A SUMMARY OF GLOBAL SMOOTHINGS OF NORMAL CROSSING COMPLEX SURFACES

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1. Introduction

This note is based on our talk in Kinosaki Algebraic Geometry Symposium 2020. We prove that there exists a family of smoothings of a simple normal crossing compact complex surface X with triple points. Since our differential geometric proof also includes the case where X is neither Kählerian nor $H^1(X, \mathcal{O}_X) = 0$, this generalizes Friedman's result on degenerations of $K3$ surfaces in algebraic geometry [Fr83]. As an application, we provide an example of a simple normal crossing surface with triple points which is smoothable to a quartic $K3$ surface. We refer the reader to the forthcoming paper [DY21] for more details.

Throughout this note, $X = \bigcup_{i=1}^N X_i$ denotes a compact connected complex surface with normal crossings with $\dim_{\mathbb{C}} X_i = 2$ for each i, unless otherwise specified. Furthermore we will assume that each X_i is smooth and X has no 4-fold intersection, which means that $X_i \cap X_j \cap X_k \cap X_\ell = \emptyset$ for distinct i, j, k and ℓ. More precisely, let X be a compact complex analytic surface with irreducible components $X_1, \ldots, X_N$. Then we say that X is a *simple normal crossing (SNC) complex surface* if X is locally embedded in $\mathbb{C}^3$ as $\{\, (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \mid \zeta_1 \cdots \zeta_\ell = 0 \,\}$ for some $\ell \in \{\, 1, 2, 3 \,\}$ and each X_i is smooth. We call a SNC compact complex surface X is *d-semistable* if

$$\left(\bigotimes_i \mathcal{I}_{X_i} / \mathcal{I}_{X_i} \mathcal{I}_D \right)^* \cong \mathcal{O}_D \tag{1.1}$$

for the singular locus D on X, where $\mathcal{I}_{X_i}$ and $\mathcal{I}_D$ are the ideal sheaves of X_i and D in X respectively. Let $D_{ij} = X_i \cap X_j$ with $i \neq j$ be the set of double curves. We will also assume that each connected component of D_{ij} defines a smooth irreducible divisor on both X_i and X_j. Let us denote N_{ij} the holomorphic normal bundle N_{D_{ij}/X_i} to D_{ij} in X_i. When X is a SNC compact complex surface with *at most double curves* (i.e. no 3-fold intersection), then d-semistablity condition (1.1) is equivalent to

$$N_{ij} \otimes N_{ji} \cong \mathcal{O}_{D_{ij}}. \tag{1.2}$$

Now we consider the case where $X = \bigcup_{i=1}^N X_i$ is a SNC compact complex surface with triple points. Let $T_{ijk} = X_i \cap X_j \cap X_k$ a set of *triple points*, $T_{ij} = \sum_{k(\neq i,j)} T_{ijk}$ a divisor on D_{ij}, and $T_i = \bigcup_{j \neq k} T_{ijk}$ the union of the set of triple points on X_i. For each D_{ij}, we consider

$$N_{ij} \otimes N_{ji} \otimes [T_{ij}] \cong \mathcal{O}_{D_{ij}} \tag{1.3}$$

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which is equivalent to the condition that X to be d-semistable (1.1). X is called a *d-smistable K3 surface* if X is a d-semistable SNC Kähler surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. It is known that d-semistable $K3$ surface are classified into Type I, II and III due to the works of Friedman [Fr83, FS86]. In particular, Friedman showed that any d-semistable $K3$ surface has a family of smoothings $\varpi : \mathscr{X} \to \Delta \subset \mathbb{C}$ of X with $K_{\mathscr{X}} = \mathcal{O}_{\mathscr{X}}$, where $\mathscr{X}$ is a 3-dimensional complex manifold and ϖ is a holomorphic map between $\mathscr{X}$ and a domain Δ in $\mathbb{C}$ (see also [KN94], Corollary 2.5). We remark that if X is a d-semistable $K3$ surface at most double curves, then X is either of Type I or of Type II. Meanwhile X is of Type III when a d-semistable $K3$ surface X admits triple points [FS86]. In 2009, Doi generalized Friedman's result in the following sense. That is, even in the case where a SNC complex surface X with at most double curves is *neither Kählerian nor* $H^1(X, \mathcal{O}_X) = 0$, there still exists a family of smoothings $\varpi : \mathscr{X} \to \Delta$ of X in a weak sense (Theorem 5.5 in [D09]). He constructed compact complex surfaces with trivial canonical bundle in a differential geometric method by gluing together two compact complex surfaces with an anticanonical divisor under suitable conditions. The purpose of our joint work [DY21] is to generalize this smoothability result to the case where X is a SNC complex surface with *triple points*. More precisely, we shall prove the following.

Theorem 1.1. *Let $X = \bigcup_{i=1}^{N} X_i$ be a simple normal crossing complex surface. Assume the following conditions:*

(i) *X is d-semistable;*
(ii) *each D_i is an anticanonical divisor on X_i; and*
(iii) *there exists a meromorphic volume form Ω_i on each X_i with a pole along D_i such that the Poincaré residue $\operatorname{res}_{D_{ij}} \Omega_i$ of Ω_i on D_{ij} is minus the Poincaré residue $\operatorname{res}_{D_{ij}} \Omega_j$ of Ω_j on D_{ij} for all i, j. (For the definition of Poincaré residues, see* [GH], pp. 147–148*).*

Then there exist $\varepsilon > 0$ and a surjective mapping $\varpi : \mathscr{X} \to \Delta = \{\, \zeta \in \mathbb{C} \mid |\zeta| < \varepsilon \,\}$ such that the following statements hold.

(a) *$\mathscr{X}$ is a smooth 6-dimensional manifold and ϖ is a smooth mapping.*
(b) *$X_0 = \varpi^{-1}(0) = X$.*
(c) *For each $\zeta \in \Delta^* = \Delta \setminus \{\, 0 \,\}$, $X_\zeta = \varpi^{-1}(\zeta)$ is a smooth compact complex surface with trivial canonical bundle.*
(d) *The complex structure on X_ζ depends continuously on ζ outside the singular locus $D = \bigcup_{i=1}^{N} D_i \subset X_0$. More precisely, for any point $p \in \mathscr{X} \setminus D$ there exist a neighborhood U of p and a diffeomophism $U \simeq V \times D$ with $D \subset \Delta$, such that the induced complex structures on V depend continuously on $\zeta \in D$.*

Note that conditions (ii) and (iii) are equivalent to the condition that the canonical bundle of the SNC complex surface X is trivial.

Comparing Theorem 1.1 with the result of R. Friedman in [Fr83], we see that even when X is not Kählerian or $H^1(X, \mathcal{O}_X)$ does not vanish, there still exists a family of smoothings $\varpi : \mathscr{X} \to \Delta$ of X in a weak sense (or a fibration), whose general fiber is a smooth compact complex surface with trivial canonical bundle. This result strongly suggests that X as in Theorem 1.1 admits a family of smoothings in the standard holomorphic sense, although the proof seems difficult.

