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Kyoto University
Interior gradient estimate for curvature flow

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Abstract

Our purpose is to understand the anisotropic curvature flow. Especially we like to prove the interior gradient estimate. We establish the interior gradient estimate for general 1-D anisotropic curvature flow. The estimate depends only on the height of the graph and not on the gradient at initial time.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. A surface given as a graph $u : \Omega \rightarrow \mathbb{R}$ is a minimal surface when $u$ satisfies

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. \quad (1.1)$$

For this equation, the following interior gradient estimates are well-known ([5, 6, 7]): Given a constant $M$ and $\bar{\Omega} \subset \subset \Omega$, there exists a constant $C$ depending only on $M$ and $\bar{\Omega}$ such that if $\sup_{\Omega} |u| \leq M$, then $\sup_{\bar{\Omega}} |\nabla u| \leq C$. The similar estimates are also known for the mean curvature flow equation ([3]). That is, if $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ satisfies

$$\frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \quad (1.2)$$
and \( \sup_{\Omega \times [0,T]} |u| \leq M, \tilde{\Omega} \subset \subset \Omega, 0 < T_0 < T \), then there exists \( C \) such that \( \sup_{\tilde{\Omega} \times [T_0,T]} |\nabla u| \leq C \). Again, \( C \) is a constant depending only on \( M, \tilde{\Omega} \) and \( T_0 \). Note that \( C \) is independent of the gradient at \( t = 0 \).

One direction to extend those results are to consider general anisotropic curvature problem, namely, to consider the variational problem corresponding to the energy functional

\[
F(u) = \int_{\Omega} a(\nu) \sqrt{1 + |\nabla u|^2},
\]

where \( \nu = (\nabla u, -1)/\sqrt{1 + |\nabla u|^2} \) is the unit normal vector to the graph of \( u \) and the function \( a : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^+ \) is the surface energy density and should satisfy certain convexity property. The Euler-Lagrange equation is

\[
(1.3) \quad \text{div}_x a_p(\nu) = 0,
\]

and the curvature flow equation is

\[
(1.4) \quad \frac{u_t}{\sqrt{1 + |\nabla u|^2}} = \text{div}_x a_p(\nu).
\]

The left-hand side of the equation (1.4) corresponds to the normal velocity of the curve \((x, u(x, \cdot))\) while the right-hand side is the weighted anisotropic curvature. This is a gradient flow of the anisotropic surface energy functional

\[
\int_{\Omega} a(\nu) \, ds,
\]

where \( ds = \sqrt{1 + |\nabla u|^2} \, dx \) and \( \nu = (-\nabla u, 1)/\sqrt{1 + |\nabla u|^2} \) with homogeneous Dirichlet \((u = 0)\) or Neumann \((a_p(-\nabla u, 1) = 0)\) boundary conditions, since

\[
\frac{d}{dt} \int_{\Omega} a(\nu) \, ds = \int_{\Omega} a_p(-\nabla u, 1) \cdot \nabla u_t \, dx = -\int_{\Omega} |\text{div}_x a_p(-\nabla u, 1)|^2 \, ds.
\]

We show the interior gradient estimates for general anisotropic curvature flow for one-dimensional case which is independent of the initial time gradient.
2 Main Theorem

Let \( r > 0 \) be given. The graph \( u : [-r,r] \times [0,T] \to \mathbb{R} \) is said to be an anisotropic curvature flow if smooth function \( u \) satisfies

\[
\frac{u_t}{\sqrt{1+u_x^2}} = (a_p(u_x, -1))_x.
\]

where \( a : \mathbb{R}^2 \to [0, \infty) \) is an anisotropic surface energy density function satisfying the following assumptions:

(a) \( a(tp, tq) = t a(p, q) \) for all \( t > 0 \),

(b) \( a \) is a convex function,

(c) there exists \( \delta_0 > 0 \) such that \( a(p, q) - \delta_0\|p, q\| \) is a convex function,

(d) \( a \) is smooth except at \( (0,0) \).

Under these assumptions, we show

Theorem 1
Suppose \( u \) is a smooth solution of (2.1) on \([-r,r] \times [0,T]\) satisfying

\[
\sup_{[-r,r] \times [0,T]} |u| \leq M.
\]

Given \( 0 < s < r \) and \( 0 < t_0 < T \), there exists a constant \( C > 0 \) depending only on \( \delta_0, M, t_0, s, r \) such that

\[
\sup_{[-(r-s), r-s] \times [t_0,T]} |u_x| \leq C.
\]

Note that the estimate is independent the gradient of the initial data. Also we point out that the dependence of \( C \) on \( a \) is only through the lower bound of the uniform convexity \( \delta_0 \), but not on the upper bound (such as \( C^1 \) bound). Thus, the result in this paper can be extended equally to the non-smooth anisotropic curvature flow problem [4] by approximations.

