Double covers and vector bundles of rank two

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Introduction

In this article, we discuss the correspondence between vector bundles of rank 2 (say 2-*bundles* for short) and line bundles on double covers.

In the study of the embedded topology of curves on the complex projective plane $\mathbb{P}^2$, it is effective to consider the irreducibility of ϕ^*C for an irreducible curve $C \subset \mathbb{P}^2$ and a Galois cover $\phi : X \to \mathbb{P}^2$ (cf. [3], [15], [16]). For example, let $B, C \subset \mathbb{P}^2$ be two plane curves such that $\deg B$ is even and C is irreducible with $\deg B \neq \deg C$, and let $\phi : X \to \mathbb{P}^2$ be the double cover branched at B; then the embedded topology of $B + C$ changes depending on whether ϕ^*C is irreducible or not. In the case where ϕ is a cyclic cover and C is smooth, a criterion for irreducibility of ϕ^*C is known in [8]. This criterion is intensively used to distinguish embedded topology of plane curves (cf. [1], [2], [4], [16]). In the case where ϕ is the double cover branched at a smooth conic and C is a nodal curve, a criterion for irreducibility of ϕ^*C is known in [5]. However, for general $C \subset \mathbb{P}^2$, it is still a problem to determine the irreducibility of ϕ^*C even if ϕ is a double cover. We consider a new approach to the problem, which is constructing various curves on X (which correspond to irreducible components of ϕ^*C) and studying property of their images (which correspond to C). The purpose of this article is a preparation for this new approach by studying a correspondence between line bundles on X and vector bundles of rank 2 (say 2-*bundles* for short) on Y in the case where ϕ is a double cover. As bi-products, we obtain approaches to studying the Picard group of double covers and to constructing 2-bundles. This article gives statements of results without proofs. For the proofs, see [17].

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1 Setting

Let $\phi : X \to Y$ be a non-singular double cover, i.e., a finite morphism of degree two between non-singular complex varieties X, Y. Let $B_\phi \subset Y$ and $R_\phi \subset X$ be the branch locus and the ramification locus of $\phi : X \to Y$, respectively. Let $F \in H^0(Y, \mathcal{O}_Y(B_\phi))$ and $t \in H^0(X, \mathcal{O}_X(R_\phi))$ be sections defining B_ϕ and R_ϕ, respectively, such that $t^2 = F$. There is a divisor L on Y such that $2L \sim B_\phi$

and $\phi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$ as $\mathcal{O}_Y$-modules. Note that we have

$$\phi_* \mathcal{O}_X \cong \left(\bigoplus_{n=0}^{\infty} t^n \mathcal{O}_Y(-nL) \right) \Big/ (t^2 - F)$$

as $\mathcal{O}_Y$-algebras. In 2017, Catanese–Perroni proved the following lemma.

Lemma 1.1 ([7])**.** *If $\mathcal{M}$ is a 2-bundle on Y, and $M : \mathcal{M}(-L) \to \mathcal{M}$ is a morphism satisfying $M^2 = F \cdot \mathrm{id}_{\mathcal{M}}$ (i.e., the composition of $M(-L) : \mathcal{M}(-2L) \to \mathcal{M}(-L)$ and $M : \mathcal{M}(-L) \to \mathcal{M}$ is the multiplication by F), then the pair $(\mathcal{M}, M)$ determines naturally a line bundle $\mathcal{L}_{(\mathcal{M},M)}$ on X such that $\phi_*\mathcal{L}_{(\mathcal{M},M)} \cong \mathcal{M}$, and M corresponds to the multiplication by t in $\mathcal{L}_{(\mathcal{M},M)}$.*

Definition 1.2. Let $(\mathcal{M}, M)$ be a pair of a 2-bundle $\mathcal{M}$ on Y and a morphism $M : \mathcal{M}(-L) \to \mathcal{M}$.

(i) We call $(\mathcal{M}, M)$ an *admissible pair for ϕ* if $M^2 = F \cdot \mathrm{id}_{\mathcal{M}} : \mathcal{M}(-2L) \to \mathcal{M}$.

(ii) Two admissible pairs $(\mathcal{M}, M)$ and $(\mathcal{N}, N)$ are *equivalent* if there exists an isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ such that $\Psi \circ M = N \circ \Psi(-L)$, and write $(\mathcal{M}, M) \sim (\mathcal{N}, N)$.

(iii) $\mathrm{AD}_\phi(Y) := \{(\mathcal{M}, M) :$ an admissible pair for $\phi\ \}/\sim$.

Remark 1.3. We have the following facts.

(i) In [7], it is shown that $(\mathcal{M}, M) \sim (\mathcal{N}, N)$ if and only if $\mathcal{L}_{(\mathcal{M},M)} \cong \mathcal{L}_{(\mathcal{N},N)}$.

(ii) $\mathrm{AD}_\phi(Y)$ has a group structure induced by $\mathrm{Pic}(X)$.