The Bogomolov-Tian-Todorov theorem states that a Calabi-Yau manifold has unobstructed deformations and the first proof of this theorem is analytic. The second proof is algebraic which were given by Ran [Ran92] and Kawamata [Kaw92] where they used T^1-lifting property effectively. However T^1-lifting property requires the cohomological condition $H^{n-1}(X, \mathcal{O}_X) = 0$ when we obtain a flat deformation $\mathscr{X}$ of a SNC variety X. Meanwhile, our differential geometric proof does not assume the cohomological condition $H^1(X, \mathcal{O}_X) = 0$, though it only works in the case of a complex surface. Bearing in mind that the advantage of differential geometric approach, it is crucial to construct examples of compact complex surfaces X with trivial canonical bundle satisfying $H^1(X, \mathcal{O}_X) \neq 0$. Hence it is natural to ask the following question.

Problem 1.2. *Can we construct either a complex torus or a primary Kodaira surface by applying Theorem* 1.1 *?*

For the moment we can construct such examples in the case where SNC varieties have only two components (i.e. doubling construction), due to Doi's work (see [D09], Example 5.3), where he used Hirzebruch surfaces as ingredients of the construction. We will deal with this example in the last section (see Example 4.4).

The proof of Theorem 1.1 is based on the results obtained in [D09] and an explicit construction of local smoothings around the double curves. We will give a sketch of proof in this article. A complete proof Theorem 1.1 and an explicit example of degenerate $K3$ surface with triple points are given in [DY21].

2. A brief review of compact complex surfaces

2.1. $\mathrm{SL}(2,\mathbb{C})$**-structures and** $\mathrm{SU}(2)$**-structures.** For later use we recall the definition of $\mathrm{SL}(2,\mathbb{C})$-structure on an oriented manifold of real dimension 4. See [G04, D09] for more details.

To begin let V be an oriented real vector space of dimension 4. Taking $\psi_0 \in \wedge^2 V^* \otimes \mathbb{C}$, we call ψ_0 an $\mathrm{SL}(2,\mathbb{C})$-*structure* on V if ψ_0 satisfies

$$\psi_0 \wedge \overline{\psi}_0 > 0, \qquad \psi_0 \wedge \psi_0 = 0.$$

Each $\mathrm{SL}(2,\mathbb{C})$-structure ψ_0 on V defines complex subspaces

$$V^{0,1} = \{\, \zeta \in V \otimes \mathbb{C} \mid \iota_\zeta \psi_0 = 0 \,\}, \qquad V^{1,0} = \overline{V^{1,0}}$$

where ι_ζ denotes the inner multiplication by ζ. Then the decomposition

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$$

gives a complex structure I_{ψ_0} on V so that ψ_0 is a complex differential form of type $(2,0)$ with respect to I_{ψ_0}.

Analogously we can extend this concept to an oriented 4-manifold M as follows. We call $\psi \in C^\infty(\wedge^2 T^*M \otimes \mathbb{C})$ an $\mathrm{SL}(2,\mathbb{C})$-*structure* on M if ψ satisfies

$$\psi \wedge \overline{\psi} > 0, \qquad \psi \wedge \psi = 0.$$

Observe that an $\mathrm{SL}(2,\mathbb{C})$-structure ψ on M induces an $\mathrm{SL}(2,\mathbb{C})$-structure on T_xM for each $x \in M$. Hence we see that ψ defines an almost complex structure I_ψ on M so that ψ is a type $(2,0)$ complex differential form with respect to I_ψ.

The following lemma gives a geometric characterization of complex surfaces with trivial canonical bundle. We refer to [D09], Lemma 2.3 for a proof.

Lemma 2.1 (Grauert, Goto [G04]). *Let M be an oriented 4-manifold and ψ be an $\mathrm{SL}(2,\mathbb{C})$-structure on M. If ψ is d-closed, then I_ψ is an integrable complex structure on M with trivial canonical bundle. Furthermore ψ is a holomorphic volume form on M with respect to I_ψ.*

The above lemma gives the following characterization of complex surfaces with trivial canonical bundle by d-closed $\mathrm{SL}(2,\mathbb{C})$-structures.

Proposition 2.2. *Let M be an oriented 4-manifold. Then M admits a complex structure with trivial canonical bundle if and only if M admits a d-closed $\mathrm{SL}(2,\mathbb{C})$-structure.*

Thus if we say that X is a complex surface with trivial canonical bundle, then we understand that X consists of an underlying oriented 4-manifold M and a d-closed $\mathrm{SL}(2,\mathbb{C})$-structure ψ on M such that ψ induces a complex structure I_ψ on M and becomes a holomorphic volume form on $X = (M, I_\psi)$.

Let X be a compact complex surface with trivial canonical bundle. If X is simply-connected or $H^1(X, \mathcal{O}_X) = 0$, then X is called a $K3$ surface. According to the Enriques Kodaira classification of compact complex surfaces, it is known that a compact complex surface with trivial canonical bundle is either a complex torus, a Kodaira surface, or a $K3$ surface (see [BHPV], Chapter 6).

Next we will give the definition of $\mathrm{SU}(2)$-structure. Again let V be an oriented real vector space of dimension 4. For each $(\psi_0, \kappa_0) \in (\wedge^2 V^* \otimes \mathbb{C}) \oplus \wedge^2 V^*$, we denote an inner product on V by $g_{(\psi_0,\kappa_0)}$ which is defined by $g_{(\psi_0,\kappa_0)}(I_\psi \cdot, \cdot) = \kappa_0(\cdot,\cdot)$. Then (ψ_0, κ_0) is said to be an $\mathrm{SU}(2)$-*structure* on V if it satisfies the following conditions:

(i) ψ_0 is an $\mathrm{SL}(2,\mathbb{C})$-structure on V (i.e. $\psi_0 \wedge \overline{\psi}_0 > 0$, $\psi_0 \wedge \psi_0 = 0$),
(ii) $\psi_0 \wedge \kappa_0 = 0$,
(iii) $g_{(\psi_0,\kappa_0)}$ is positive definite, and
(iv) $2\kappa_0^2 = \psi_0 \wedge \overline{\psi}_0$.

Definition 2.3. Let M be an oriented 4-manifold. Then

$$(\psi, \kappa) \in C^\infty(\wedge^2 T^*M \otimes \mathbb{C}) \oplus C^\infty(\wedge^2 T^*M)$$

is said to be an $\mathrm{SU}(2)$-*structure* on M if the restriction $(\psi,\kappa)|_{T_xM}$ is an $\mathrm{SU}(2)$-structure on T_xM for any $x \in M$.

If ψ and κ of an $\mathrm{SU}(2)$-structure on M are both d-closed, then $X = (M, I_\psi, \kappa)$ is a Kähler surface with trivial canonical bundle by Lemma 2.1. Hence $\mathrm{SU}(2)$-structures have an important role in the proof of Theorem 1.1.

