Title: Interior gradient estimate for curvature flow (Variational Problems and Related Topics)

Author(s): Nagase, Yuko; Tonegawa, Yoshihiro

Citation: 数理解析研究所講究録 (2004), 1405: 147-154

Issue Date: 2004-11

URL: http://hdl.handle.net/2433/26100

Type: Departmental Bulletin Paper

Textversion: publisher

Kyoto University
Interior gradient estimate for curvature flow

Yuko Nagase
Yoshihiro Tonegawa
Department of mathematics
Hokkaido university
Sapporo 060-0810, Japan

Abstract

Our purpose is to understand the anisotropic curvature flow. Especially we like to prove the interior gradient estimate. We establish the interior gradient estimate for general 1-D anisotropic curvature flow. The estimate depends only on the height of the graph and not on the gradient at initial time.

1 Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). A surface given as a graph \( u : \Omega \rightarrow \mathbb{R} \) is a minimal surface when \( u \) satisfies

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.
\]

For this equation, the following interior gradient estimates are well-known ([5, 6, 7]): Given a constant \( M \) and \( \tilde{\Omega} \subset \subset \Omega \), there exists a constant \( C \) depending only on \( M \) and \( \tilde{\Omega} \) such that if \( \sup_{\Omega} |u| \leq M \), then \( \sup_{\tilde{\Omega}} |\nabla u| \leq C \). The similar estimates are also known for the mean curvature flow equation ([3]). That is, if \( u : \Omega \times (0, T) \rightarrow \mathbb{R} \) satisfies

\[
\frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right),
\]
and \( \sup_{\Omega \times [0,T]} |u| \leq M, \tilde{\Omega} \subset \subset \Omega, 0 < T_0 < T \), then there exists \( C \) such that \( \sup_{\tilde{\Omega} \times [t_0,T]} |\nabla u| \leq C \). Again, \( C \) is a constant depending only on \( M, \tilde{\Omega} \) and \( T_0 \). Note that \( C \) is independent of the gradient at \( t = 0 \).

One direction to extend those results are to consider general anisotropic curvature problem, namely, to consider the variational problem corresponding to the energy functional \[
F(u) = \int_{\Omega} a(\nu) \sqrt{1 + |\nabla u|^2},
\]
where \( \nu = (\nabla u, -1)/\sqrt{1 + |\nabla u|^2} \) is the unit normal vector to the graph of \( u \) and the function \( a : \mathbb{R}^{n+1} \to \mathbb{R}^+ \) is the surface energy density and should satisfy certain convexity property. The Euler-Lagrange equation is

\[
(1.3) \quad \text{div}_x a_p(\nu) = 0,
\]
and the curvature flow equation is

\[
(1.4) \quad \frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div}_x a_p(\nu).
\]

The left-hand side of the equation (1.4) corresponds to the normal velocity of the curve \((x, u(x, \cdot))\) while the right-hand side is the weighted anisotropic curvature. This is a gradient flow of the anisotropic surface energy functional \[
\int_{\Omega} a(\nu) ds,
\]
where \( ds = \sqrt{1 + |\nabla u|^2} dx \) and \( \nu = (-\nabla u, 1)/\sqrt{1 + |\nabla u|^2} \) with homogeneous Dirichlet (\( u = 0 \)) or Neumann (\( a_p(-\nabla u, 1) = 0 \)) boundary conditions, since

\[
\frac{d}{dt} \int_{\Omega} a(\nu) ds = \int_{\Omega} a_p(-\nabla u, 1) \cdot \nabla u_t dx = - \int_{\Omega} |\text{div}_x a_p(-\nabla u, 1)|^2 ds.
\]

We show the interior gradient estimates for general anisotropic curvature flow for one-dimensional case which is independent of the initial time gradient.
2 Main Theorem

Let $r > 0$ be given. The graph $u : [-r, r] \times [0, T] \to \mathbb{R}$ is said to be an anisotropic curvature flow if smooth function $u$ satisfies

\begin{equation}
\frac{u_t}{\sqrt{1+u_x^2}} = (a_p(u_x, -1))_x.
\end{equation}

where $a : \mathbb{R}^2 \to [0, \infty)$ is an anisotropic surface energy density function satisfying the following assumptions:

(a) $a(tp, tq) = ta(p, q)$ for all $t > 0$,

(b) $a$ is a convex function,

(c) there exists $\delta_0 > 0$ such that $a(p, q) - \delta_0 |(p, q)|$ is a convex function,

(d) $a$ is smooth except at $(0, 0)$.

