$ 

In the last line, we used $z_{xx} \geq 0$, which follows by the same reason for $w_{xx} \geq 0$ before, and $a_{qq}(-1, w_x) \geq \delta$. We next claim that for $t_0 \leq t$, there exists $c = c(t_0, s, \delta) > 0$ such that $z_x \geq c$ on $[-M - 1, M + 1] \times [t_0, T]$. $z_x$ satisfies again the heat equation with non-negative initial data and the homogeneous Neumann data, and thus by the strong maximum principle (or extending the solution to $\mathbb{R}$ by a suitable reflection argument and then using the representation formula with the heat kernel) we have such $c$. Since $w_{xx} \geq 0$, for $(x, t)$ with $t \geq t_0$, we have

$$w_x(x, t) \geq w_x(-M - 1, t) \geq z_x(-M - 1, t) \geq c$$

as the result. Note that we are using $w \geq z$ and $w = z$ on the boundary $x = -M - 1$. This completes the proof.
$t = 0$

$\chi = \gamma(y)$
$= W(y,0)$

$\chi = W(y,t),$ \text{ figure 1}

$y = u(x,0)$

$y = V(x,t)$
$= V(W(y,t),t)$

$\chi = Z(y,t)$

$y = U(x,0)$

$\chi = W(y,t),$ \text{ figure 2}

$\chi = \gamma(y)$
$= W(y,0)$

$y = u(x,0)$

$y = V(x,t)$
$= V(W(y,t),t)$

$\chi = Z(y,t)$

$y = U(x,0)$

$\chi = W(y,t),$ \text{ figure 3}

$y = V(x,t)$
$= V(W(y,t),t)$

$\chi = Z(y,t)$

$y = U(x,0)$

$\chi = W(y,t),$ \text{ figure 4}
References