2.2. **Compact complex surfaces with anticanonical divisors.** Next we recall some results on compact complex manifolds with an anticanonical divisor which were already used in [D09] and [DY14]. For simplicity, we only consider the case of complex surfaces although it is possible to extend the most part of results into arbitrary dimension.

Let X be a compact complex surface and D a smooth anticanonical divisor on X. Taking an open covering $\{ U_\alpha \}$ of X, we define $V_\alpha = U_\alpha \cap D$. Then $\{ V_\alpha \}$ is an open covering of D. Furthermore, we can show the following.

Lemma 2.4. *There is a local coordinate system $\{ U_\alpha, (z_\alpha, w_\alpha) \}$ on X such that*

(i) *w_α is a local defining function of D on U_α, i.e. $V_\alpha = \{ w_\alpha = 0 \}$.*

(ii) *the* 2*-forms* $\Omega_\alpha = \dfrac{dw_\alpha}{w_\alpha} \wedge dz_\alpha$ *on* U_α *together yield a holomorphic volume form* Ω *on* $X \setminus D$*.*

Proof. The statement (i) is obvious. Hence it suffices to prove (ii).

Let $\phi_{\alpha\beta}$ and $f_{\alpha\beta}$ be non-vanishing holomorphic functions on $U_\alpha \cap U_\beta$ which determine the coordinate transformation of X by

$$z_\alpha = \phi_{\alpha\beta}(z_\beta, w_\beta) \qquad \text{and} \qquad w_\alpha = f_{\alpha\beta}(z_\beta, w_\beta)w_\beta. \tag{2.1}$$

On $U_\alpha \cap U_\beta$, we recall that the canonical bundle K_X is given by transition function

$$h_{\alpha\beta}(z_\beta, w_\beta) = \frac{dw_\beta \wedge dz_\beta}{dw_\alpha \wedge dz_\alpha}. \tag{2.2}$$

Also the line bundle $[D]$ on X is given by transition functions

$$f_{\alpha\beta} = \frac{w_\alpha}{w_\beta}. \tag{2.3}$$

(See [GH], p.145). Since we take $[D]$ to be an anticanonical divisor on X, we can choose the local coordinates (z_α, w_α) satisfying

$$f_{\alpha\beta}(z_\beta, w_\beta)h_{\alpha\beta}(z_\beta, w_\beta) = 1. \tag{2.4}$$

Substituting (2.2) and (2.3) into (2.4), we see that the local holomorphic volume forms

$$\Omega_\alpha = \frac{dw_\alpha}{w_\alpha} \wedge dz_\alpha$$

together yield a holomorphic volume form Ω on $X \setminus D$. $\square$

Next we shall consider the holomorphic normal bundle $N_{D/X}$ to D in X which is defined as the quotient line bundle

$$N_{D/X} = \frac{T'_X|_D}{T'_D}$$

where T'_X (resp. T'_D) is the holomorphic tangent bundle of X (resp. D). We often denote $N_{D/X}$ by N for simplicity. Let $\pi : N \to D$ be the projection and $i_0 : X \to N$ the zero section. We may identify $i_0(D)$ of N with D in X. Restricting z_α to $V_\alpha = U_\alpha \cap D$, we obtain a local coordinate system $\{ (V_\alpha, x_\alpha) \}$ on D, with $x_\alpha = z_\alpha|_{V_\alpha}$. On $\pi^{-1}(V_\alpha) \simeq V_\alpha \times \mathbb{C}$, we have local coordinates (x_α, y_α) of the normal bundle $N = [D]|_D$ where $x_\alpha \in V_\alpha$ and $y_\alpha \in \mathbb{C}$ is the fiber coordinate. Analogous to (2.1), the coordinate transformation of N is given by

$$x_\alpha = \psi_{\alpha\beta}(x_\beta) \qquad \text{and} \qquad y_\alpha = g_{\alpha\beta}(x_\beta)y_\beta, \tag{2.5}$$

where $\psi_{\alpha\beta}$ and $g_{\alpha\beta}$ are holomorphic functions on $\pi^{-1}(V_\alpha \cap V_\beta)$ defined by

$$\psi_{\alpha\beta}(x_\beta) = \phi_{\alpha\beta}(x_\beta, 0) \qquad \text{and} \qquad g_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta, 0) \tag{2.6}$$

respectively. By restricting (2.2) to $V_\alpha \cap V_\beta$, we see that

$$h_{\alpha\beta}(z_\beta, w_\beta)\Big|_{V_\alpha \cap V_\beta} = \frac{dw_\beta \wedge dz_\beta}{dw_\alpha \wedge dz_\alpha}\Big|_{V_\alpha \cap V_\beta}$$

which becomes

$$h_{\alpha\beta}(x_\beta, 0) = g_{\alpha\beta}(x_\beta)^{-1}\frac{dx_\beta}{dx_\alpha} \tag{2.7}$$

because $g_{\alpha\beta}(x_\beta)^{-1} = y_\beta/y_\alpha$ by (2.5).

On the other hand, restricting (2.4) to $V_\alpha \cap V_\beta$, we have

$$f_{\alpha\beta}(x_\beta, 0)h_{\alpha\beta}(x_\beta, 0) = 1$$

which yields

$$\frac{dx_\beta}{dx_\alpha} = 1$$

by (2.6) and (2.7). Hence we showed that the local holomorphic volume form $\Omega_{D,\alpha} = dx_\alpha$ on V_α together yield a holomorphic volume form Ω_D on D so that the canonical bundle K_D of D is trivial. Note that this agrees with a consequence of the adjunction formula $K_D = (K_X \otimes [D])|_D \cong \mathcal{O}_D$.

As in [GH] p.147, we consider Ω as a meromorphic 2-form on X with a single pole along D. Then the holomorphic volume form Ω_D obtained from Ω in the above is said to be the *Poincaré residue* of Ω which is denoted by $\mathrm{res}(\Omega)$. We readily see that $\mathrm{res}(\Omega)$ is not depend on the choice of local coordinates of Ω.

2.3. **Semistable degenerations of $K3$ surfaces.** Next we reall a summary of the classification of degenerations of $K3$ surfaces. Let $\varpi : \mathscr{X} \to \Delta$ be a proper map from a compact complex 3-dimensional manifold $\mathscr{X}$ to a domain $\Delta = \{\, \zeta \in \mathbb{C} \mid |\zeta| < \varepsilon \,\}$ such that

(1) $\mathscr{X} \setminus \varpi^{-1}(0)$ is smooth, and
(2) the fiber $X_\zeta = \varpi^{-1}(\zeta)$ is a smooth compact Kähler surface for each $\zeta \in \Delta^* = \Delta \setminus \{\, 0 \,\}$.

We call ϖ a *degeneration* of complex surfaces. Furthermore, a degeneration ϖ is said to be *semistable* if

(3) the total space $\mathscr{X}$ is smooth, and
(4) the central fiber $X_0 = \varpi^{-1}(0)$ is a Kähler surface with simple normal crossings.

In the study of the degenerations of $K3$ surfaces, the following results due to Kulikov and Personn-Pinkham are important.