Under these assumptions, we show

**Theorem 1**

Suppose $u$ is a smooth solution of (2.1) on $[-r, r] \times [0, T]$ satisfying

\[ \sup_{[-r, r] \times [0, T]} |u| \leq M. \]

Given $0 < s < r$ and $0 < t_0 < T$, there exists a constant $C > 0$ depending only on $\delta_0, M, t_0, s, r$ such that

\[ \sup_{[-(r-s), r-s] \times [t_0, T]} |u_x| \leq C. \]

Note that the estimate is independent the gradient of the initial data. Also we point out that the dependence of $C$ on $a$ is only through the lower bound of the uniform convexity $\delta_0$, but not on the upper bound (such as $C^1$ bound). Thus, the result in this paper can be extended equally to the non-smooth anisotropic curvature flow problem [4] by approximations.

**Remark 1** For example, $a(p, q) = (p^2 + q^2)^{1/2}$ is isotropic curvature flow (mean curvature flow) and satisfies above assumptions. $a(p, q) = (|p|^r + |q|^r)^{1/2}$ ($1 < r < \infty$) is anisotropic curvature flow and also satisfies assumptions.

**Remark 2** In general dimension, if we assume the axis symmetry of the graph of $u$, we expect to prove the same interior gradient estimate.
3 Proof

We cite the following theorem due to Angenent [2] which says that the number of zeros of the solution of parabolic equations is nonincreasing as time increases.

Lemma 1 (Angenent [2])
Suppose \( u \in C^\infty([x_1, x_2] \times [0, T]) \) satisfies the equation

\[
(3.1) \quad u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u
\]
on \([x_1, x_2] \times [0, T]\) and

\[
u(x_j, t) = 0 \text{ for } t \in [0, T] \quad j = 1, 2.
\]

Here, \( a, b, c \) are smooth functions of \((x, t)\) and \( a > 0\). Then for all \( t \in (0, T)\), the zero set of \( x \to u(x,t) \) will be finite, even when counted with multiplicity. The number of zeros of \( x \to u(x,t) \) counted with multiplicity is nonincreasing function of \( t \).

Proof of Theorem. Given \( 0 < s < r \) and \( 0 < t_0 < T \), we construct a solution \( v \) for (2.1) on \([-s, s] \times (0, T]\) with the following properties:

(a) \( v(-s, t) = -M - 1 \) and \( v(s, t) = M + 1 \) for \( 0 < t \leq T \),

(b) \( v_x > 0 \) on \([-s, s] \times (0, T]\),

(c) for any \(-s < x \leq s\), \( \lim_{t \to 0} v(x, t) > M \).

The property (c) means that \( v \) has an initial data which is vertical at \( x = -s \). We show that the function \( v \) has a gradient bound \( 0 < v_x \leq C \) on \([-s, s] \times [t_0, T]\), where \( C \) depends only on \( M, \delta_0, s, t_0 \). We show the existence of such \( v \) later in the proof. Assuming such \( v \) exists for now, we then prove that any solution with \( \sup_{[-(r-s), r-s] \times [0, T]} |u| \leq M \) satisfies \( \sup_{[-(r-s), r-s] \times [t_0, T]} u_x \leq C \). The same argument using \(-u\) will show \( \sup_{[-(r-s), r-s] \times [t_0, T]} |u_x| \leq C \). For a contradiction, assume that there exists a point \((\bar{x}, \bar{t})\) in \([-r-s, r-s] \times [t_0, T]\) with \( u_x(\bar{x}, \bar{t}) > C \). Since \( \sup |u| \leq M \) and by (a), we may choose \( \lambda \) so that \( |\bar{x} - \lambda| < s \) and \( v(\bar{x} - \lambda, \bar{t}) = u(\bar{x}, \bar{t}) \). With this \( \lambda \), define \( v_\lambda(x, t) = v(x - \lambda, t) \). Since \( u_x(\bar{x}, \bar{t}) > C \geq (v_\lambda)_x(\bar{x}, \bar{t}) \) and \( v_\lambda(\lambda + s, \bar{t}) = v(s, \bar{t}) = M + 1 > u(\lambda + s, \bar{t}) \), there has to be at least another point \( \bar{x} < \tilde{x} < \lambda + s \) such that \( u(\bar{x}, \tilde{t}) = v_\lambda(\tilde{x}, \bar{t}) \). Thus \( u - v_\lambda \) has at least two zeros at \( t = \bar{t} \) on
$\lambda - s < x < \lambda + s$. Function $u - v_{\lambda}$ satisfies the equation of the type (3.1) on $[\lambda - s, \lambda + s] \times (0, T]$, with non-zero boundary values for all $t > 0$ due to $\text{sup} |u| \leq M$ and (a). Thus we may use Lemma 1 and conclude that $u - v_{\lambda}$ has at least two zeros in $x$ variable for all $\ell > t > 0$. Since $v_{\lambda} > M$ for $x$ away from $\lambda - s$ and all small $t$, and since we assume that $u$ is a smooth function up to $t = 0$, this is impossible to satisfy for all small enough $t$. (See fig. 3 and 4.)

