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Stable transition layers in a balanced bistable equation with degeneracy

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1 Introduction and Main Results

In this paper, we consider steady-state solutions for the following problem:

$$\begin{cases} u_t - \varepsilon^2 u_{xx} = f(x,u), & (x,t) \in (0,1)\times(0,\infty), \\ u_x(0,t) = u_x(1,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases}$$

where ε is a positive number and $f(x,u)$ is given by

$$f(x,u) = -u(u-\alpha(x))(u+\alpha(x)).$$

Here $\alpha : [0,1] \to \mathbb{R}$ is a positive C^1 function and a C^2 function except for a finite number of points on $[0,1]$. Such $f(x,u)$ is a typical example of the so-called bistable nonlinearity and we note that $f(x,u)$ satisfies that

$$\int_{-\alpha(x)}^{\alpha(x)} f(x,u)du = 0.$$

In this sense, we call the bistable function f to be *balanced.*

Since we are interested in the stationary problem, we consider the following problem:

$$(\mathrm{P}_\varepsilon)\begin{cases} -\varepsilon^2 u_{xx} = f(x,u) & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0. \end{cases}$$

It is easily shown that there exist stable solutions $u_\varepsilon^{(+)}$, $u_\varepsilon^{(-)}$ for (P_ε) such that $\lim_{\varepsilon\to 0} u_\varepsilon^{(+)}(x) = \alpha(x)$, $\lim_{\varepsilon\to 0} u_\varepsilon^{(-)}(x) = -\alpha(x)$ uniformly in $x \in [0,1]$ (see [7, Proposition 2.2]). The aim of this paper is to find stable solutions u_ε with transition layers.

Nakashima [7] has studied the problem (P_ε) when α is smooth and nondegenerate, i.e. $\alpha'' \neq 0$ at each local minimum of α. In this paper we consider the case that α degenerates on an interval I of positive measure where α takes its local minimum, that is, $\alpha'(x) = 0$ on I. Nakashima and Tanaka [9] also have studied such a degenerate case and obtain solutions with a single layer and multi-layers(clustering layers) by using a variational method. However the stability of these solutions were not discussed. In this paper we obtain a stable solution with transition layers by a sub-supersolution method of Brezis and Nirenberg type (see [2]) and precise profile of the solution near the interval where α degenerates by using a blow up argument inspired by the arguments in Dancer and Shusen Yan [3].

Now we state precise conditions on α.

Conditions . (C1) α is a positive function on $[0,1]$ and $\alpha \in C^1[0,1]$.

(C2) There exist a finite number of points $x_1, x_2, \cdots, x_{2m} \in (0,1)$ $(m \geq 1)$ such that

(i) $\alpha'(x) = 0$ on $I_i := [x_{2i-1}, x_{2i}]$ for $i = 1, \cdots, m$;

(ii) $\alpha \in C^2((x_{2i}, x_{2i+1}))$ for each $i = 0, 1, \cdots, m$, and there exist limits

$$\alpha''(x_{2i-1} - 0) = \lim_{h \downarrow 0} \frac{\alpha'(x_{2i-1}) - \alpha'(x_{2i-1} - h)}{h}$$

and

$$\alpha''(x_{2i} + 0) = \lim_{h \downarrow 0} \frac{\alpha'(x_{2i-1} + h) - \alpha'(x_{2i})}{h}$$

for each $i = 1, \cdots, m$;

(iii) $\alpha''(x_{2i-1} - 0) > 0$ and $\alpha''(x_{2i} + 0) > 0$ for $i = 1, \cdots, m$.

Hereafter we denote $\alpha''(x_{2i-1})$, $\alpha''(x_{2i})$ instead of $\alpha''(x_{2i-1} - 0)$, $\alpha''(x_{2i} + 0)$.

Remark . The condition (i) of (C2) implies that $\alpha(x) = const.$ on I_i and if $x_{2i-1} = x_{2i}$ for $i = 1, \cdots, m$, this is the case as in Nakashima [7].

We set $L = \{x_1, x_2, \cdots, x_{2m}\}$ and $K = \{I_1, I_2, \cdots, I_m\}$. We choose any subset $\hat{K}$ of K. We denote $\hat{K} = \{\hat{I}_1, \hat{I}_2, \cdots, \hat{I}_l\}$ with $1 \leq l \leq m$ and $\hat{I}_i = [x'_{2i-1}, x'_{2i}]$ for each $i = 1, \cdots l$ where we use the notation $x'_0 = 0$, $x'_{2l+1} = 1$. We consider the following two cases:

$$\text{(I)}\quad \Omega_1 = \bigcup_{i=0}^{[\frac{2l-1}{2}]} (x'_{4i}, x'_{4i+1}),\ \Omega_2 = \bigcup_{i=0}^{[\frac{2l-3}{2}]} (x'_{4i+2}, x'_{4i+3})$$

$$\text{(II)}\quad \Omega_1 = \bigcup_{i=0}^{[\frac{2l-3}{2}]} (x'_{4i+2}, x'_{4i+3}),\ \Omega_2 = \bigcup_{i=0}^{[\frac{2l-1}{2}]} (x'_{4i}, x'_{4i+1})$$

and we set

$$\Omega_i^\delta = \{x \in (0,1) | \mathrm{dist}(x, \partial\Omega_i \backslash \{0,1\}) > \delta\}.$$

First, we construct a solution to (P_ε) that may have transition layers.

Theorem 1.1. *Assume that* (C1) *and* (C2) *hold. Then for sufficiently small $\varepsilon > 0$, there exists a family of stable solutions $\{u_\varepsilon\}$ of (P_ε) such that*

$$|-\alpha(x) - u_\varepsilon(x)| < \sigma \text{ in } \Omega_1^\delta,$$
$$|\alpha(x) - u_\varepsilon(x)| < \sigma \text{ in } \Omega_2^\delta,$$

where $\sigma = \sigma(\varepsilon) = o_\varepsilon(1)$, $\delta = \delta(\varepsilon) = o_\varepsilon(1)$.

Moreover u_ε is a local minimizer of the functional

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2}|u_x|^2 - F(x,u)dx,$$

where $F(x,u) = \int_0^u f(x,s)ds$.

Next theorem describes the precise profile of u_ε near the intervals where α degenerates.

Theorem 1.2. *Consider the case (I). Let u_ε be the solution of (P_ε) obtained in Theorem 1.1. Then u_ε has exactly one layer in $[x'_{2i-1} - 2\varepsilon^{1-\rho}, x'_{2i} + 2\varepsilon^{1-\rho}]$ for any small $0 < \rho < 1$ and for each $i = 1, 2, \cdots, l$. That is for any small $\eta > 0$, there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, the followings hold.*

(1) *For each $i = 1, 2, \cdots, l$, there exists the unique pair of numbers $\{t_{\varepsilon,1,i}, t_{\varepsilon,2,i}\}$ such that $x'_{2i-1} - 2\varepsilon^{1-\rho} < t_{\varepsilon,1,i} < t_{\varepsilon,2,i} < x'_{2i} + 2\varepsilon^{1-\rho}$ and the followings hold.*

(a) *If i is odd number, the followings hold;*

$$\begin{cases} u_\varepsilon < -\overline{\alpha}_i + \eta \text{ on } [x'_{2i-1} - 2\varepsilon^{1-\rho}, t_{\varepsilon,1,i}), \\ u_\varepsilon(t_{\varepsilon,1,i}) = -\overline{\alpha}_i + \eta, \\ u_\varepsilon(t_{\varepsilon,2,i}) = \overline{\alpha}_i - \eta, \\ u_\varepsilon > \overline{\alpha}_i - \eta \text{ on } (t_{\varepsilon,2,i}, x'_{2i} + 2\varepsilon^{1-\rho}]. \end{cases}$$

(b) *If i is even number, the followings hold;*

$$\begin{cases} u_\varepsilon > \overline{\alpha}_i - \eta \text{ on } [x'_{2i-1} - 2\varepsilon^{1-\rho}, t_{\varepsilon,1,i}), \\ u_\varepsilon(t_{\varepsilon,1,i}) = \overline{\alpha}_i - \eta, \\ u_\varepsilon(t_{\varepsilon,2,i}) = -\overline{\alpha}_i + \eta, \\ u_\varepsilon < -\overline{\alpha}_i + \eta \text{ on } (t_{\varepsilon,2,i}, x'_{2i} + 2\varepsilon^{1-\rho}]. \end{cases}$$

Here $\overline{\alpha_i} = \alpha(x_{2i-1}) = \alpha(x_{2i})$.

(2) *If i is odd number, u_ε is increasing on $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$ and if i is even number, u_ε is decreasing on $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$.*

(3) $0 < R_1 \le \frac{t_{\varepsilon,2,i} - t_{\varepsilon,1,i}}{\varepsilon} \le R_2$, *where R_1 and R_2 are two constants independent of $\varepsilon > 0$.*

Remark . If we take the case (II), the statement (a) of (1) holds if i is odd number and statement (b) of (1) holds if i is even number. And if i is odd number, u_ε is increasing on $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$ and if i is even number, u_ε is decreasing on $(t_{\varepsilon,1,i}, t_{\varepsilon,2,i})$.

Remark . Since $\{t_{\varepsilon,1,i}\}_{0<\varepsilon<\varepsilon_0}$ and $\{t_{\varepsilon,2,i}\}_{0<\varepsilon<\varepsilon_0}$ are bounded sequences, from the part (3) of Theorem 1.2, we may assume that there exists $t_i \in [x'_{2i-1}, x'_{2i}]$ such that $t_{\varepsilon,1,i}, t_{\varepsilon,2,i} \to t_i$ as $\varepsilon \to 0$. But the exact location of t_i is not yet known when $x'_{2i} - x'_{2i-1} > 0$ and this is an open problem.

This paper is organized as follows. To prove Theorem 1.1 and 1.2, we take a sub-supersolution method of Brezis and Nirenberg Type ([2]). Hence in section 2, we prepare the sub-supersolution method. In section 3 we construct a subsolution and a supersolution and prove Theorem 1.1. In section 4 we prove Theorem 1.2.

2 Preliminaries

In this section we prepare the sub-supersolution method of Brezis and Nirenberg type under the Neumann boundary condition. In [2], Brezis and Nirenberg developed such method under the Dirichlet boundary condition. Nacimento [6] pointed out their method also works under the Neumann boundary condition without a precise proof. Now we give the definition of a subsolution and a supersolution in the form suitable for our problem.