Theorem 2.5 ([Fr83], Theorem 5.1)**.** *Let $\varpi : \mathscr{X} \to \Delta$ be a semistable degeneration of $K3$ surfaces. If all components X_i of the central fiber $X_0 = \varpi^{-1}(0)$ are algebraic, then there exists a birational isomorphism $\rho : \mathscr{X} \to \mathscr{X}'$ with a commutative diagram*

$$\begin{tikzcd} \mathscr{X} \arrow[rr, dashed, "\rho"] \arrow[dr, "\varpi"'] & & \mathscr{X}' \arrow[dl, "\varpi'"] \\ & \Delta & \end{tikzcd}$$

such that

(a) *ρ is an isomorphism $\mathscr{X} \setminus \varpi^{-1}(0) \cong \mathscr{X}' \setminus (\varpi')^{-1}(0)$, and*
(b) *$\varpi' : \mathscr{X}' \to \Delta'$ is a semistable degeneration with $K_{\mathscr{X}'} = \mathcal{O}_{\mathscr{X}'}$ where $K_{\mathscr{X}'}$ is the canonical line bundle of $\mathscr{X}'$.*

Theorem 2.6 ([Fr83], Theorem 5.2). *Let $\varpi : \mathscr{X} \to \Delta$ be a semistable degeneration of $K3$ surfaces with $K_{\mathscr{X}} = \mathcal{O}_{\mathscr{X}}$ as in Theorem 2.5. Then $X_0 = \varpi^{-1}(0)$ is one of the following three types:*

Type I: *X is a smooth $K3$ surface.*

Type II: *$X = X_1 \cup \cdots \cup X_N$ is a chain of surfaces, where X_1 and X_N are rational surfaces, $X_2, \cdots, X_{N-1}$ are elliptic ruled surfaces and $X_i \cap X_{i+1}$, $i = 1, \cdots N-1$ are smooth elliptic curves.*

Type III: *$X = \bigcup_{i=1}^N X_i$, where each X_i is a rational surface and the double curves $D_{ij} = X_i \cap X_j \subseteq X_i$ are cycles of rational curves.*

We call X a *d-semistable $K3$ surface* if

- X is a d-semistable SNC compact Kähler surface with trivial canonical bundle, and
- $H^1(X, \mathcal{O}_X) = 0$.

It is well-known that any d-semistable $K3$ surfaces are classified into Type I–III in Theorem 2.6 (cf. [Fr83], Definition 5.5).

3. Global smoothings of simple normal crossing complex surfaces

3.1. **Local coordinates on SNC complex surfaces.** Let $X = \bigcup_{i=1}^N X_i$ be a SNC complex surface satisfying conditions (i)–(iii) of Theorem 1.1. We can find a local holomorphic coordinate system $\{ U_{i,\alpha}, (z^1_{i,\alpha}, z^2_{i,\alpha}) \}$ on $X_i = \bigcup_{\alpha \in \Lambda_i} U_{i,\alpha}$ with Λ_i a finite subset of $\mathbb{N}$, satisfying the following conditions:

(A) $U_{i,\alpha} = \left\{ (z^1_{i,\alpha}, z^2_{i,\alpha}) \in \mathbb{C}^2 \,\middle|\, \left|z^1_{i,\alpha}\right| < 1, \left|z^2_{i,\alpha}\right| < 1 \right\}$;
(B) if $U_{i,\alpha} \cap D_i \neq \emptyset$ and $U_{i,\alpha} \cap T_i = \emptyset$, then $U_{i,\alpha} \cap D_i = \{ z^2_{i,\alpha} = 0 \}$; and
(C) if $U_{i,\alpha} \cap T_i \neq \emptyset$, then $U_{i,\alpha} \cap D_i = \{ z^1_{i,\alpha} z^2_{i,\alpha} = 0 \}$, so that $U_{i,\alpha} \cap T_i = \{ z^1_{i,\alpha} = z^2_{i,\alpha} = 0 \}$.

In particular, each $U_{i,\alpha}$ contains at most one triple point. Now we set

$$\Lambda_i^{(0)} = \{ \alpha \in \Lambda_i \,|\, U_{i,\alpha} \cap D_i = \emptyset \},$$
$$\Lambda_i^{(1)} = \{ \alpha \in \Lambda_i \,|\, U_{i,\alpha} \cap D_i \neq \emptyset \text{ and } U_{i,\alpha} \cap T_i = \emptyset \},$$
$$\Lambda_i^{(2)} = \{ \alpha \in \Lambda_i \,|\, U_{i,\alpha} \cap T_i \neq \emptyset \},$$
$$\Lambda_{ij} = \{ \alpha \in \Lambda_i \,\big|\, U_{i,\alpha} \cap D_{ij} \neq \emptyset \}, \quad \Lambda_{ij}^{(1)} = \Lambda_{ij} \cap \Lambda_i^{(1)}, \quad \Lambda_{ij}^{(2)} = \Lambda_{ij} \cap \Lambda_i^{(2)}, \quad \text{and}$$
$$\Lambda_{ijk} = \Lambda_{ij} \cap \Lambda_{ik} \subset \Lambda_i^{(2)}.$$

From condition (ii) of Theorem 1.1, we can choose the above coordinate system so that

(D) the meromorphic volume form Ω_i in (iii) of Theorem 1.1 can be locally represented as

$$\Omega_i = \begin{cases} \mathrm{d}z^1_{i,\alpha} \wedge \mathrm{d}z^2_{i,\alpha} & \text{if } \alpha \in \Lambda_i^{(0)}, \\ \mathrm{d}z^1_{i,\alpha} \wedge \dfrac{\mathrm{d}z^2_{i,\alpha}}{z^2_{i,\alpha}} & \text{if } \alpha \in \Lambda_i^{(1)}, \\ \sigma_{i,\alpha} \dfrac{\mathrm{d}z^1_{i,\alpha}}{z^1_{i,\alpha}} \wedge \dfrac{\mathrm{d}z^2_{i,\alpha}}{z^2_{i,\alpha}} & \text{if } \alpha \in \Lambda_i^{(2)}, \end{cases}$$

where $\sigma_{i,\alpha}$ for $\alpha \in \Lambda_i^{(2)}$ is a complex number.

In terms of the above coordinate system, we define a new one $\{U_{i,\alpha}, (z_{ij,\alpha}, w_{ij,\alpha})\}$ associated with D_{ij} as follows:

(E) if $\alpha \notin \Lambda_{ij}^{(2)}$, then we set $z_{ij,\alpha} = z_{i,\alpha}^1,\ w_{ij,\alpha} = z_{i,\alpha}^2$;

(F) if $\alpha \in \Lambda_{ij}^{(2)}$, then between $z_{i,\alpha}^1$ and $z_{i,\alpha}^2$, we choose as $w_{ij,\alpha}$ the coordinate which is a defining function of D_{ij} on $U_{i,\alpha}$, and define $z_{ij,\alpha}$ as the remainder, so that $U_{i,\alpha} \cap D_{ij} = \{w_{ij,\alpha} = 0\}$ and $U_{i,\alpha} \cap D_{ik} = \{z_{ij,\alpha} = 0\}$ for some $k \neq i, j$.