Remark 1 For example, \( a(p, q) = (p^2 + q^2)^{\frac{1}{2}} \) is isotropic curvature flow (mean curvature flow) and satisfies above assumptions. \( a(p, q) = (|p|^r + |q|^r)^{\frac{1}{r}} \) \((1 < r < \infty)\) is anisotropic curvature flow and also satisfies assumptions.

Remark 2 In general dimension, if we assume the axis symmetry of the graph of \( u \), we expect to prove the same interior gradient estimate.
3 Proof

We cite the following theorem due to Angenent [2] which says that the number of zeros of the solution of parabolic equations is nonincreasing as time increases.

**Lemma 1 (Angenent [2])**

Suppose \( u \in C^\infty([x_1, x_2] \times [0, T]) \) satisfies the equation

\[
  u_t = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u
\]
on \([x_1, x_2] \times [0, T]\) and

\[
  u(x_j, t) = 0 \quad \text{for} \quad t \in [0, T] \quad j = 1, 2.
\]

Here, \( a, b, c \) are smooth functions of \((x, t)\) and \( a > 0 \). Then for all \( t \in (0, T] \), the zero set of \( x \rightarrow u(x, t) \) will be finite, even when counted with multiplicity. The number of zeros of \( x \rightarrow u(x, t) \) counted with multiplicity is nonincreasing function of \( t \).

**Proof of Theorem.** Given \( 0 < s < r \) and \( 0 < t_0 < T \), we construct a solution \( v \) for (2.1) on \([-s, s] \times (0, T]\) with the following properties:

(a) \( v(-s, t) = -M - 1 \) and \( v(s, t) = M + 1 \) for \( 0 < t \leq T \),

(b) \( v_x > 0 \) on \([-s, s] \times (0, T]\),

(c) for any \(-s < x \leq s\), \( \lim_{t \to 0} v(x, t) > M \).

The property (c) means that \( v \) has an initial data which is vertical at \( x = -s \). We show that the function \( v \) has a gradient bound \( 0 < v_x \leq C \) on \([-s, s] \times [t_0, T] \), where \( C \) depends only on \( M, \delta_0, s, t_0 \). We show the existence of such \( v \) later in the proof. Assuming such \( v \) exists for now, we then prove that any solution with \( \sup_{[-r, r] \times [0, T]} |u| \leq M \) satisfies \( \sup_{[-(r-s), r-s] \times [t_0, T]} u_x \leq C \). The same argument using \(-u\) will show \( \sup_{[-(r-s), r-s] \times [t_0, T]} |u_x| \leq C \). For a contradiction, assume that there exists a point \((\bar{x}, \bar{t}) \in [-r, r] \times [t_0, T] \) with \( u_x(\bar{x}, \bar{t}) > C \). Since \( \sup |u| \leq M \) and by (a), we may choose \( \lambda \) so that \( |\bar{x} - \lambda| < s \) and \( v(\bar{x} - \lambda, \bar{t}) = u(\bar{x}, \bar{t}) \). With this \( \lambda \), define \( v_\lambda(x, t) = v(x - \lambda, t) \).

Since \( u_x(\bar{x}, \bar{t}) > C \geq (v_\lambda)_{x}(\bar{x}, \bar{t}) \) and \( v_\lambda(\lambda + s, \bar{t}) = v(s, \bar{t}) = M + 1 > u(\lambda + s, \bar{t}) \), there has to be at least another point \( \bar{x} < \bar{x} < \lambda + s \) such that \( u(\bar{x}, \bar{t}) = v_\lambda(\bar{x}, \bar{t}) \). Thus \( u - v_\lambda \) has at least two zeros at \( t = \bar{t} \) on
\( \lambda - s < x < \lambda + s \). Function \( u - v_\lambda \) satisfies the equation of the type (3.1) on \([\lambda - s, \lambda + s] \times (0, T]\), with non-zero boundary values for all \( t > 0 \) due to \( \sup |u| \leq M \) and (a). Thus we may use Lemma 1 and conclude that \( u - v_\lambda \) has at least two zeros in \( x \) variable for all \( t > 0 \). Since \( v_\lambda > M \) for \( x \) away from \( \lambda - s \) and all small \( t \), and since we assume that \( u \) is a smooth function up to \( t = 0 \), this is impossible to satisfy for all small enough \( t \). (See fig. 3 and 4.)