Consider the following problem:

$$\begin{cases} u_{xx} + g(x, u) = 0, \quad 0 < x < 1, \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{2.1}$$

where $g(x, s)$ is a C^1 function with respect to (x, s) and we assume the following growth condition

$$|g(x, s)| \le C(1 + |s|^p)$$

for some $1 < p < +\infty$ and for some $C > 0$. Moreover we assume that for some $k \geq 0$ the function $g(x,s) + ks$ is nondecreasing in s for each x. We remark that $\overline{g}(x,s) := g(x,s) + s$ also satisfies

$$\begin{aligned} |\overline{g}(x,s)| &\leq C(1+|s|^p) + |s| \\ &\leq C(1+|s|^p) + \frac{|s|^p}{p} + \frac{1^q}{q} \\ &\leq C'(1+|s|^p) \end{aligned}$$

for some $C' > 0$. We define the subsolution and the supersolution for (2.1) as follows.

Definition 2.1. Let u^*(resp. u_*) : $[0,1] \to \mathbb{R}$ be a continuous function. The function u^* (resp. u_*) is called a *supersolution*(resp. *subsolution*) of (2.1) if

(S1) there exists $\delta_0 > 0$ such that u^*(resp. u_*) $\in C^2((0,\delta_0) \cup (1-\delta_0, 1)) \cap C^1([0,\delta_0) \cup (1-\delta_0, 1])$,

(S2) for all $\varphi \in C_0^\infty(0,1)$ with $\varphi \geq 0$, we have

$$\begin{aligned} &\int_0^1 (-u^* \varphi_{xx} - g(x,u^*)\varphi)dx \geq 0 \\ &\left(\text{resp. } \int_0^1 (-u_* \varphi_{xx} - g(x,u_*)\varphi)dx \leq 0\right), \end{aligned} \tag{2.2}$$

(S3) $u^*_x(0) \leq 0$ and $u^*_x(1) \geq 0$ (resp. $u_{*x}(0) \geq 0$ and $u_{*x}(1) \leq 0$).

Before we state the existence of a solution to (2.1), we have to define the energy functional I of (2.1):

$$I(u) = \int_0^1 \frac{1}{2}|u_x|^2 - G(x,u)dx, \; G(x,u) = \int_0^u g(x,s)ds.$$

Note that if we define

$$\overline{G}(x,u) = \int_0^u \overline{g}(x,s)ds$$

the energy functional I can be written as follows

$$I(u) = \int_0^1 \frac{1}{2}|u_x|^2 + \frac{1}{2}u^2 - \overline{G}(x,u)dx.$$

The next proposition is the existence result of a solution to (2.1) between a subsolution and a supersolution for the Neumann boundary condition.

Proposition 2.2. *If there exists a supersolution u^* to (2.1) and a subsolution u_* to (2.1) with $u_* < u^*$ and neither u_* nor u^* is a solution of (2.1). Then there exists a solution u_0 to (2.1) such that $u_* \leq u_0 \leq u^*$ and u_0 is a local minimizer of I on $H^1(0,1)$. Moreover u_0 is a global minimizer of the following functional $\tilde{I}$:*

$$\tilde{I}(u) = \int_0^1 \frac{1}{2}|u_x|^2 - \tilde{G}(x,u)dx, \; \tilde{G}(x,u) = \int_0^u \tilde{g}(x,s)ds$$

where

$$\tilde{g}(x,s) = \begin{cases} g(x,u_*(x)), & s < u_*(x), \\ g(x,s), & u_*(x) \leq s \leq u^*(x), \\ g(x,u^*(x)), & u^*(x) < s. \end{cases}$$

We need some lemmas to prove Proposition 2.2.

Lemma 2.3. (cf.[2, Theorem 1]) *Assume that* $u_0 \in H^1(0,1)$ *is a local minimizer of* I *in the* C^1 *topology; this means that there exists some* $r > 0$ *such that*

$$I(u_0) \leq I(u_0 + v) \quad \textit{for } v \in C^1[0,1] \quad \textit{with } \|v\|_{C^1} \leq r. \tag{2.3}$$

Then u_0 *is a local minimizer of* I *in the* H^1 *topology; i.e. there exists* $\varepsilon_0 > 0$ *such that*

$$I(u_0) \leq I(u_0 + v) \quad \textit{for } v \in H^1(0,1) \quad \textit{with } \|v\|_{H^1} \leq \varepsilon_0.$$

Proof. The proof is divided in 2 steps.

Step 1. We claim that $u_0 \in C^{1,\gamma}[0,1]$ for all $0 < \gamma < 1$.

Recall that u_0 is a weak solution of

$$\begin{cases} -u_0'' = g(x,u_0), & \text{in } (0,1), \\ u_0'(0) = u_0'(1) = 0, \end{cases}$$

where $'$ denote the derivative in x. First by the Sobolev imbedding we have $u_0 \in C[0,1]$ and hence $u_0 \in L^p(0,1)$ for any $1 < p < \infty$. Next by the standard regularity result, we have $u_0 \in W^{2,p}(0,1)$ for any $1 < p < \infty$. Again by the Sobolev imbedding, we have $u_0 \in C^{1,\gamma}[0,1]$ for any $0 < \gamma < 1$.

Step 2. Without loss of generality we may now assume that $u_0 = 0$. Suppose that the conclusion does not hold. Then for every $\varepsilon > 0$, there exists $v_\varepsilon \in B_\varepsilon(0) := \{w \in H^1(0,1) | \|w\|_{H^1} \leq \varepsilon\}$ such that

$$I(v_\varepsilon) < I(0). \tag{2.4}$$

By a standard lower semicontinuity argument $\min_{B_\varepsilon} I$ is attained at some point which we may still denote by v_ε. We shall prove that $v_\varepsilon \to 0$ in C^1 and this contradict to (2.3) and (2.4). The corresponding Euler equation for v_ε involves a Lagrange multiplier $\mu_\varepsilon \leq 0$, namely, v_ε satisfies

$$\langle DI(v_\varepsilon), \zeta \rangle_{(H^1)^*,H^1} = \mu_\varepsilon (v_\varepsilon, \zeta)_{H^1} \quad \text{for } \zeta \in H^1(0,1)$$

i.e.

$$\int_0^1 v_\varepsilon' \zeta' + v_\varepsilon \zeta - \overline{g}(x,v_\varepsilon)\zeta dx = \mu_\varepsilon \int_0^1 v_\varepsilon' \zeta' + v_\varepsilon \zeta dx \quad \text{for } \zeta \in H^1(0,1),$$

where $DI(v_\varepsilon)$ denotes the Frećhet derivative of I at v_ε. This means

$$-v_\varepsilon'' + v_\varepsilon = \frac{1}{1-\mu_\varepsilon} \overline{g}(x,v_\varepsilon). \tag{2.5}$$

Using (2.5) together with the remark $|\overline{g}(x,u)| \leq C(1 + |u|^p)$ and the essential fact $\mu_\varepsilon \leq 0$, one may obtain from the bound $\|v\|_{H^1} \leq C$ to $\|v\|_{C^{1,\gamma}} \leq C$ by the bootstrap argument as in step 1(independent of $\varepsilon > 0$). Since $v_\varepsilon \to 0$ in H^1, $v_\varepsilon \to 0$ in C^1. The proof is completed. □

Next lemma is due to [2].

Lemma 2.4. ([2, Theorem 2]) *Let* $u \in L^1_{\text{loc}}(0,1)$ *and assume that for some* $k \geq 0$, u *satisfies*

$$\begin{cases} -u'' + ku \geq 0 & \textit{in } \mathcal{D}'(0,1), \\ u \geq 0 & \textit{in } (0,1). \end{cases}$$

Then either $u \equiv 0$, *or there exists* $\varepsilon > 0$ *such that*

$$u(x) \geq \varepsilon \text{dist}(x, \partial(0,1)).$$

Proof. See [2]. □

Next lemma is the strong maximum principle for a subharmonic function in the distribution sense.

Lemma 2.5. *Let $a < b$ and $u \in C[a,b]$ be a subharmonic function, i.e.*

$$\int_a^b -u\varphi'' dx \leq 0$$

for $\varphi \in C_0^\infty(a,b)$ with $\varphi \geq 0$. Then if u is not a constant function, the maximum of u over $[a,b]$ is attained at $x = a$ or $x = b$. Moreover we have

$$u(x) < \max_{[a,b]} u = \max\{u(a), u(b)\} \quad \textit{for } x \in (a,b).$$

Proof. See for example [4]. □

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Let u_0 be a minimizer of $\tilde{I}$ on $H^1(0,1)$, it is easily seen the minimum is achieved and satisfies

$$-u_0'' = \tilde{g}(x, u_0) \ \text{ in } (0,1).$$

By the bootstrap argument, we have that $u_0 \in C^{1,\gamma}[0,1]$. We claim that $u_* \leq u_0 \leq u^*$. We will just prove the $u_* \leq u_0$. Set $A = \{x \in (0,1) | u_0(x) < u_*(x)\}$ and we will show $A = \emptyset$. First we have

$$-(u_* - u_0)'' \leq g(x, u_*) - \tilde{g}(x, u_0) \tag{2.6}$$

and in particular

$$-(u_* - u_0)'' \leq 0 \ \text{ in } \mathcal{D}'(A).$$

First we assume that $u_* - u_0 \leq 0$ at $x = 0, 1$. Then $A \subset (0,1)$ and since $u_* - u_0 \leq 0$ on ∂A, it follows from Lemma 2.5 that $u_* - u_0 \leq 0$ in A. Hence we can conclude $A = \emptyset$. Next we prove that $u_* - u_0 \leq 0$ at $x = 0, 1$. Let us assume that $u_*(0) - u_0(0) > 0$ and $u_*(1) - u_0(1) \leq 0$. Similarly $w := u_* - u_0$ satisfies

$$-w'' \leq 0 \ \text{ in } \mathcal{D}'(A)$$

and w attain its strict maximum on $x = 0$ by Lemma 2.5. Note that since u_* is not solution for (2.1), w is not constant. Indeed if w is constant, i.e., $u_* - u_0 = C$ for some constant $C > 0$, then u_* satisfies that

$$-u_*'' = -u_0'' = \tilde{g}(x, u_* - C) = g(x, u_*)$$

and this contradict to the assumption that u_* is not a solution to (2.2). Since

$$-u_0'' = g(x, u_*(x))$$

and u_* is C^2 near $x = 0$ and g is C^1, we have $u_0 \in C^2$ and $w = u_* - u_0$ is C^2 near $x = 0$. Moreover w satisfies

$$-w'' \leq 0 \quad \text{for } x > 0 \text{ small,}$$
$$w(0) > w(x) \quad \text{for } x > 0 \text{ small.}$$

Hence by Hopf's Lemma we have $w'(0) = u_*'(0) - u_0'(0) < 0$ and this contradict to the (S3) in the Definition 2.1. Similarly we can obtain the contradiction if we assume that $u_*(0) - u_0(0) > 0$ and $u_*(1) - u_0(1) > 0$ or $u_*(0) - u_0(0) \leq 0$ and $u_*(1) - u_0(1) > 0$.