In particular, we have

$$z_{ik,\alpha} = w_{ij,\alpha}, \quad w_{ik,\alpha} = z_{ij,\alpha} \quad \text{for } \alpha \in \Lambda_{ijk}. \tag{3.1}$$

We can further choose the coordinate system so that the following condition holds.

(G) Let $V_{ij,\alpha} = U_{i,\alpha} \cap D_{ij}$ and $x_{ij,\alpha} = z_{ij,\alpha}|_{V_{ij,\alpha}}$ for $\alpha \in \Lambda_{ij}$. Then we have $\Lambda_{ij} = \Lambda_{ji}$, $V_{ij,\alpha} = V_{ji,\alpha}$ and $x_{ij,\alpha} = x_{ji,\alpha}$ for all $i \neq j$ and $\alpha \in \Lambda_{ij}$.

Let $(x_{ij,\alpha}, y_{ij,\alpha})$ be local coordinates of $\pi_{ij}^{-1}(V_{ij,\alpha}) \subset N_{ij}$, where π_{ij} is the projection from N_{ij} to D_{ij} and $y_{ij,\alpha}$ are fiber coordinates. Then from condition (i) of Theorem 1.1, we may further assume that

(H) the map $h_{ij,\zeta} : N_{ij} \setminus (D_{ij} \cup \pi_{ij}^{-1}(T_{ij})) \to N_{ji} \setminus (D_{ji} \cup \pi_{ji}^{-1}(T_{ji}))$ locally defined by

$$h_{ij,\zeta} : (x_{ij,\alpha}, y_{ij,\alpha}) \mapsto (x_{ji,\alpha}, y_{ji,\alpha}) = \begin{cases} (x_{ij,\alpha}, \zeta/y_{ij,\alpha}) & \text{for } \alpha \in \Lambda_{ij}^{(1)}, \\ (x_{ij,\alpha}, \zeta/(x_{ij,\alpha} y_{ij,\alpha})) & \text{for } \alpha \in \Lambda_{ij}^{(2)}. \end{cases} \tag{3.2}$$

is a well-defined isomorphism for $\zeta \in \mathbb{C}^*$.

Now by the tubular neighborhood theorem, there exists a diffeomorphism Φ_{ij} from a neighborhood V_{ij} of the zero section of N_{ij} to a neighborhood $U_{ij} \subset \bigcup_{\alpha \in \Lambda_{ij}} U_{i,\alpha}$ of D_{ij} in X_i such that Φ_{ij} is locally represented as

$$\begin{gathered} z_{ij,\alpha} = x_{ij,\alpha} + O(|y_{ij,\alpha}|^2), \quad w_{ij,\alpha} = y_{ij,\alpha} + O(|y_{ij,\alpha}|^2) \quad \text{for } \alpha \in \Lambda_{ij}^{(1)}, \text{ and} \\ z_{ij,\alpha} = x_{ij,\alpha}, \quad w_{ij,\alpha} = y_{ij,\alpha} \quad \text{for } \alpha \in \Lambda_{ij}^{(2)}. \end{gathered} \tag{3.3}$$

3.2. **Local smoothings of $X_i \cup X_j$ around D_{ij} without a triple point.** Here we suppose $D_{12} \neq \emptyset$ is a double curve without a triple point, so that $\Lambda_{12}^{(2)} = \emptyset$. The indices i, j will take 1 or 2. For D_{ij} with $i < j$, we replace 1, 2 with i, j respectively. We have chosen the coordinate system $\{U_{i,\alpha}, (z_{ij,\alpha}, w_{ij,\alpha})\}$ on X_i so that $w_{ij,\alpha}$ is a defining function of D_{ij} on $U_{i,\alpha}$ and $\Omega_i = \epsilon_{ij} \mathrm{d}z_{ij,\alpha} \wedge \dfrac{\mathrm{d}w_{ij,\alpha}}{w_{ij,\alpha}}$ on $U_{i,\alpha}$ for $\alpha \in \Lambda_{ij}$, where $\epsilon_{ij} = (i-j)/|i-j|$. By condition (ii) of Theorem 1.1 and the adjunction formula,

$$K_{D_{ij}} = (K_{X_i} \otimes [D_{ij}])|_{D_{ij}} \cong \mathcal{O}_{D_{ij}}.$$

Thus $\mathrm{d}x_{ij,\alpha}$ defines a holomorphic volume form and $\dfrac{\sqrt{-1}}{2} \mathrm{d}x_{ij,\alpha} \wedge \mathrm{d}\overline{x}_{ij,\alpha}$ a Hermitian form on D_{ij}. We define a complex 2-form Ω_{ij}^∞ and a real 2-form ω_{ij}^∞ on $N_{ij} \setminus D_{ij}$ by

$$\begin{aligned} \Omega_{ij}^\infty &= \epsilon_{ij} \pi_{ij}^* \mathrm{d}x_{ij,\alpha} \wedge \frac{\mathrm{d}y_{ij,\alpha}}{y_{ij,\alpha}}, \\ \omega_{ij}^\infty &= \pi_{ij}^* (\mathrm{d}x_{ij,\alpha} \wedge \mathrm{d}\overline{x}_{ij,\alpha}) + \frac{\sqrt{-1}}{2} \partial t_{ij} \wedge \overline{\partial} t_{ij}. \end{aligned}$$

By Lemma 3.2 in [D09], $(\Omega_{ij}^{\infty}, \omega_{ij}^{\infty})$ defines an $\mathrm{SU}(2)$-structure on $N_{ij} \setminus D_{ij}$ such that the associated metric is cylindrical. If we regard Ω_{ij}^{∞} and ω_{ij}^{∞} as defined on t_{ij}, then we can prove that

$$\left|\Omega_i - \Omega_{ij}^{\infty}\right| = O(e^{-t_{ij}/2}), \quad \left|\omega_{ij} - \omega_{ij}^{\infty}\right| = O(e^{-t_{ij}/2}), \tag{3.4}$$

where ω_{ij} is the $(1,1)$-part of ω_{ij}^{∞}, normalized so that $\Omega_i \wedge \overline{\Omega}_i = 2\omega_{ij} \wedge \omega_{ij}$, and the norm is measured by the cylindrical metric associated with $(\Omega_{ij}^{\infty}, \omega_{ij}^{\infty})$. We also see that

$$h_{ij,\zeta}^{*}\Omega_{ji}^{\infty} = \Omega_{ij}^{\infty}, \quad h_{ij,\zeta}^{*}\omega_{ji}^{\infty} = \omega_{ij}^{\infty}.$$

Then one can construct local smoothings of $X_1 \cup X_2$ around D_{12} in the same manner as [D09], Section 5.3.