Thus it remains to prove the existence of such $v$. To do this, we invert the role of independent variable $x$ and dependent variable $y = v(x, t)$. Let $y = w(x, t)$ be the inverse function of $v$ with respect to the space variables, i.e., $w$ satisfies $y = v(w(y, t), t)$ identically. Since the equation is geometric, $w$ should satisfy the similar equation to (2.1) on $[-M - 1, M + 1] \times (0, T]$ with the role of $y$ and $x$ exchanged. Now, the conditions on $v$ in terms of $w$ are

(a') $w(-M - 1, t) = -s$ and $w(M + 1, t) = s$ for $0 < t \leq T$, 
(b') $w_{x} > 0$ on $[-M - 1, M + 1] \times (0, T]$, 
(c') for any $-M - 1 \leq x \leq M$, $\lim_{t \to 0} w(x, t) = -s$.

Furthermore, on $[-M - 1, M + 1] \times (0, T]$, $w$ should satisfy

\begin{equation}
\frac{w_{t}}{\sqrt{1 + w_{x}^{2}}} = (a_{q}(1, w_{x}))_{x}.
\end{equation}

Since $\frac{\partial w}{\partial x} = 1/\frac{\partial x}{\partial y}$, we need to show that there exists a constant $C > 0$ such that $w_{x} > C$ on $[-M, M] \times [t_{0}, T]$. We solve (3.2) with the following convex initial data. Let $\Gamma \in C^{\infty}([-M - 1, M + 1])$ (See fig.2 and 4.) be

- $\Gamma(x) = -s$ for $x \in [-M - 1, M]$, 
- $\Gamma(M + 1) = s$, $\Gamma''(M + 1) = 0$, 
- $\Gamma(x) \geq -s$, $\Gamma'(x) \leq 3s$, $\Gamma''(x) \geq 0$ for $x \in [M, M + 1]$.

Let $w$ be the unique smooth solution of (3.2) with the initial data $\Gamma$ and the boundary data (a'). Since any functions $c_{1} + c_{2}x$ are solutions of (3.2), one obtains the gradient estimate

\begin{equation}
0 \leq w_{x} \leq 3s
\end{equation}
on $[-M-1, M+1] \times [0, T]$, by using these functions as barriers and the standard maximum principle applied to $w_x$. Also, note that the convexity of $w$ is preserved, i.e., $w_{xx} \geq 0$. This is seen by differentiating the equation with respect to $t$ and then applying the maximum principle to $w_t$. $w_t = 0$ on the boundary and $w_t = a_{qq}w_{xx} \geq 0$ for $t = 0$ imply $w_t \geq 0$. The equation then yields $w_{xx} \geq 0$ on $[-M-1, M+1] \times [0, T]$.

Now, (3.3) implies that $a_{qq}(-1, w_x) \geq c(s, \delta_0) (\text{call this } \delta > 0 \text{ by assumption (c)}).$ We claim that the solution of

$$
\begin{cases}
  z_t = \delta z_{xx} & [-M-1, M+1] \times [0, T], \\
  z(\pm(M+1), t) = \pm s & t \in [0, T], \\
  z(x, 0) = \Gamma(x) & x \in [-M-1, M+1]
\end{cases}
$$

satisfies $w \geq z$ on $[-M-1, M+1] \times [0, T].$ (See fig.2) This is because of the following combined with the standard maximum principle:

$$(w-z)_t = a_{qq}(-1, w_x)w_{xx} - \delta z_{xx} = a_{qq}(-1, w_x)(w-z)_{xx} + (a_{qq}(-1, w_x) - \delta)z_{xx} \geq a_{qq}(-1, w_x)(w-z)_{xx}.$$ 

In the last line, we used $z_{xx} \geq 0$, which follows by the same reason for $w_{xx} \geq 0$ before, and $a_{qq}(-1, w_x) \geq \delta$. We next claim that for $t_0 \leq t$, there exists $c = c(t_0, s, \delta) > 0$ such that $z_x \geq c$ on $[-M-1, M+1] \times [t_0, T]$. $z_x$ satisfies again the heat equation with non-negative initial data and the homogeneous Neumann data, and thus by the strong maximum principle (or extending the solution to $\mathbb{R}$ by a suitable reflection argument and then using the representation formula with the heat kernel) we have such $c$. Since $w_{xx} \geq 0$, for $(x, t)$ with $t \geq t_0$, we have

$$w_x(x, t) \geq w_x(-M-1, t) \geq z_x(-M-1, t) \geq c$$

as the result. Note that we are using $w \geq z$ and $w = z$ on the boundary $x = -M-1$. This completes the proof.
\[ t = 0 \]

\[ \chi = \gamma (y) = w(y, 0) \]

\[ t > t_0 \]

\[ \chi = w(y, t) \]

\[ \chi = z(y, t) \]

\[ y = u(x, 0) \]

\[ y = v(x, t) = v(w(y, t), t) \]
References