Returning to (2.6) we have

$$-(u_* - u_0)'' + k(u_* - u_0) \leq (g(x, u_*) + ku_*) - (g(x, u_0) + ku_0) \leq 0.$$

Since u_* is not a solution, it follows from Proposition 2.4 that there is some $\varepsilon > 0$ such that

$$u_*(x) - u_0(x) \leq -\varepsilon \text{dist}(x, \partial(0,1)) \quad \text{for} \quad x \in (0,1).$$

Similarly for u^* we have

$$u_*(x) + \varepsilon \text{dist}(x, \partial(0,1)) \leq u_0(x) \leq u^*(x) - \varepsilon \text{dist}(x, \partial(0,1)) \quad \text{for } x \in (0,1).$$

It follows that if $u \in C^1[0,1]$ and $||u - u_0||_{C^1} \leq \varepsilon$ then

$$u_* \leq u \leq u^* \quad \text{in } (0,1).$$

By the remark following this proof, $I(u) - \tilde{I}(u)$ is constant for $||u - u_0||_{C^1} \leq \varepsilon$. Hence u_0 is a local minimizer for I in the C^1 topology (since it is global minimizer for $\tilde{I}$). Now, we invoke Proposition 2.3 to claim that u_0 is also a local minimizer of I in the H^1 topology. This completes the proof of Proposition 2.2. □

Remark . If we take a function $u \in H^1(0,1)$ satisfies $u_* \leq u \leq u^*$, we have

$$\begin{aligned}
\tilde{G}(x,u) &= \int_0^u \tilde{g}(x,s)ds \\
&= \int_{u_*}^u \tilde{g}(x,s)ds + \int_0^{u_*} \tilde{g}(x,s)ds \\
&= \int_{u_*}^u g(x,s)ds + \int_0^{u_*} \tilde{g}(x,s)ds \\
&= G(x,u) - G(x,u_*(x)) + \tilde{G}(x,u_*(x)).
\end{aligned}$$

Thus the functional $\tilde{I}$ is

$$\begin{aligned}
\tilde{I}(u) &= \int_0^1 \frac{1}{2}|u_x|^2 - \tilde{G}(x,u)dx \\
&= \int_0^1 \frac{1}{2}|u_x|^2 - G(x,u)dx + const.
\end{aligned}$$

and we can replace I by $\tilde{I}$ for the function $u \in H^1(0,1)$ satisfies $u_* \leq u \leq u^*$.

Next we give the sufficient condition for functions becoming subsolutions and supersolutions. This condition is due to Nakashima [7]. First we state a notation.

Let $u : [0,1] \to \mathbb{R}$ be a continuous function and $u \in C^1([0,\delta_0) \cup (1-\delta_0, 1])$ for some $\delta_0 > 0$ such that for a finite number of points $a_1, a_2, \cdots, a_k \in (0,1)$

(i) $u : [0,1] \to \mathbb{R}$ is class C^2 in $(a_0, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_k, a_{k+1})$ with $a_0 = 0$, $a_{k+1} = 1$.

(ii) There exists $\lim_{x \to a_i + 0} u_x(x)$, $\lim_{x \to a_i - 0} u_x(x)$ for each $i = 1, 2, \cdots, k$.

We denote $P(a_1, a_2, \cdots, a_k)$ the set of function u satisfies (i) and (ii).

Proposition 2.6. *Let $u^* \in P(a_1, a_2, \cdots, a_k)$ satisfies the following conditions:*

(S1)′ *For each* $i = 0, 1, \cdots, k$
$$-u^*_{xx} - g(x, u^*) \geq 0 \text{ in } (a_i, a_{i+1}).$$

(S2)′ *For each* $i = 1, 2, \cdots k$
$$\lim_{x \to a_i + 0} u^*_x(x) \leq \lim_{x \to a_i - 0} u^*_x(x).$$

(S3)′ $u^*_x(0) \leq 0$ *and* $u^*_x(1) \geq 0$.

Then u^* *is a supersolution for* (2.1). *If* $u_* \in P(a_1, a_2, \cdots, a_k)$ *satisfies* (S1)′, (S2)′ *and* (S3)′ *which reversed the inequality sign, then* u_* *is a subsolution for* (2.1).

Proof. It suffices to show that
$$\int_0^1 (-u^* \varphi_{xx} - g(x, u^*)\varphi) dx \geq 0$$
for any $\varphi \in C_0^\infty(0, 1)$, with $\varphi \geq 0$. Indeed, using the integration by parts, we have
$$\begin{aligned}
&\int_0^1 (-u^* \varphi_{xx} - g(x, u^*)\varphi) dx \\
= \;& \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} (-u^* \varphi_{xx} - g(x, u^*)\varphi) dx \\
= \;& \sum_{i=0}^{k} [u^*_x(a_{i+1} - 0)\varphi(a_{i+1}) - u^*_x(a_i + 0)\varphi(a_i)] \\
& \quad + \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} (-u^*_{xx} - g(x, u^*))\varphi dx \\
= \;& [u^*_x(1)\varphi(1) - u^*_x(0)\varphi(0)] + \sum_{i=1}^{k} [u^*_x(a_i - 0) - u^*_x(a_i + 0)]\varphi(a_i) \\
& \quad + \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} (-u^*_{xx} - g(x, u^*))\varphi dx \geq 0.
\end{aligned}$$
Here we have used the assumption (S1)′, (S2)′ and (S3)′ and $a_0 = 0$ and $a_{k+1} = 1$. □

Finally, we consider an auxiliary problem for each positive number γ:
$$u_{zz} + u(\gamma - u)(\gamma + u) = 0, u(-\infty) = -\gamma, u(+\infty) = \gamma. \tag{2.7}$$
By the phase plane method we can obtain some properties of the solution for (2.7).

Lemma 2.7. *For each* $\gamma > 0$, *there exists a unique solution* $U(z; \gamma)$ *of* (2.7) *with* $U(0; \gamma) = 0$. *Moreover, it has the following properties:*

(1) $\frac{d}{dz} U(z, \gamma) > 0$ *for* $z \in \mathbb{R}$.

(2) *There exist positive constants* C_1 *and* C_2 *independent of* γ *such that*
$$|U(z; \gamma) - \gamma| < C_1 \gamma \exp(-C_2 \gamma z) \quad z \geq 0,$$
$$|U(z; \gamma) + \gamma| < C_1 \gamma \exp(C_2 \gamma z) \quad z \leq 0.$$

(3) *There exist positive constants* C_3 *and* C_4 *independent of* γ *such that*

$$|U'(z;\gamma)| < C_3\gamma^2 \exp(-C_4\gamma|z|) \quad z \in \mathbb{R}.$$

(4) $\frac{d^2}{dz^2}U(z;\gamma) \geq 0$ *for* $z \leq 0$ *and* $\frac{d^2}{dz^2}U(z;\gamma) \leq 0$ *for* $z \geq 0$.

(5) $U(z;\gamma) = \gamma U(\gamma z; 1)$.

Although the following lemma is elementary, this is very important in our argument.

Lemma 2.8. *Let* $\gamma > 0$ *and* $0 < \eta < \gamma$ *be fixed constants and* v *satisfies*

$$\begin{cases} -v_{zz} = v(\gamma^2 - v^2) & \text{on } \mathbb{R}, \\ v(0) = -\gamma + \eta, & \\ v(z) \leq -\gamma + \eta & \text{for } z \leq 0, \\ v \text{ is bounded on } \mathbb{R}. & \end{cases}$$

Then v *is a unique solution of*

$$\begin{cases} -v_{zz} = v(\gamma^2 - v^2) & \text{on } \mathbb{R}, \\ v(0) = -\gamma + \eta, & \\ v'(z) > 0 & z \in \mathbb{R}, \\ v(z) \to \pm\gamma & \text{as } z \to \pm\infty. \end{cases}$$

Proof. Since v is bounded, by using the phase plane analysis, v is a periodic solution or a unique heteroclinic solution joining $-\gamma$ and γ. Since $v(z) \leq -\gamma + \eta < 0$ for $z \leq 0$, we can conclude that v is the unique heteroclinic solution joining $-\gamma$ and γ. □

Making use of Lemma 2.7, we will construct a supersolution and a subsolution for (P_ε) and we obtain a solution u_ε to (P_ε) by using Proposition 2.2 in the following sections.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Although the construction of a subsolution and a supersolution is almost same as in Nakashima [7], we give the proof of Theorem 1.1 in details for reader's convenience.

For the sake of simplicity, we first assume $L = \{x_1, x_2\}$ and $K = \hat{K} = \{I_1\}$; so that $\alpha'(x) = 0$ for $x \in [x_1, x_2]$ and $\alpha''(x_1) > 0$, $\alpha''(x_2) > 0$ and we set $\alpha(x_1) = \alpha(x_2) = \overline{\alpha}$. First we construct a subsolution and a supersolution in the case when $\alpha'(0) \geq 0$ and $\alpha'(1) \geq 0$.

Proposition 3.1. *Assume* $\alpha'(0) \geq 0$ *and* $\alpha'(1) \geq 0$. *Let* σ_1 *and* σ_2 *be positive numbers satisfying* $\sigma_1 < 1$ *and* $\sigma_1 < \sigma_2 < 2\sigma_1$. *For sufficiently small* $\varepsilon > 0$, *there exist* $\zeta_1 = \zeta_1(\varepsilon)$ *and* $\zeta_2 = \zeta_2(\varepsilon)$ *such that* $-2\varepsilon^{1-\sigma_1} < \zeta_1 < -\varepsilon^{1-\sigma_1} < \zeta_2 < 0$ *and the following function*

$$u_\varepsilon^*(x) = \begin{cases} -\alpha(x) + \varepsilon^{2-\sigma_2}, & 0 \leq x \leq x_1 + \zeta_1, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})), & x_1 + \zeta_1 \leq x \leq x_1 + \zeta_2, \\ \alpha(x) + \varepsilon^{2-\sigma_2}, & x_1 + \zeta_2 \leq x \leq 1 \end{cases}$$

is a supersolution solution for (P_ε).