3.3. **Local smoothings of $X_i \cup X_j \cap X_k$ around $D_{ij} \cap D_{jk} \cap D_{ki}$.** Here we suppose $T_{123} \neq \emptyset$ and consider local smoothing of $X_1 \cup X_2 \cup X_3$ around $D_{12} \cup D_{23} \cup D_{31}$. The indices i, j, k will take 1, 2 or 3. For general i, j, k with $i < j < k$, we will be done if we replace $1, 2, 3$ with i, j, k respectively. For later convenience, let ϵ_{ijk} denote the Levi-Civita symbol $\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$, $\epsilon_{ij} = (i-j)/\left|i-j\right|$ as before, and define ν_{ij} by $\nu_{ij} = \sum_{k=1}^{3} k\left|\epsilon_{ijk}\right|$, so that $\nu_{ij} \in \{1,2,3\}$ is the unique number such that $\epsilon_{ij\nu_{ij}} \neq 0$. By condition (ii) of Theorem 1.1 and the adjunction formula, we have

$$K_{D_{ij}} = (K_{X_i} \otimes [D_{ij}])|_{D_{ij}} = \left[-\sum_{\ell \neq i} D_{i\ell}\right]\Bigg|_{D_{ij}} \otimes [D_{ij}]|_{D_{ij}} \cong \left[-\sum_{\ell \neq i,j} D_{i\ell}\right]\Bigg|_{D_{ij}} = [-T_{ij}].$$

Thus $\{\mathrm{d}x_{ij,\alpha}\}_{\alpha \in \Lambda_{ij}^{(1)}}$ and $\left\{\dfrac{\mathrm{d}x_{ij,\alpha}}{x_{ij,\alpha}}\right\}_{\alpha \in \Lambda_{ij}^{(2)}}$ together define a holomorphic volume form $\psi_{D_{ij}^0}$ on $D_{ij}^0 = D_{ij} \setminus T_{ij}$. We also have a Hermitian form $\omega_{D_{ij}^0} = \dfrac{\sqrt{-1}}{2}\psi_{D_{ij}^0} \wedge \overline{\psi}_{D_{ij}^0}$ on D_{ij}^0. Let $N_{ij}^0 = N_{ij}|_{D_{ij}^0}$. We define a holomorphic volume form Ω_{ij}^{∞} and a Hermitian form ω_{ij}^{∞} on $N_{ij}^0 \setminus D_{ij}^0$ by

$$\Omega_{ij}^{\infty} = -\sigma_{ij}\pi_{ij}^{*}\psi_{D_{ij}^0} \wedge \partial t_{ij}, \tag{3.5}$$

$$\begin{aligned}\omega_{ij}^{\infty} =& \frac{|\sigma_{ij}|}{\sqrt{3}}\Bigg\{\pi_{ij}^{*}\omega_{D_{ij}^0} + \frac{\sqrt{-1}}{2}\partial t_{ij} \wedge \overline{\partial} t_{ij} \\ &+ \frac{\sqrt{-1}}{2}(\pi_{ij}^{*}\psi_{D_{ij}^0} - \epsilon_{ij}\partial t_{ij}) \wedge (\pi_{ij}^{*}\overline{\psi}_{D_{ij}^0} - \epsilon_{ij}\overline{\partial} t_{ij})\Bigg\},\end{aligned} \tag{3.6}$$

so that $\Omega_{ij}^{\infty} \wedge \overline{\Omega}_{ij}^{\infty} = 2\omega_{ij}^{\infty} \wedge \omega_{ij}^{\infty}$. In particular, if $\alpha \in \Lambda_{ij}^{(2)}$, then Ω_{ij}^{∞} and ω_{ij}^{∞} are locally represented as

$$\Omega_{ij}^{\infty} = \sigma_{ij}\frac{\mathrm{d}x_{ij,\alpha}}{x_{ij,\alpha}} \wedge \frac{\mathrm{d}y_{ij,\alpha}}{y_{ij,\alpha}}, \tag{3.7}$$

$$\begin{aligned}\omega_{ij}^{\infty} =& \frac{\sqrt{-1}}{2\sqrt{3}}\left|\sigma_{ij}\right|\Bigg\{\frac{\mathrm{d}x_{ij,\alpha}}{x_{ij,\alpha}} \wedge \frac{\mathrm{d}\overline{x}_{ij,\alpha}}{\overline{x}_{ij,\alpha}} + \frac{\mathrm{d}y_{ij,\alpha}}{y_{ij,\alpha}} \wedge \frac{\mathrm{d}\overline{y}_{ij,\alpha}}{\overline{y}_{ij,\alpha}} \\ &+ \left(\frac{\mathrm{d}x_{ij,\alpha}}{x_{ij,\alpha}} + \frac{\mathrm{d}y_{ij,\alpha}}{y_{ij,\alpha}}\right) \wedge \left(\frac{\mathrm{d}\overline{x}_{ij,\alpha}}{\overline{x}_{ij,\alpha}} + \frac{\mathrm{d}\overline{y}_{ij,\alpha}}{\overline{y}_{ij,\alpha}}\right)\Bigg\}.\end{aligned} \tag{3.8}$$

We regard Ω_{ij}^{∞} and ω_{ij}^{∞} as defined on $\Phi_{ij}(t_{ij}^{-1}((0,\infty))\cap V_{ij})\setminus D_i$ via Φ_{ij}. Then we see from (3.1), (3.3), (3.7) and (3.8) that

$$\left|\Omega_i - \Omega_{ij}^{\infty}\right| = O(e^{-t_{ij}/2}), \quad \left|\omega_{ij} - \omega_{ij}^{\infty}\right| = O(e^{-t_{ij}/2}) \quad \text{and} \tag{3.9}$$

$$\Omega_{ij}^{\infty} = \Omega_i = \Omega_{ik}^{\infty}, \quad \omega_{ij}^{\infty} = \omega_{ij} = \omega_{ik} = \omega_{ik}^{\infty} \quad \text{on } U_{i,\alpha} \text{ for } \alpha \in \Lambda_{ijk}, \tag{3.10}$$

where ω_{ij} is the $(1,1)$-part of ω_{ij}^{∞}, normalized so that $\Omega_i \wedge \overline{\Omega}_i = 2\omega_{ij} \wedge \omega_{ij}$, and $|\cdot|$ is measured by the cylindrical metric g_{ij}^{∞} associated with ω_{ij}^{∞}. We also see from (3.2), (3.5) and (3.6) that

$$h_{ij,\zeta}^{*}\Omega_{ji}^{\infty} = \Omega_{ij}^{\infty}, \quad h_{ij,\zeta}^{*}\omega_{ji}^{\infty} = \omega_{ij}^{\infty}.$$

We are now ready to construct a family of local smoothings of $X_1 \cup X_2 \cup X_3$ around $D_{12} \cup D_{23} \cup D_{31}$. The construction consists of the following three steps:

Step 1. Following Section 3.2, we consider local smoothings of $X_i \cup X_j$ around D_{ij} to obtain a family of local smoothings $\varpi_{ij} : \mathcal{V}_{ij} \to \Delta$.