Thus it remains to prove the existence of such \( v \). To do this, we invert the role of independent variable \( x \) and dependent variable \( y = v(x, t) \). Let \( y = w(x, t) \) be the inverse function of \( v \) with respect to the space variables, i.e., \( w \) satisfies \( y = v(w(y, t), t) \) identically. Since the equation is geometric, \( w \) should satisfy the similar equation to (2.1) on \([-M - 1, M + 1] \times (0, T]\) with the role of \( y \) and \( x \) exchanged. Now, the conditions on \( v \) in terms of \( w \) are

(a') \( w(-M - 1, t) = -s \) and \( w(M + 1, t) = s \) for \( 0 < t \leq T \),

(b') \( w_x > 0 \) on \([-M - 1, M + 1] \times (0, T]\),

(c') for any \(-M - 1 \leq x \leq M \), \( \lim_{t \to 0} w(x, t) = -s \).

Furthermore, on \([-M - 1, M + 1] \times (0, T]\), \( w \) should satisfy

\[
\frac{w_t}{\sqrt{1 + w_x^2}} = (a_q(1, w_x))_x.
\]

Since \( \frac{\partial w}{\partial x} = 1/\frac{\partial x}{\partial y} \), we need to show that there exists a constant \( C > 0 \) such that \( w_x > C \) on \([-M, M] \times [t_0, T]\). We solve (3.2) with the following convex initial data. Let \( \Gamma \in C^\infty([-M - 1, M + 1]) \) (See fig.2 and 4.) be

- \( \Gamma(x) = -s \) for \( x \in [-M - 1, M] \),
- \( \Gamma(M + 1) = s \), \( \Gamma''(M + 1) = 0 \),
- \( \Gamma(x) \geq -s \), \( \Gamma'(x) \leq 3s \), \( \Gamma''(x) \geq 0 \) for \( x \in [M, M + 1] \).

Let \( w \) be the unique smooth solution of (3.2) with the initial data \( \Gamma \) and the boundary data (a'). Since any functions \( c_1 + c_2 x \) are solutions of (3.2), one obtains the gradient estimate

\[
0 \leq w_x \leq 3s
\]
on \([-M-1, M+1] \times [0,T]\), by using these functions as barriers and the standard maximum principle applied to \(w_x\). Also, note that the convexity of \(w\) is preserved, i.e., \(w_{xx} \geq 0\). This is seen by differentiating the equation with respect to \(t\) and then applying the maximum principle to \(w_t\). \(w_t = 0\) on the boundary and \(w_t = a_{qq} w_{xx} \geq 0\) for \(t = 0\) imply \(w_t \geq 0\). The equation then yields \(w_{xx} \geq 0\) on \([-M-1, M+1] \times [0,T]\).

Now, (3.3) implies that \(a_{qq}(-1,w_x) \geq c(s,\delta_0)(\text{call this } \delta) > 0\) by assumption (c). We claim that the solution of

\[
\begin{cases}
   z_t = \delta z_{xx} & \text{on } [-M-1, M+1] \times [0,T], \\
   z(\pm(M+1),t) = \pm s & t \in [0,T], \\
   z(x,0) = \Gamma(x) & x \in [-M-1, M+1]
\end{cases}
\]

satisfies \(w \geq z\) on \([-M-1, M+1] \times [0,T]\). (See fig.2) This is because of the following combined with the standard maximum principle:

\[
(w-z)_t = a_{qq}(-1,w_x)w_{xx} - \delta z_{xx} = a_{qq}(-1,w_x)(w-z)_{xx} + (a_{qq}(-1,w_x) - \delta)z_{xx} \geq a_{qq}(-1,w_x)(w-z)_{xx}.
\]

In the last line, we used \(z_{xx} \geq 0\), which follows by the same reason for \(w_{xx} \geq 0\) before, and \(a_{qq}(-1,w_x) \geq \delta\). We next claim that for \(t_0 \leq t\), there exists \(c = c(t_0,s,\delta) > 0\) such that \(z_x \geq c\) on \([-M-1, M+1] \times [t_0,T]\).

\(z_x\) satisfies again the heat equation with non-negative initial data and the homogeneous Neumann data, and thus by the strong maximum principle (or extending the solution to \(\mathbb{R}\) by a suitable reflection argument and then using the representation formula with the heat kernel) we have such \(c\). Since \(w_{xx} \geq 0\), for \((x,t)\) with \(t \geq t_0\), we have

\[
w_x(x,t) \geq w_x(-M-1,t) \geq z_x(-M-1,t) \geq c
\]

as the result. Note that we are using \(w \geq z\) and \(w = z\) on the boundary \(x = -M-1\). This completes the proof.
\[
\begin{aligned}
\text{figure 1} & \quad \text{figure 2} \\
\text{figure 3} & \quad \text{figure 4}
\end{aligned}
\]
References