In the following proposition we give a subsolution for (P_ε). From the condition (S3)$'$ of Proposition 2.6 for subsolutions, we need slight modification near $x = 0$ and 1.

Proposition 3.2. *Assume that $\alpha'(0) > 0$ and $\alpha'(1) > 0$. Let σ_1 and σ_2 be the same numbers as in Proposition 3.1 and let σ_3 and σ_4 be positive numbers satisfying $\sigma_4 < \sigma_3 < 2\sigma_4$ and $\sigma_3 < \sigma_2$. For sufficiently small $\varepsilon > 0$, there exist $\zeta_3 = \zeta_3(\varepsilon)$ and $\zeta_4 = \zeta_4(\varepsilon)$ such that the following function $u_{*,\varepsilon}$ is a subsolution for (P_ε).*

$$u_{*,\varepsilon}(x) = \begin{cases} -\alpha(x) - \varepsilon^{2-\sigma_2} - \varepsilon^{-\sigma_3}(x - \varepsilon^{\sigma_4})^2, & 0 \le x \le \varepsilon^{\sigma_4}, \\ -\alpha(x) - \varepsilon^{2-\sigma_2}, & \varepsilon^{\sigma_4} \le x \le x_2 + \zeta_3, \\ U(\varepsilon^{-1}(x - x_2) - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1})), & x_2 + \zeta_3 \le x \le x_2 + \zeta_4, \\ \alpha(x) - \varepsilon^{2-\sigma_2}, & x_2 + \zeta_4 \le x \le 1 - \varepsilon^{\sigma_4}, \\ \alpha(x) - \varepsilon^{2-\sigma_2} - \varepsilon^{-\sigma_3}(x - 1 + \varepsilon^{\sigma_4})^2, & 1 - \varepsilon^{\sigma_4} \le x \le 1. \end{cases}$$

In Propositions 3.1 and 3.2 we should say that numbers ζ_1, ζ_2, ζ_3 and ζ_4 are chosen so that $u^*_\varepsilon \in P(x_1 + \zeta_1, x_1 + \zeta_2)$ and $u_{*,\varepsilon} \in P(x_2 + \zeta_3, x_2 + \zeta_4)$ satisfies the condition (S1)$'$ and (S2)$'$ of Proposition 2.6. More precisely, ζ_1 is determined from

$$\begin{aligned} \zeta_1 = \min\{\zeta \in (-x_1, -\varepsilon^{1-\sigma_1}) : \\ -\alpha(x_1 + \zeta) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1}))\} \end{aligned} \tag{3.1}$$

and ζ_2 is a negative number satisfying $\zeta_2 \in (-\varepsilon^{1-\sigma_1}, 0)$ and

$$\alpha(\zeta_2 + x_1) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta_2 + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) \tag{3.2}$$

and ζ_3 is a positive number satisfying

$$-\alpha(x_2 + \zeta_3) + \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta_3 - \varepsilon^{\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1})), \tag{3.3}$$

and ζ_4 is satisfying $\zeta_4 \in (\varepsilon^{1-\sigma_1}, 1 - x_2)$ and

$$\begin{aligned} \zeta_4 = \max\{\zeta \in (\varepsilon^{1-\sigma_1}, 1 - x_2) : \\ \alpha(x_2 + \zeta) - \varepsilon^{2-\sigma_2} = U(\varepsilon^{-1}\zeta - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1}))\}. \end{aligned} \tag{3.4}$$

The following two lemmas assure the unique existence of such ζ_1, ζ_2, ζ_3 and ζ_4.

Lemma 3.3. *For sufficiently small $\varepsilon > 0$, ζ_1 and ζ_4 are uniquely determined from* (3.1) *and* (3.4). *Moreover, they satisfy;*

(1) (i) $-\varepsilon^{1+\sigma_1-\sigma_2-\delta(\varepsilon)} < \zeta_1 + \varepsilon^{1-\sigma_1} < -\varepsilon^{1+\sigma_1-\sigma_2+\delta(\varepsilon)}$,
(ii) $\varepsilon^{1+\sigma_1-\sigma_2-\delta(\varepsilon)} < \zeta_4 - \varepsilon^{1-\sigma_1} < \varepsilon^{1+\sigma_1-\sigma_2+\delta(\varepsilon)}$,
where δ is a positive number such that $\delta = \delta(\varepsilon) = o_\varepsilon(1)$.

(2) (i) $-\alpha'(\zeta_1 + x_1) \ge \varepsilon^{-1}U_z(\varepsilon^{-1}\zeta_1 + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1}))$.
(ii) $\alpha'(x_2 + \zeta_4) \le \varepsilon^{-1}U_z(\varepsilon^{-1}\zeta_4 - \varepsilon^{-\sigma_1}; \alpha(x_2 + \varepsilon^{1-\sigma_1}))$.

Proof. We only prove for ζ_1. Set $F_1(x) := U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) + \alpha(x) - \varepsilon^{2-\sigma_2}$. It follows from (4) of Lemma 2.7 that $U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma}))$ is convex for $x \le x_1 - \varepsilon^{1-\sigma_1}$, so that $F_1(x)$ is also convex in $(x_1 - 2\varepsilon^{1-\sigma_1}, x_1 - \varepsilon^{1-\sigma_1})$. Therefore, $F_1(x)$ has at most two zero-points in $(x_1 - 2\varepsilon^{1-\sigma_1}, x_1 - \varepsilon^{1-\sigma_1})$.

We will prove that a zero-point of $F_1(x)$ indeed exists. Let k be any number such that $k < \sigma_2 - \sigma_1$ and $|k| < 2\sigma_1 - \sigma_2$. Setting $x_k = x_1 - \varepsilon^{1-\sigma_1} - \varepsilon^{1+\sigma_1-\sigma_2+k}$, we get $F_1(x_k) = U(-\varepsilon^{\sigma_1-\sigma_2+k}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) + \alpha(x_k) - \varepsilon^{2-\sigma_2}$. Using the Taylor expansion of α extended for $x > x_1$ in C^2 at x_1 we obtain

$$\begin{aligned} \alpha(x_1 - \varepsilon^{1-\sigma_1}) &= \alpha(x_1) - \alpha'(x_1)\varepsilon^{1-\sigma_1} + \frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_1} + o(\varepsilon^{2-2\sigma_1}) \\ &= \overline{\alpha} + \frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_1} + o(\varepsilon^{2-2\sigma_1}). \end{aligned}$$

If $\varepsilon > 0$ is sufficiently small, then $\varepsilon^{\sigma_1-\sigma_2+k}$ becomes large; so that (2) and (5) of Lemma 2.7 implies

$$\begin{aligned} & U(-\varepsilon^{\sigma_1-\sigma_2+k};\alpha(x_1-\varepsilon^{1-\sigma_1})) \\ = \; & \alpha(x_1-\varepsilon^{1-\sigma_1})U(-\alpha(x_1-\varepsilon^{1-\sigma_1})\varepsilon^{\sigma_1-\sigma_2+k};1) \\ = \; & \alpha(x_1-\varepsilon^{1-\sigma_1})U\left(-\left(\overline{\alpha}+\frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_1}+o(\varepsilon^{2-2\sigma})\right)\varepsilon^{\sigma_1-\sigma_2+k};1\right) \\ = \; & \alpha(x_1-\varepsilon^{1-\sigma_1})(-1+o(\varepsilon^2)). \end{aligned}$$

Hence we have

$$F_1(x_k) = -\alpha(x_1-\varepsilon^{1-\sigma_1})+\alpha(x_1-\varepsilon^{1-\sigma_1}-\varepsilon^{1+\sigma_1-\sigma_2+k})-\varepsilon^{2-\sigma_2}+o(\varepsilon^2).$$

Using the Taylor expansion of α at $x = x_1-\varepsilon^{1-\sigma_1}$ we can show

$$F_1(x_k) = -\alpha'(x_1-\varepsilon^{1-\sigma_1})\varepsilon^{1+\sigma_1-\sigma_2+k}-\varepsilon^{2-\sigma_2}+o(\varepsilon^{2+2\sigma_1-2\sigma_2+2k}).$$

Here we should note

$$-\alpha'(x_1-\varepsilon^{1-\sigma_1})\varepsilon^{1+\sigma_1-\sigma_2+k} = \alpha''(x_1)\varepsilon^{2-\sigma_2+k}+o(\varepsilon^{2-\sigma_2+k}).$$

Therefore, in view of $|k| < 2\sigma_1-\sigma_2$, one can deduce

$$F_1(x_k) = \alpha''(x_1)\varepsilon^{2-\sigma_2+k}-\varepsilon^{2-\sigma_2}+o(\varepsilon^{2-\sigma_2+k}).$$

Let $k > 0$ be fixed. Since $\alpha''(x_1) > 0$, it is easy to see that

$$F_1(x_k) < 0 \text{ and } F_1(x_{-k}) > 0 \tag{3.5}$$

with $x_{-k} < x_k$. Hence we can find ζ_1 such that $F_1(x_1+\zeta_1) = 0$ and $x_1+\zeta_1 \in (x_{-k}, x_k)$. From (3.5),we can see that $\frac{dF_1}{dx}(\zeta_1+x_1) \le 0$, which means that $x_1+\zeta_1$ is the smallest zero-point of F_1 in $(x_1-2\varepsilon^{1-\sigma_1}, x_1-\varepsilon^{1-\sigma_1})$. Clearly ζ_1 satisfies (2). The proof of Lemma 3.3 is completed. □

Lemma 3.4. *For sufficiently small $\varepsilon > 0$, there exist a unique $\zeta_2 \in (-\varepsilon^{1-\sigma_1}, 0)$ and a unique $\zeta_3 \in (0, \varepsilon^{1-\sigma_1})$ satisfying* (3.2) *and* (3.3).