Step 2. To consider local smoothings of $X_1 \cup X_2 \cup X_3$ around T_{123}, we define projections $\varpi_{123,\alpha} : \mathcal{V}_{123,\alpha} \to \Delta$.

Step 3. Using appropriate injective diffeomorphisms $\Psi_{ij,\alpha} : \mathcal{V}_{123,\alpha} \to \mathcal{V}_{ij}$ compatible with the projections to Δ, we glue together $\mathcal{V}_{ij}$ in Step 1 along $\mathcal{V}_{123,\alpha}$ in Step 2.

For more details, we refer the reader to [DY21] Sections 3.1–3.3.

3.4. **Existence of holomorphic volume forms on global smoothings.**

Sketch of the Proof of Theorem 1.1. In the previous two sections we obtained partial smoothings of X around each normal crossing. Now we glue all pieces together and construct a family $\varpi : \mathscr{X} = \{ X_\zeta \mid \zeta \in \Delta \} \to \Delta$ of global smoothings of $X = X_0$.

For each double curve $D_{ij} \subset X_i$, we obtained a Hermitian form ω_{ij} on $\{ 0 < t_{ij} \} \subset U_{ij}$ satisfying $\omega_{ij} = \omega_{ik}$ on $U_{i,\alpha}$ for $\alpha \in \Lambda_{ijk}$. Thus there exists a Hermitian form ω_i on $X_i \setminus D_i$ such that $\Omega_i \wedge \overline{\Omega}_i = 2\omega_i \wedge \omega_i$ on $X_i \setminus D_i$ and $\omega_i = \omega_{ij}$ on $\{ 1 \leqslant t_{ij} \}$ for all j. Then it follows from (3.4), (3.9) and [D09], Proposition 3.4 that there exists a complex 1-form ξ_{ij} on $\{ 0 < t_{ij} \}$ such that

$$\Omega_i - \Omega_{ij}^{\infty} = \mathrm{d}\xi_{ij}, \quad \text{and } \left|\nabla^k \xi_{ij}\right| = O(e^{-t_{ij}/2}) \quad \text{for all } k \geqslant 0.$$

As a differentiable manifold X_ζ is constructed from the ingredients

- $X_i \setminus \bigcup_{j(\neq i)} \{ t_{ij} \geqslant T+1 \}$ with the pair $(\Omega_{i,\zeta}, \omega_{i,\zeta})$ of 2-forms,
- $\{ t_{ij} > T-1 \} \subset N_{ij}^0 \subset \mathcal{V}_{ij}$ with the $\mathrm{SU}(2)$-structure $(\Omega_{ij}^{\infty}, \omega_{ij}^{\infty})$, and
- $\mathcal{V}_{ijk,\alpha}$ with the $\mathrm{SU}(2)$-structure $(\Omega_{ijk,\alpha}^{\infty}, \omega_{ijk,\alpha}^{\infty})$ for $\alpha \in \Lambda_{ijk}$

via the appropriate gluing maps. Since the gluing maps preserve the associated forms, they together define a pair $(\widetilde{\Omega}_\zeta, \widetilde{\omega}_\zeta)$ of 2-forms on X_ζ. Let $\mathscr{A}_{\mathrm{SU}(2)}(X_\zeta)$ be the set of $\mathrm{SU}(2)$-structures on X_ζ. We take $\mathscr{T}_{\mathrm{SU}(2)}(X_\zeta)$ as a neighborhood of $\mathscr{A}_{\mathrm{SU}(2)}(X_\zeta)$ so that the projection $\Theta : \mathscr{T}_{\mathrm{SU}(2)}(X_\zeta) \to \mathscr{A}_{\mathrm{SU}(2)}(X_\zeta)$ is well-defined (see [D09], Lemma 2.8). Then by a similar argument as in [D09], Section 3, one can define an $\mathrm{SU}(2)$-structure on X_ζ by

$$(\psi_\zeta, \kappa_\zeta) = \Theta(\widetilde{\Omega}_\zeta, \widetilde{\omega}_\zeta).$$

For the main estimates of ψ_ζ and κ_ζ, we will discuss in [DY21], Section 3.4. ☐

4. EXAMPLES

In this section, we apply Theorem 1.1 to a normal crossing Y to produce compact complex surfaces with trivial canonical bundle.

Example 4.1. (A $K3$ surface) Let Y_i $(i = 1, 2)$ be two hyperplanes in $\mathbb{C}P^3$, and Y_3 a quartic surface in $\mathbb{C}P^3$. For a SNC complex surface $Y = Y_1 \cup Y_2 \cup Y_3$, let us denote $Y_{ij} = Y_i \cap Y_j$ and $Y_{ijk} = Y_i \cap Y_j \cap Y_k$ respectively. Let $D_1 = Y_2 \cap Y_3$, $D_2 = Y_3 \cap Y_1$, $D_3 = Y_1 \cap Y_2$ and $\tau = Y_1 \cap Y_2 \cap Y_3$. Then we choose smooth points $P_i \in |\mathcal{O}_{D_i}(4)|$ for $i = 1, 2, 3$, satisfying the condition

$$P_i \cap \tau = \emptyset \qquad\qquad i \in \{1, 2, 3\}$$

so that each P_i and τ are distinct points. Next we consider the blow-ups of Y_i at P_j and take the proper transform of D_i. This is divided into the following steps.

Step 1. For $\{i, j\} = \{1, 2\}$, let $\pi_i : Y_i' := \mathrm{Bl}_{P_j}(Y_i) \dashrightarrow Y_i$ be the blow-up of Y_i at P_j in D_j. Let us take the proper transform Y_{3i}' of Y_{3i} and Y_{ji}' of Y_{ji} under the blow-up π_i. Let P_3' be the proper transform of $P_3 \in D_3$ under the blow-up π_1.

Step 2. Next we take the blow-up of Y_1' at P_3':

$$\pi_1' : Y_1'' := \mathrm{Bl}_{P_3'} Y_1' \dashrightarrow Y_1'.$$

Then we construct a SNC complex surface by gluing Y_1'', Y_2' and Y_3 along their intersection. Consequently we obtain a SNC complex surface $\widetilde{Y} = \widetilde{Y}_1 \cup \widetilde{Y}_2 \cup \widetilde{Y}_3$ with a normalization $\nu : Y_1'' \cap Y_2' \cap Y_3 \to \widetilde{Y}$ such that $\nu(Y_1'') = \widetilde{Y}_1$, $\nu(Y_2') = \widetilde{Y}_2$ and $\nu(Y_3) = \widetilde{Y}_3$. Then we can prove the following.

Proposition 4.2 ([DY21], Proposition 4.3)**.** *The above $\widetilde{Y}$ is d-semistable.*

By applying Theorem 1.1 to $\widetilde{Y}$, we obtain a family of smoothings $\varpi : \mathscr{X} \to \Delta$ of $\widetilde{Y}$ whose general fibers $M_\zeta = \varpi^{-1}(\zeta)$ are compact complex surfaces with trivial canonical bundle. Moreover we prove the following.