Proof. We only prove for ζ_2. Setting $F_2(x) = \alpha(x)+\varepsilon^{2-\sigma_2}-U(\varepsilon^{-1}(x-x_1)+\varepsilon^{-\sigma_1};\alpha(x_1-\varepsilon^{1-\sigma_1}))$, we have only to show $F_2(x_1) < 0$, $F_2(x_1-\varepsilon^{1-\sigma_1}) > 0$ and $\frac{dF_2}{dx} < 0$ in $(x_1-\varepsilon^{1-\sigma_1}, x_1)$ to get conclusion. By (2) of Lemma 2.7,

$$\begin{aligned} F_2(x_1) &= \alpha(x_1)+\varepsilon^{2-\sigma_2}-U(\varepsilon^{-\sigma_1};\alpha(x_1-\varepsilon^{1-\sigma_1})) \\ &= \alpha(x_1)+\varepsilon^{2-\sigma_2}-\alpha(x_1-\varepsilon^{1-\sigma_1})U(\alpha(x_1-\varepsilon^{1-\sigma_1})\varepsilon^{-\sigma_1};1) \\ &= \alpha(x_1)+\varepsilon^{2-\sigma_2}-\alpha(x_1-\varepsilon^{1-\sigma_1})+O(\exp(-C_2'\varepsilon^{-\sigma_1})) \end{aligned} \tag{3.6}$$

for some $C_2' > 0$ if $\varepsilon > 0$ is sufficiently small. The Taylor expansion of α extended in C^2 for $x_1 > 0$ at x_1 gives

$$F_2(x_1) = \varepsilon^{2-\sigma_2}-\frac{1}{2}\alpha''(x_1)\varepsilon^{2-2\sigma_1}+o(\varepsilon^{2-2\sigma_1}) < 0$$

for sufficiently small $\varepsilon > 0$ because $2\sigma_1 > \sigma_2$ and $\alpha''(x_0) > 0$. And we have

$$\begin{aligned} F_2(x_1-\varepsilon^{1-\sigma_1}) &= \alpha(x_1-\varepsilon^{1-\sigma_1})+\varepsilon^{2-\sigma_2} \\ &= \alpha(x_1)+\frac{\alpha''(x_1)}{2}\varepsilon^{2-2\sigma_2}+\varepsilon^{2-\sigma_1}+o(\varepsilon^{2-2\sigma_1}) > 0. \end{aligned}$$

Finally it follows from (1) of Lemma 2.7 that

$$\frac{d}{dx}F_2(x) = -\alpha'(x) - \varepsilon^{-1}U_z(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})) < 0 \tag{3.7}$$

for $0 < x < x_1$. Thus the proof is completed. □

Proof of Proposition 3.1. First we will show that u_ε^* satisfies the condition (S1)′ of Proposition 2.6 in each interval $J_1 := (0, \zeta_1 + x_1)$, $J_2 := (\zeta_1 + x_1, \zeta_2 + x_1)$, $J_3 := (\zeta_2 + x_1, x_1)$, $J_4 := (x_1, x_2)$, $J_5 := (x_2, 1)$. We set $\Phi_1(u)(x) := \varepsilon^2 u_{xx} + u(\alpha(x)^2 - u^2)$. In J_1 we have

$$\Phi_1(u_\varepsilon^*)(x) = -\varepsilon^2\alpha''(x) - 2\alpha(x)^2\varepsilon^{2-\sigma_2} + O(\varepsilon^{4-2\sigma_2}) < 0$$

if $\varepsilon > 0$ is sufficiently small. The similar inequality holds in J_3, J_4 and J_5. In the remaining interval J_2 we have

$$\Phi_1(u_\varepsilon^*) = U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_0 - \varepsilon^{1-\sigma_1}))\{\alpha(x)^2 - \alpha(x_1 - \varepsilon^{1-\sigma_1})^2\}. \tag{3.8}$$

Here we observe

$$\begin{cases} U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{-\sigma_1})) > 0 & \text{and } \alpha(x) < \alpha(x_1 - \varepsilon^{1-\sigma_1}), \\ & \text{if } x_1 - \varepsilon^{1-\sigma_1} < x < x_1 + \zeta_2, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{-\sigma_1})) < 0 & \text{and } \alpha(x) > \alpha(x_1 - \varepsilon^{1-\sigma_1}), \\ & \text{if } x_1 + \zeta_1 < x < x_1 - \varepsilon^{1-\sigma_1}, \end{cases}$$

for sufficiently small $\varepsilon > 0$. Therefore, the right-hand side of (3.8) is negative, so that (S1)′ is verified in each J_i ($i = 1, 2, 3, 4, 5$). By (2) of Lemma 3.3 and (3.7), it is easy to verify (S2)′ at $x = x_1 + \zeta_1$ and $x = x_1 + \zeta_2$ from Lemmas 3.3 and 3.4 and at $x = x_1$ and $x = x_2$ since α is C^1 function. Finally, (S3)′ comes from the assumption that $\alpha'(0) \geq 0$ and $\alpha'(1) \geq 0$. Thus we have proved that u_ε^* is a supersolution for (P_ε). □

When α satisfies $\alpha'(0) < 0$ or $\alpha'(1) < 0$, u_ε^* in Proposition 3.1 does not satisfy (S3) or (S3)′. Therefore, we have to modify u_ε^* near $x = 0$ or $x = 1$ as in Proposition 3.2. The following proposition deals with typical case $\alpha'(0) < 0$ and $\alpha'(1) < 0$.

Proposition 3.5. *Assume $\alpha'(0) < 0$ and $\alpha'(1) < 0$. Let σ_1 and σ_2 be the same numbers as in Proposition 3.1 and let σ_3 and σ_4 be positive numbers satisfying $\sigma_4 < \sigma_3 < 2\sigma_4$ and $\sigma_3 < \sigma_2$. For sufficiently small $\varepsilon > 0$, the following function*

$$u_\varepsilon^*(x) = \begin{cases} -\alpha(x) + \varepsilon^{2-\sigma_2} + \varepsilon^{-\sigma_3}(x - \varepsilon^{\sigma_4})^2, & 0 \leq x \leq \varepsilon^{\sigma_4}, \\ -\alpha(x) + \varepsilon^{2-\sigma_2}, & \varepsilon^{\sigma_4} \leq x \leq x_1 + \zeta_1, \\ U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1}; \alpha(x_1 - \varepsilon^{1-\sigma_1})), & x_1 + \zeta_1 \leq x \leq x_1 + \zeta_2, \\ \alpha(x) + \varepsilon^{2-\sigma_2}, & x_1 + \zeta_2 \leq x \leq 1 - \varepsilon^{\sigma_4}, \\ \alpha(x) + \varepsilon^{2-\sigma_2} + \varepsilon^{-\sigma_3}(x - 1 + \varepsilon^{\sigma_4})^2, & 1 - \varepsilon^{\sigma_4} \leq x \leq 1 \end{cases}$$

is a supersolution for (P_ε). *Here ζ_1 and ζ_2 are the same constants as in Proposition 3.1.*

Remark . The same conclusion is valid with obvious modification in case $\alpha'(0) < 0$ and $\alpha'(1) \geq 0$ or $\alpha'(0) \geq 0$ and $\alpha'(1) < 0$.

Since for the subsolution, we can prove similarly, we omit the proof for the subsolution. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We begin with the proof for the case $L = \{x_1, x_2\}$, $K = \hat{K} = \{I_1\}$. Let $\Omega_1 = (0, x_1)$, $\Omega_2 = (x_2, 1)$ (Case(I)). By Proposition 3.1 and Proposition 3.5, we can show that there exists a supersolution u_ε^* with the following properties.

$$\begin{cases} \alpha(x) < u_\varepsilon^*(x) < \alpha(x) + \varepsilon^\theta & \text{for} \quad x_2 \le x \le 1, \\ -\alpha(x) < u_\varepsilon^*(x) < -\alpha(x) + \varepsilon^\theta & \text{for} \quad 0 \le x \le x_1 - 2\varepsilon^{1-\sigma_1}, \end{cases} \tag{3.9}$$

for sufficiently small $\varepsilon > 0$ with $\theta = \min\{2 - \sigma_2, 2\sigma_4 - \sigma_3\}$. Similarly, Proposition 3.2 allows us to find a subsolution $u_{*,\varepsilon}$ satisfying

$$\begin{cases} \alpha(x) - \varepsilon^\theta < u_{*,\varepsilon}(x) < \alpha(x) & \text{for} \quad x_2 + 2\varepsilon^{1-\sigma_1} \le x \le 1, \\ -\alpha(x) - \varepsilon^\theta < u_{*,\varepsilon}(x) < -\alpha(x) & \text{for} \quad 0 \le x \le x_1. \end{cases} \tag{3.10}$$

Since Proposition 2.2 assures the existence of a solution u_ε such that $u_{*,\varepsilon} < u_\varepsilon < u_\varepsilon^*$, it follows that u_ε satisfies

$$\begin{cases} |u_\varepsilon(x) - \alpha(x)| < \varepsilon^\theta & \text{for} \quad x_2 + 2\varepsilon^{1-\sigma_1} \le x \le 1, \\ |u_\varepsilon(x) + \alpha(x)| < \varepsilon^\theta & \text{for} \quad 0 \le x \le x_1 - 2\varepsilon^{1-\sigma_1}. \end{cases} \tag{3.11}$$

Moreover u_ε is a local minimizer of the functional J_ε on $H^1(0,1)$. Thus the proof is completed for the case (I).

For the case (II), we have only to consider (P_ε) with u and α replaced by $v(x) := u(1-x)$ and $\beta(x) := \alpha(1-x)$. Repeating the above procedure we see the existence of solution v for

$$\begin{cases} \varepsilon^2 v_{xx}(x) + v(\beta(x)^2 - v^2) = 0 \quad \text{in } (0,1), \\ v_x(0) = v_x(1) = 0, \end{cases}$$

such that v satisfies (3.11) with α, x_1 and x_2 replaced by β, $1 - x_1$ and $1 - x_2$. Therefore $u(x) = v(1-x)$ becomes the desired solution. Moreover u is a local minimizer of the functional J_ε on $H^1(0,1)$.

Finally we study the case when the number of elements of the set $\hat{K}$ is larger than or equal to 2. Set $\hat{K} = \{I_1', I_2', \cdots, I_l'\}$, $I_i' = [x_{2i}', x_{2i+1}']$ and $x_0' = 0$ and $x_{2l+1}' = 1$. We prove for the case (I), that is the case when

$$\Omega_1 = \bigcup_{i=0}^{[\frac{2l-1}{2}]} (x_{4i}', x_{4i+1}'), \ \Omega_2 = \bigcup_{i=0}^{[\frac{2l-3}{2}]} (x_{4i+2}', x_{4i+3}').$$

The preceding arguments enable us to construct a suitable supersolution u_ε^* and a suitable subsolution $u_{*,\varepsilon}$. For example, in (x_1', x_2'), the supersolution u_ε^* is defined by $u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$ and the subsolution $u_{*,\varepsilon}$ is defined by $u_{*,\varepsilon} = -\alpha(x) - \varepsilon^{2-\sigma_2}$ and in (x_2', x_3') the supersolution u_ε^* is defined by $u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$ and the subsolution $u_{*,\varepsilon}$ is defined by $u_{*,\varepsilon}(x) = \alpha(x) - \varepsilon^{2-\sigma_2}$ in $(x_2' + 2\varepsilon^{1-\sigma_1}, x_3' - 2\varepsilon^{1-\sigma_1})$ with some $\sigma_1 > 0$ and $\sigma_2 > 0$. We observe that, if x lies in the neighborhood of x_{4i}' $(i \ge 1)$ and x_{4i+1}' $(i \ge 0)$, then u_ε^* is defined with use of suitable dilation, translation or reflection of the solution U of (2.7) and if x lies in a neighborhoods of x_{4i+2}' and x_{4i+3}' $(i \ge 0)$, $u_{*,\varepsilon}$ is defined with use of suitable dilation, translation or reflection of solution U of (2.7). Thus the proof is completed. □

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Proof of Theorem 1.2. For the sake of simplicity, we only prove for the case when $K = \hat{K} = \{I_1\}$ and $I_1 = [x_1, x_2]$ and $\Omega_1 = (0, x_1)$ (Case(I)).

Let u_ε be the solution obtained by Theorem 1.1 and fix $\eta > 0$ sufficiently small. We put $\overline{\alpha} = \alpha(x_1) = \alpha(x_2)$. Since $u_\varepsilon(x) < -\alpha(x) + \varepsilon^{2-\sigma_2} < -\overline{\alpha} + \varepsilon^{2-\sigma_2}$ for $x \in [0, x_1 - 2\varepsilon^{1-\sigma_1}]$, for sufficiently small $\varepsilon > 0$ we have $u_\varepsilon(x) < -\overline{\alpha} + \eta$ for $x \in [0, x_1 - 2\varepsilon^{1-\sigma_1}]$. Similarly we have $u_\varepsilon(x) > \overline{\alpha} - \eta$ for $x \in [x_2 + 2\varepsilon^{1-\sigma_1}, 1]$. Hence we can define the followings

$$\begin{aligned} \overline{x}_\varepsilon &= \inf\{x > x_1 - 2\varepsilon^{1-\sigma_1} | u_\varepsilon(x) = -\overline{\alpha} + \eta\}, \\ \widetilde{x}_\varepsilon &= \sup\{x < x_2 + 2\varepsilon^{1-\sigma_1} | u_\varepsilon(x) = \overline{\alpha} - \eta\}. \end{aligned}$$

We may assume that $\overline{x}_\varepsilon \to \overline{x} \in [x_1, x_2]$ and $\widetilde{x}_\varepsilon \to \widetilde{x} \in [x_1, x_2]$. Now we let $v_\varepsilon(t) = u_\varepsilon(\overline{x}_\varepsilon + \varepsilon t)$. Then we have

$$\begin{aligned} -v_\varepsilon'' &= v(\alpha(\overline{x}_\varepsilon + \varepsilon t)^2 - v^2), \\ v_\varepsilon(0) &= -\overline{\alpha} + \eta. \end{aligned}$$

Since $\{v_\varepsilon\}$ are uniformly bounded in L^∞ and $\overline{x}_\varepsilon \to \overline{x} \in [x_1, x_2]$, it is easy to see that $v_\varepsilon \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R})$, and

$$-v'' = v(\overline{\alpha}^2 - v^2) \quad t \in \mathbb{R}.$$

Since it is easily seen that for any $t \leq 0$, $u_\varepsilon(\overline{x}_\varepsilon + \varepsilon t) \leq -\overline{\alpha} + \eta$ for sufficiently small $\varepsilon > 0$, we can obtain $v \leq -\overline{\alpha} + \eta$ for $t \leq 0$. Hence by Lemma 2.8, v satisfies $v'(t) > 0$ and $v(t) \to \pm\overline{\alpha}$ as $t \to \pm\infty$. As a result, we can find a $R > 0$ large, such that $v(R) = \overline{\alpha} - \eta$. Thus, there is a $R_\varepsilon \in (R-1, R+1)$, such that $v_\varepsilon'(t) < 0$ if $t \in [0, R_\varepsilon]$ and $v_\varepsilon(R_\varepsilon) = -\overline{\alpha} - \eta$. Indeed, since $v(R+1) > \overline{\alpha} - \eta$, $v(R-1) < \overline{\alpha} - \eta$, $v'(t) > 0$ on $\mathbb{R}$ and v_ε to v in $C^1([R-1, R+1])$, for sufficiently small $\varepsilon > 0$, $v_\varepsilon(R+1) > \overline{\alpha} - \eta$, $v_\varepsilon(R-1) < \overline{\alpha} - \eta$ and $v_\varepsilon' > 0$ on $[R-1, R+1]$. Hence there exists the desired $R_\varepsilon \in (R-1, R+1)$. We may assume that $R_\varepsilon \to R$. Therefore, $u_\varepsilon'(x) > 0$ if $x \in [\overline{x}_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon]$ and $u_\varepsilon(\overline{x}_\varepsilon + \varepsilon R_\varepsilon) = \overline{\alpha} - \eta$.

Claim. $\widetilde{x}_\varepsilon = \overline{x}_\varepsilon + \varepsilon R_\varepsilon$.

Suppose that the claim is not true. Then we can find a $t_\varepsilon > \overline{x}_\varepsilon + \varepsilon R_\varepsilon$, such that $u_\varepsilon(x) > \overline{\alpha} - \eta$ for $x \in (\overline{x}_\varepsilon + \varepsilon R_\varepsilon, t_\varepsilon)$, $u_\varepsilon(t_\varepsilon) = \overline{\alpha} - \eta$ and $u_\varepsilon'(t_\varepsilon) \leq 0$. Note that $\lim_{\varepsilon\to 0} \varepsilon^{-1}(t_\varepsilon - (\overline{x}_\varepsilon + \varepsilon R_\varepsilon)) = +\infty$. Indeed if it is not satisfied, there exist $R_\varepsilon' \geq 0$ such that $t_\varepsilon - (\overline{x}_\varepsilon + \varepsilon R_\varepsilon) = \varepsilon R_\varepsilon'$ and $\sup_\varepsilon R_\varepsilon' < +\infty$. We may assume that $R_\varepsilon' \to R'$ for some $R' \geq 0$. Since $v' > 0$ on $\mathbb{R}$ and $v_\varepsilon \to v$ in $C^1([0, R+R'+1])$, for sufficiently small $\varepsilon > 0$, $v_\varepsilon(R_\varepsilon + R_\varepsilon') = \overline{\alpha} - \eta$ and $v_\varepsilon'(R_\varepsilon + R_\varepsilon') > 0$. Hence we have $u_\varepsilon'(t_\varepsilon) = u_\varepsilon'(\overline{x}_\varepsilon + \varepsilon(R_\varepsilon + R_\varepsilon')) = \varepsilon^{-1} v_\varepsilon'(R_\varepsilon + R_\varepsilon') > 0$. This contradict to $u_\varepsilon'(t_\varepsilon) \leq 0$. Let $\overline{v}_\varepsilon(t) = u_\varepsilon(t_\varepsilon + \varepsilon t)$. It is easy to check that $\overline{v}_\varepsilon \to \overline{v}$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ and $\overline{v}$ satisfies

$$\begin{cases} -\overline{v}'' = \overline{v}(\overline{\alpha}^2 - \overline{v}^2) & \text{in } \mathbb{R}, \\ \overline{v}(0) = \overline{\alpha} - \eta. \end{cases}$$

Let $t \leq 0$. Since $\lim_{\varepsilon\to 0} \varepsilon^{-1}(t_\varepsilon - (\overline{x}_\varepsilon + \varepsilon R_\varepsilon)) = +\infty$, $t_\varepsilon + \varepsilon t > \overline{x}_\varepsilon + \varepsilon R_\varepsilon$ for sufficiently small $\varepsilon > 0$. Hence we obtain $\overline{v}(t) \geq \overline{\alpha} - \eta$ for $t \leq 0$ and $\overline{v} \to \mp\alpha$ as $t \to \pm\infty$ by Lemma 2.8. Hence there exists $\widetilde{R} > 0$ such that $\overline{v}(\widetilde{R}) = -\overline{\alpha} + \eta$. Thus, there is $\widetilde{R}_\varepsilon \in (R-1, R+1)$, such that $\overline{v}'(t) < 0$ if $t \in [0, \widetilde{R}_\varepsilon]$ and $\overline{v}(R_\varepsilon) = -\overline{\alpha} + \eta$. Therefore, $u_\varepsilon'(x) < 0$ if $x \in [t_\varepsilon, t_\varepsilon + \varepsilon \widetilde{R}_\varepsilon]$ and $u_\varepsilon(t_\varepsilon + \varepsilon R_\varepsilon) = -\overline{\alpha} + \eta$. We will analyze the energy of u_ε on $[\overline{x}_\varepsilon, \widetilde{x}_\varepsilon]$ to lead to a contradiction.