Proposition 4.3 ([DY21], Proposition 4.5)**.** *$\widetilde{Y}$ is a d-semistable $K3$ surface of type III. In particular the Euler characteristic of M_ζ is 24.*

For more details on this example, see [DY21] Section 4.

Example 4.4 ([D09], Example 5.3)**.** This example is due to Doi [D09]. Let Σ_n denote the n-the Hirzebruch surface. Recall that the Hirzebruch surface is a toric surface which inherits the corresponding moment polytope (see Figure 1). In particular, Σ_n is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$ having the form $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1})$. Let $E_0 \subset \Sigma_n$ be the image of the section $(0, 1)$ of $\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1}$, that is, the *zero section* of Σ_n. On the other hand, letting σ be any section of $\mathcal{O}_{\mathbb{C}P^1}(n)$, we consider the section $(\sigma, 0)$ of $\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathcal{O}_{\mathbb{C}P^1}$. Away from zeros of σ, we take the image of $(\sigma, 0)$ in Σ_n. Then $(\sigma, 0)$ gives a curve $C_\sigma \subset \Sigma_n$. The *infinity section* E_∞ is the closure of the curve C_σ which is independent of the choice of σ. Then we readily see that E_0 and E_∞ are irreducible curves with self-intersection

$$E_0 \cdot E_0 = n, \qquad E_\infty \cdot E_\infty = -n \tag{4.1}$$

respectively. Let C be a cubic curve in $\mathbb{C}P^2$ and $Y_n = \Sigma_n\big|_C$ the restriction of Σ_n to C. Then we see that Y_n is a $\mathbb{C}P^1$-bundle over the elliptic curve C. By taking the restriction of E_0 and E_∞ to C, we obtain the zero section $D_0 = E_0\big|_C$ and the infinity section $D_\infty = E_\infty\big|_C$

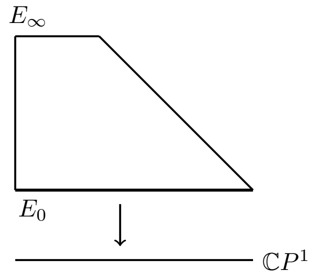

FIGURE 1

of Y_n. For example, in the case of $n = 0$, $Y_0 = C \times \mathbb{C}P^1$ is the trivial bundle while D_0 and D_∞ correspond to the points $[1:0]$ and $[0:1]$ in $\mathbb{C}P^1$ respectively.

In particular, D_0 and D_∞ are naturally isomorphic to the curve C, and $D := D_0 + D_\infty$ defines an anticanonical divisor on Y_n. Since Y_n is a $\mathbb{C}P^1$-bundle over C, $Y_n \setminus D$ is a $\mathbb{C}^*$-bundle over C. Let (x, y) be a local coordinate of $Y_n \setminus D$. Then the fiber coordinate y can be written as

$$\mathbb{C}^* \xrightarrow{\cong} S^1 \times (0, \infty), \qquad y = \exp(-T - \sqrt{-1}\theta) \mapsto (\theta, T)$$

for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $T \in (0, \infty)$. As mentioned in the above, we see that $D_0 = \{\, T = 0 \,\}$ and $D_\infty = \{\, T = \infty \,\}$. Hence we have the isomorphism

$$Y_n \setminus D \cong S_n \times (0, \infty)$$

where S_n is the S^1-bundle over C. Recall that D_0 is a curve in Y_n with $D_0 \cdot D_0 = n$ by (4.1). Since $N_{D_0/Y_n} \cong \mathcal{O}_{Y_n}(D_0)\big|_{D_0}$ from [BHPV, Proposition 6.3], we see that $N_{D_0/Y_n} \cong \mathcal{O}_C(n)$. Similarly the normal bundle of D_∞ is computed as $N_{D_\infty/Y_n} \cong \mathcal{O}_C(-n)$. Then the gluing map h_T is locally given by

$$\begin{array}{ccccc} h_T : Y_n \setminus D \cong & S_n \times (0,\infty) & \longrightarrow & Y_n \setminus D. \\ & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ & (z, T) & \longmapsto & (z', T') = (z, 1/T) \end{array}$$

as in [D09], Remark 3.5. Hence we can glue two copies of $Y_n \setminus D$ along a neighborhood U_1 of 0 and a neighborhood U_2 of ∞ to construct a compact complex surface

$$M_n = (Y_n \setminus D) \cup_{h_T} (Y_n \setminus D) \tag{4.2}$$

with trivial canonical bundle. The above construction shows that M_n is homeomorphic to $S_n \times S^1$ (see [BHPV], p.196).

Let us compute the Betti numbers of the resulting compact complex manifold M_n. We use the following lemma on elliptic fiber bundles.

Lemma 4.5 ([BHPV] p.196, Proposition 5.3)**.** *Let $X \to B$ be a fiber bundle with a smooth compact connected curve B. Suppose that X is homeomorphic to $S \times S^1$, where S is the S^1-bundle over B. Then:*

(A) $b_1(X) = b_1(B) + 2$ *and* $b_2(X) = 2b_1(B) + 2$ *if the bundle $X \to B$ is topologically trivial.*
(B) $b_1(X) = b_1(B) + 1$ *and* $b_2(X) = 2b_1(B)$ *if the bundle $X \to B$ is not topologically trivial.*

Claim 4.6. *Let M_n be a compact complex surface constructed in* (4.2).

(A) *For $n = 0$, we find $b_1(M_0) = 4$, $b_2(M_0) = 6$. Thus the resulting complex manifold M_0 is a complex torus.*

(B) *For $n > 0$, we find $b_1(M_n) = 3$, $b_2(M_n) = 4$. Hence M_n is a primary Kodaira surface.*

Proof. (i) For $n = 0$, we see that $M_0 = C \times \mathbb{C}P^1$. Hence M_0 is the trivial $\mathbb{C}P^1$-bundle over C where $C \in |\mathcal{O}_{\mathbb{C}P^2(3)}|$ is the elliptic curve. Then Lemma 4.5 (i) implies that

$$\begin{aligned} b_1(M_0) &= b_1(C) + 2 = 2 + 2 = 4, \\ b_2(M_0) &= 2b_1(C) + 2 = 2 \cdot 2 + 2 = 6. \end{aligned} \tag{4.3}$$

By the classification of compact complex surfaces with trivial canonical bundle (see [BHPV], p.244 Table 10), we know that such a surface is a complex torus $\mathbb{C}^2/\Lambda$. In particular $h^{p,q}(\mathbb{C}^2/\Lambda) = \binom{2}{p}\binom{2}{q}$ which is consistent with (4.3).

(ii) For $n > 0$, we know that M_n is a $\mathbb{C}P^1$-bundle over C which is not trivial bundle. Hence we see that

$$b_1(M_n) = b_1(C) + 1 = 3, \qquad \text{and} \qquad b_2(M_n) = 2b_1(C) = 4$$

by Lemma 4.5 (ii). Consequently the resulting compact complex manifold is a primary Kodaira surface. See [BHPV], p.197 for their invariants. □

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