Since the energy functional correspond to the problem (P_ε) is

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2}|u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4} dx - \int_0^1 \frac{\alpha(x)^4}{4} dx$$

and the term $\int_0^1 \frac{\alpha(x)^4}{4} dx$ is independent of u, we can replace the energy functional J_ε by

$$J_\varepsilon(u) = \int_0^1 \frac{\varepsilon^2}{2}|u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4} dx.$$

Since by Proposition 2.2, we have

$$J_\varepsilon(u_\varepsilon) \le J_\varepsilon(\varphi)$$

for $\varphi \in H^1(0,1)$ with $u_{*,\varepsilon} \le \varphi \le u_\varepsilon^*$, for any $y_2 > y_1$, u_ε is a minimizer of the following problem

$$\inf\{J_\varepsilon(u,(y_1,y_2)) : u - u_\varepsilon \in H_0^1(y_1,y_2), u_{*,\varepsilon} \le u \le u_\varepsilon^*\}, \tag{4.1}$$

where

$$J_\varepsilon(u,M) = \int_M \frac{\varepsilon^2}{2}|u'|^2 + \frac{(\alpha(x)^2 - u^2)^2}{4}dx$$

for any open set M. Let m_{ε,y_1,y_2} denote the minimum value of the problem (4.1). We will obtain the lower bound and the upper bound for $m_{\varepsilon,\overline{x}_\varepsilon,\overline{x}_\varepsilon}$. At first we obtain the lower bound. First we have

$$\begin{aligned} & J_\varepsilon(u_\varepsilon,(\overline{x}_\varepsilon,\overline{x}_\varepsilon+\varepsilon R_\varepsilon)) \\ = & \int_{\overline{x}_\varepsilon}^{\overline{x}_\varepsilon+\varepsilon R_\varepsilon} \left(\frac{\varepsilon^2}{2}|u_\varepsilon'|^2 + \frac{(\alpha(x)^2 - u_\varepsilon^2)^2}{4}\right) dx \\ = & \varepsilon \int_0^{R_\varepsilon} \left(\frac{1}{2}|v_\varepsilon'|^2 + \frac{(\alpha(\overline{x}_\varepsilon+\varepsilon t)^2 - v_\varepsilon^2)^2}{4}\right) dt \\ = & (\beta + O(\eta) + o_\varepsilon(1))\varepsilon \end{aligned} \tag{4.2}$$

where

$$\beta = \int_{-\infty}^{+\infty} \left(\frac{1}{2}|v'(t)|^2 + \frac{(\overline{\alpha}^2 - v^2)^2}{4}\right) dt.$$

Indeed, we have

$$\begin{aligned} & \int_0^{R_\varepsilon} \frac{1}{2}|v_\varepsilon'|^2 + \frac{(\alpha(\overline{x}_\varepsilon+\varepsilon t)^2 - v_\varepsilon^2)^2}{4}dt \\ = & \int_0^{R} \frac{1}{2}|v'|^2 + \frac{(\overline{\alpha}^2 - v^2)^2}{4}dt + o_\varepsilon(1). \end{aligned} \tag{4.3}$$

Next we remark that

$$\int_0^R \frac{1}{2}|v'|^2 + \frac{(\overline{\alpha}^2 - v^2)^2}{4}dt = \int_{-R(\eta)}^{R(\eta)} \frac{1}{2}U'(t;\overline{\alpha})^2 + \frac{(\overline{\alpha}^2 - U(t;\overline{\alpha})^2)^2}{4}dt,$$

where $R(\eta)$ is the unique positive number satisfies $U(R(\eta);\overline{\alpha}) = \overline{\alpha} - \eta$. We claim that

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_{R(\eta)}^{+\infty} \frac{1}{2}U'(t;\overline{\alpha})^2 + \frac{(\overline{\alpha}^2 - U(t;\overline{\alpha})^2)^2}{4}dt < +\infty.$$

By using L'Hospital's rule, we only have to show is that for

$$\begin{aligned} S(\eta) \; &:= \; \frac{\partial}{\partial\eta}\int_{R(\eta)}^{+\infty} \frac{1}{2}U'(t;\overline{\alpha})^2 + \frac{(\overline{\alpha}^2 - U(t;\overline{\alpha})^2)^2}{4}dt \\ &= \; -\left(\frac{U'(R(\eta);\overline{\alpha})^2}{2} + \frac{(\overline{\alpha}^2 - U(R(\eta);\overline{\alpha})^2)^2}{4}\right) R_\eta(\eta), \end{aligned} \tag{4.4}$$

the following limit exists

$$\lim_{\eta\to 0} S(\eta) < +\infty.$$

Multiplying the equation

$$-U''(t;\overline{\alpha}) = (\overline{\alpha}^2 U(t;\overline{\alpha}) - U(t;\overline{\alpha})^3)$$

by $U'(t;\overline{\alpha})$ and integrating over $[R(\eta), +\infty)$ we have

$$\frac{U'(R(\eta);\overline{\alpha})^2}{2} = \frac{(\overline{\alpha}^2 - U(R(\eta);\overline{\alpha})^2)^2}{4} = \frac{\eta^2(2\overline{\alpha}-\eta)^2}{4}.$$

Hence we obtain

$$|U'(R(\eta);\overline{\alpha})| = \frac{1}{2}\eta|2\overline{\alpha}-\eta|. \tag{4.5}$$

Next we remark that from $U(R(\eta);\overline{\alpha}) = \overline{\alpha} - \eta$ we have

$$U'(R(\eta);\overline{\alpha})R_\eta(\eta) = -1. \tag{4.6}$$

From (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned}
|S(\eta)| &\le \frac{\eta^2(2\overline{\alpha}-\eta)^2}{2}|R_\eta(\eta)| \\
&= \frac{\eta^2(2\overline{\alpha}-\eta)^2}{2}\cdot\frac{1}{|U'(R(\eta);\overline{\alpha})|} \\
&= \frac{\eta^2(2\overline{\alpha}-\eta)^2}{2}\cdot\frac{2}{\eta|2\overline{\alpha}-\eta|} \\
&= \eta|2\overline{\alpha}-\eta|.
\end{aligned}$$

From (4.3) and above estimate we obtain (4.2).

Similarly we have

$$J_\varepsilon(u_\varepsilon, (\tilde{t}_\varepsilon, \tilde{t}_\varepsilon + \varepsilon\widetilde{R}_\varepsilon)) = (\beta + O(\eta) + o_\varepsilon(1))\varepsilon \tag{4.7}$$

Hence from (4.2), (4.7), we obtain

$$\begin{aligned}
m_{\varepsilon,\overline{x}_\varepsilon,\widetilde{x}_\varepsilon} &= J_\varepsilon(u_\varepsilon, (\overline{x}_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon)) + J(u_\varepsilon, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon, t_\varepsilon)) \\
&\quad + J_\varepsilon(u_\varepsilon, (t_\varepsilon, t_\varepsilon + \varepsilon\widetilde{R}_\varepsilon)) + J_\varepsilon(u_\varepsilon, (t_\varepsilon + \varepsilon\widetilde{R}_\varepsilon, \widetilde{x}_\varepsilon)) \\
&\ge 2(\beta + O(\eta) + o_\varepsilon(1))\varepsilon.
\end{aligned}$$

Now we give the upper bound for $m_{\varepsilon,\overline{x}_\varepsilon,\widetilde{x}_\varepsilon}$. We define the following function $\overline{u}_\varepsilon$

$$\overline{u}_\varepsilon(x) := \begin{cases}
u_\varepsilon(x) & \text{if } x \in [\overline{x}_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon], \\
\frac{u_\varepsilon^*(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) - (\overline{\alpha} - \eta)}{\varepsilon}(x - (\overline{x}_\varepsilon + \varepsilon R_\varepsilon)) + \overline{\alpha} - \eta, & \\
\qquad\qquad \text{if } x \in [\overline{x}_\varepsilon + \varepsilon R_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon], & \\
u_\varepsilon^*(x) & \text{if } x \in [\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \widetilde{x}_\varepsilon - \varepsilon]. \\
\frac{(\overline{\alpha} - \eta) - u_\varepsilon^*(\widetilde{x}_\varepsilon - \varepsilon)}{\varepsilon}(x - \widetilde{x}_\varepsilon) + u_\varepsilon^*(\widetilde{x}_\varepsilon - \varepsilon) & \text{if } x \in [\widetilde{x}_\varepsilon - \varepsilon, \widetilde{x}_\varepsilon]
\end{cases}$$

We note that the function $\overline{u}_\varepsilon$ satisfies $u_{\varepsilon,*} \le u_\varepsilon \le \overline{u}_\varepsilon^*$ and $\overline{x}_\varepsilon + \varepsilon R_\varepsilon > x_1 - \varepsilon^{1-\sigma_1}$ hold. We estimate $J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon, \widetilde{x}_\varepsilon))$. We only consider the most delicate case when $\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon < x_1 + \zeta_2 < \widetilde{x}_\varepsilon - \varepsilon$. In other case, it can be estimated more easily. First we note that from (4.2) we have

$$J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon)) = J_\varepsilon(u_\varepsilon, (\overline{x}_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon)) = \varepsilon(\beta + O(\eta) + o_\varepsilon(1)). \tag{4.8}$$

Next we estimate $J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon))$. Since $\overline{\alpha} - \eta < u_\varepsilon^*(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) < \alpha(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) + \varepsilon^{2-\sigma_2}$, we have $0 < u_\varepsilon^*(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon) - (\overline{\alpha} - \eta) < \eta + O(\varepsilon^{2-2\sigma_1}) + \varepsilon^{2-\sigma_2}$. Hence

$\overline{u}_\varepsilon'(x) = \frac{\eta+O(\varepsilon^{2-2\sigma_1})+\varepsilon^{2-\sigma_2}}{\varepsilon}$. Thus we have

$$\begin{aligned}\int_{\overline{x}_\varepsilon+\varepsilon R_\varepsilon}^{\overline{x}_\varepsilon+\varepsilon R_\varepsilon+\varepsilon} \frac{\varepsilon^2}{2}|\overline{u}_\varepsilon'|^2 dx &= O(\varepsilon\eta) + \varepsilon O(\varepsilon^{2-2\sigma_1}) + \varepsilon \cdot \varepsilon^{2-\sigma_2} \\ &= O(\varepsilon\eta) + o(\varepsilon).\end{aligned}$$

Since $\overline{\alpha} - \eta \le \overline{u}_\varepsilon \le \alpha(x) + \varepsilon^{2-\sigma_2}$ on $[\overline{x}_\varepsilon + \varepsilon R_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon]$, we have

$$\begin{aligned}(\overline{\alpha} - \eta)^2 - \alpha(x)^2 \le \overline{u}_\varepsilon(x)^2 - \alpha(x)^2 \le O(\varepsilon^{2-\sigma_2}), \\ -2\eta\overline{\alpha} + \eta^2 + O(\varepsilon^{2-2\sigma_1}) \le \overline{u}_\varepsilon(x)^2 - \alpha(x)^2 \le O(\varepsilon^{2-\sigma_2}).\end{aligned}$$

Hence we have

$$(\alpha(x)^2 - \overline{u}_\varepsilon(x)^2)^2 \le O(\varepsilon^{4-2\sigma_2}) + O(\eta) + O(\varepsilon^{2-2\sigma_1})$$

and

$$\begin{aligned}\int_{\overline{x}_\varepsilon+\varepsilon R_\varepsilon}^{\overline{x}_\varepsilon+\varepsilon R_\varepsilon+\varepsilon} \frac{(\alpha(x)^2 - \overline{u}_\varepsilon(x)^2)^2}{4} dx &= O(\varepsilon\eta) + \varepsilon(O(\varepsilon^{4-2\sigma_2}) + O(\varepsilon^{2-2\sigma_1})) \\ &= O(\varepsilon\eta) + o(\varepsilon).\end{aligned}$$

Thus we obtain

$$J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon, \overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon)) \le O(\varepsilon\eta) + o(\varepsilon). \tag{4.9}$$

Next we estimate $J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \widetilde{x}_\varepsilon - \varepsilon))$. Since we assume $\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon < x_1 + \zeta_2 < \widetilde{x}_\varepsilon - \varepsilon$, we divide the interval $(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, \widetilde{x}_\varepsilon - \varepsilon)$ to $(\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)$ and $(x_1 + \zeta_2, \widetilde{x}_\varepsilon - \varepsilon)$. We set $V(x) = U(\varepsilon^{-1}(x - x_1) + \varepsilon^{-\sigma_1} : \alpha(x_1 - \varepsilon^{1-\sigma}))$. So we have

$$\begin{aligned}&J_\varepsilon(\overline{u}_\varepsilon, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ =\ & J_\varepsilon(u_\varepsilon^*, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ =\ & J_\varepsilon(V, (\overline{x}_\varepsilon + \varepsilon R_\varepsilon + \varepsilon, x_1 + \zeta_2)) \\ \le\ & \int_{x_1-\varepsilon^{1-\sigma_1}}^{x_1} \frac{\varepsilon^2 V'(x)^2}{2} + \frac{(\alpha(x)^2 - V(x)^2)^2}{4} dx \\ =\ & \varepsilon \int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2}{2} dt \\ & +\varepsilon \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1 + \varepsilon t - \varepsilon^{1-\sigma_1})^2 - U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1})^2)^2}{4} dt \\ =\ & \varepsilon\left(\frac{\beta}{2} + o_\varepsilon(1)\right).\end{aligned} \tag{4.10}$$

Last equality follows from (5) of Lemma 2.7 if $0 < \sigma_1 < \frac{2}{3}$. Indeed, first we note that from (5) of Lemma 2.7 we have $U(t, \alpha(x_1 - \varepsilon^{1-\sigma_1})) = \alpha(x_1 - \varepsilon^{1-\sigma_1})U(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)$ and $U'(t, \alpha(x_1 - \varepsilon^{1-\sigma_1}))^2 = \alpha(x_1 - \varepsilon^{1-\sigma_1})^4 U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)$. Similarly $U'(t, \overline{\alpha})^2 = \overline{\alpha}^4 U'(\overline{\alpha}t, 1)^2$. Hence we obtain

$$\begin{aligned}&U'(t, \alpha(x_1 - \varepsilon))^2 - U'(t, \overline{\alpha})^2 \\ =\ & \alpha(x_1 - \varepsilon^{1-\sigma_4})^4 (U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)^2 - U'(\overline{\alpha}t, 1)^2) \\ & +(\alpha(x_1 - \varepsilon^{1-\sigma_1})^4 - \overline{\alpha}^4)U'(\overline{\alpha}t, 1)^2.\end{aligned}$$

From (3) of Lemma 2.7, we have

$$|U'(\alpha(x_1 - \varepsilon^{1-\sigma_1})t, 1)^2 - U'(\overline{\alpha}t, 1)^2| \le C\exp(-C't)$$

for some $C, C' > 0$ independent of $\varepsilon > 0$. Hence we obtain

$$\int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t,\alpha(x_1-\varepsilon^{1-\sigma_1}))^2}{2}dt \to \int_0^{\infty} \frac{U'(t,\overline{\alpha})^2}{2}dt \text{ as } \varepsilon \to 0.$$

Next we estimate

$$\int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^2 - U(t,\alpha(x_1-\varepsilon^{1-\sigma_1})^2)^2}{4}dx.$$

First we note that

$$\begin{aligned}
& (\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^2 - U(t,\alpha(x_1-\varepsilon^{1-\sigma_1})^2)^2 - (\overline{\alpha}^2 - U(t,\overline{\alpha})^2)^2 \\
= \; & \alpha(\#)^4 - 2\alpha(\#)^2 U(t,\alpha(\natural))^2 + U(t,\alpha(\natural))^4 \\
& -(\overline{\alpha}^4 - 2\overline{\alpha}^2 U(t,\overline{\alpha})^2 + U(t,\overline{\alpha})^4) \\
= \; & (\alpha(\#)^4 - \overline{\alpha}^4) - 2\alpha(\#)^2(U(t,\alpha(\natural))^2 - U(t,\overline{\alpha})^2) \\
& -2(\alpha(\#)^2 - \overline{\alpha}^2)U(t,\overline{\alpha})^2 + (U(t,\alpha(\natural))^4 - U(t,\overline{\alpha})^4),
\end{aligned}$$

where $\alpha(\#) = \alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})$ and $\alpha(\natural) = \alpha(x_1-\varepsilon^{1-\sigma_1})$. Since by (2) of Lemma 2.7, we only have to estimate the term $\alpha(\#)^4 - \overline{\alpha}^4$.

Since $\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1}) = \overline{\alpha} + O(\varepsilon^{2-2\sigma_1})$ for $t \in [0, \varepsilon^{-\sigma_1}]$, we have $\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^4 - \overline{\alpha}^4 = O(\varepsilon^{2-2\sigma_1})$ and

$$\int_0^{\varepsilon^{-\sigma_1}} |\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^4 - \overline{\alpha}^4| dt = O(\varepsilon^{2-3\sigma_1}).$$

If we take $0 < \sigma_1 < \frac{2}{3}$, we obtain

$$\int_0^{\varepsilon^{-\sigma_1}} |\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^4 - \overline{\alpha}^4| dt = o_\varepsilon(1).$$

Hence we can conclude that

$$\begin{aligned}
& \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^2 - U(t,\alpha(x_1-\varepsilon^{1-\sigma_1})^2)^2}{2}dt \\
= \; & \int_0^{\infty} \frac{(\overline{\alpha}^2 - U(t,\overline{\alpha})^2)^2}{4}dt + o_\varepsilon(1)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{\varepsilon^{-\sigma_1}} \frac{U'(t,\alpha(x_1-\varepsilon^{1-\sigma_1}))^2}{2}dt \\
& \quad + \int_0^{\varepsilon^{-\sigma_1}} \frac{(\alpha(x_1+\varepsilon t-\varepsilon^{1-\sigma_1})^2 - U(t,\alpha(x_1-\varepsilon^{1-\sigma_1})^2)^2}{4}dt \\
= \; & \left(\frac{\beta}{2} + o_\varepsilon(1)\right).
\end{aligned}$$

Next we estimate $J_\varepsilon(\overline{u}_\varepsilon, (x_1+\zeta_2, \widetilde{x}_\varepsilon - \varepsilon))$. We note that on $(x_1+\zeta_2, \widetilde{x}_\varepsilon - \varepsilon)$, $\overline{u}_\varepsilon(x) = u_\varepsilon^*(x) = \alpha(x) + \varepsilon^{2-\sigma_2}$ and $\alpha'(x) = O(\varepsilon^{2-2\sigma_1})$. Hence we have

$$J_\varepsilon(\overline{u}_\varepsilon, (x_1+\zeta_2, \widetilde{x}_\varepsilon - \varepsilon))$$

$$
\begin{aligned}
&= J_\varepsilon(\alpha(x)+\varepsilon^{2-\sigma_2},(x_1+\zeta_2,\widetilde{x}_\varepsilon-\varepsilon))\\
&= \int_{x_1+\zeta_2}^{\widetilde{x}_\varepsilon-\varepsilon}\frac{\varepsilon^2}{2}\alpha'(x)^2+\frac{(\alpha(x)^2-(\alpha(x)+\varepsilon^{2-\sigma_2})^2)^2}{4}dx\\
&= \int_{x_1+\zeta_2}^{\widetilde{x}_\varepsilon-\varepsilon}O(\varepsilon^{6-4\sigma_1})+O(\varepsilon^{4-2\sigma_2})dx\\
&= O(\varepsilon^{6-4\sigma_1})+O(\varepsilon^{4-2\sigma_2})\\
&= o(\varepsilon) \qquad (4.11)
\end{aligned}
$$

for $0<\sigma_2<\frac{3}{2}$. Finally we can estimate $J_\varepsilon(\overline{u}_\varepsilon,(\widetilde{x}_\varepsilon-\varepsilon,\widetilde{x}_\varepsilon))$ similarly as in $J_\varepsilon(\overline{u}_\varepsilon,(\overline{x}_\varepsilon+\varepsilon R_\varepsilon+\varepsilon))$, that is we can obtain

$$J_\varepsilon(\overline{u}_\varepsilon,(\widetilde{x}_\varepsilon-\varepsilon,\widetilde{x}_\varepsilon))\le O(\varepsilon\eta)+o(\varepsilon). \qquad (4.12)$$

As a result, from (4.8), (4.9), (4.10), (4.11) and (4.12) we obtain

$$m_{\varepsilon,\overline{x}_\varepsilon,\widetilde{x}_\varepsilon}\le\varepsilon\left(\beta+\frac{\beta}{2}\right)+O(\varepsilon\eta)+o(\varepsilon). \qquad (4.13)$$

Combining (4.8) and (4.13), we are led to

$$2\beta\varepsilon+O(\varepsilon\eta)+o(\varepsilon)\le\frac{3}{2}\beta\varepsilon+O(\varepsilon\eta)+o(\varepsilon)$$

and

$$\frac{\beta}{2}\le\frac{O(\varepsilon\eta)}{\varepsilon\eta}\eta+\frac{o(\varepsilon)}{\varepsilon}.$$

Since $O(\varepsilon\eta)/\varepsilon\eta$ is bounded, we can take $\eta>0$ so small that

$$\frac{O(\varepsilon\eta)}{\varepsilon\eta}\eta<\frac{\beta}{3}.$$

This is a contradiction for $\varepsilon>0$ small. So we can conclude $\overline{x}_\varepsilon+\varepsilon R_\varepsilon=\widetilde{x}_\varepsilon$ and we can set $t_{\varepsilon,1,1}=\overline{x}_\varepsilon$ and $t_{\varepsilon,2,1}=\widetilde{x}_\varepsilon$. In other case, it can be estimated more easily. Thus, the proof is completed. □

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