2 The group structure of $\mathrm{AD}_\phi(Y)$

To describe a correspondence between admissible pairs for a non-singular double cover $\phi : X \to Y$ and line bundles on X, we introduce some notation. Let $(\mathcal{M}, M)$ be an admissible pair for ϕ, and let $\mathfrak{U} := \{U_i\}_{i \in I}$ be an affine open covering of Y such that

$$\phi_* \mathcal{O}_X |_{U_i} \cong \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} t_i \qquad \text{as } \mathcal{O}_{U_i}\text{-algebras,}$$
$$\varphi_i : \mathcal{M}_i := \mathcal{M}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus 2} \qquad \text{as } \mathcal{O}_{U_i}\text{-modules}$$

for any $i \in I$, where $t_i := t|_{U_i}$. Note that $t_j = t_i \xi_{ij}$ for $i, j \in I$, where $\xi_{ij} \in \mathcal{O}_{U_i \cap U_j}^{\times}$ correspond to transition functions of $\mathcal{O}_Y(-L)$:

$$\begin{array}{ccc} \mathcal{O}_Y(-L)|_{U_j} & = & \mathcal{O}_Y(-L)|_{U_i} \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{O}_{U_j} & \xrightarrow{\times \xi_{ij}} & \mathcal{O}_{U_i} \end{array}$$

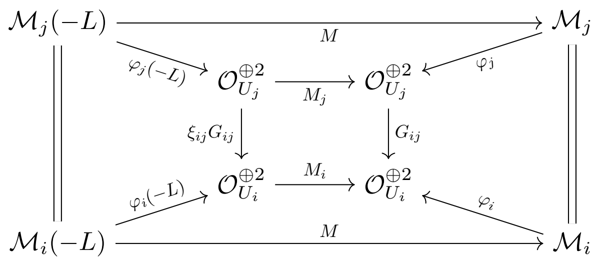
Then we have transition functions $G_{ij} \in \mathrm{GL}(2, \mathcal{O}_{U_i \cap U_j})$ of $\mathcal{M}$ for $i, j \in I$:

$$G_{ij} = \begin{pmatrix} g_{ij,11} & g_{ij,12} \\ g_{ij,21} & g_{ij,22} \end{pmatrix} := \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_{U_j}^{\oplus 2}|_{U_i \cap U_j} \to \mathcal{O}_{U_i}^{\oplus 2}|_{U_i \cap U_j} \tag{1}$$

satisfying $G_{ik} = G_{ij}G_{jk}$ and $G_{ii} = E$ for each $i, j, k \in I$, where E is the identity matrix. The restriction of $M : \mathcal{M}(-L) \to \mathcal{M}$ to U_i corresponds to a matrix M_i:

$$M_i = \begin{pmatrix} a_{i0} & a_{i2} \\ a_{i1} & -a_{i0} \end{pmatrix} := \varphi_i \circ \big(\varphi_i(-L)\big)^{-1} : \mathcal{O}_{U_i}^{\oplus 2} \to \mathcal{O}_{U_i}^{\oplus 2} \tag{2}$$

satisfying $a_{i0}^2 + a_{i1}a_{i2} = F_i := F|_{U_i}$ and $M_j = \xi_{ij} G_{ij}^{-1} M_i G_{ij}$ as elements of $\mathrm{GL}(2, \mathbb{C}(X))$ for each $i, j \in I$:



Definition 2.1. With the above notation, we call $(\{G_{ij}\}, \{M_i\})_{\mathfrak{U}}$ a *representation* of the admissible pair $(\mathcal{M}, M)$. A representation $(\{G_{ij}\}, \{M_i\})_{\mathfrak{U}}$ of $(\mathcal{M}, M)$ is said to be *good* if a_{i1} is a unit on U_i for each $i \in I$.

We can prove the following lemma by taking a ‘fine’ affine open covering of Y and a certain local basis of $\mathcal{M}$.

Lemma 2.2 ([17, Lemma 1.3])**.** *Any admissible pair $(\mathcal{M}, M)$ for ϕ has a good representation.*

We describe the group structure of $\mathrm{AD}_\phi(Y)$. Let $\big(\mathcal{M}^{(k)}, M^{(k)}\big)$ be an admissible pair for a non-singular double cover $\phi : X \to Y$ for each $k = 1, \ldots, m$. Let $\big(\{G_{ij}^{(k)}\}, \{M_i^{(k)}\}\big)_{\mathfrak{U}}$ be a good representation of $\big(\mathcal{M}^{(k)}, M^{(k)}\big)$ for each $k = 1, \ldots, m$, where

$$G_{ij}^{(k)} = \begin{pmatrix} g_{ij,11}^{(k)} & g_{ij,12}^{(k)} \\ g_{ij,21}^{(k)} & g_{ij,22}^{(k)} \end{pmatrix}, \qquad M_i^{(k)} = \begin{pmatrix} a_{i0}^{(k)} & a_{i2}^{(k)} \\ a_{i1}^{(k)} & -a_{i0}^{(k)} \end{pmatrix} \quad (k = 1, \ldots, m).$$

Let $\big(\{G_{ij}^{(0)}\}, \{M_i^{(0)}\}\big)_{\mathfrak{U}}$ be the good representation corresponding to $\mathcal{O}_X$, where

$$G_{ij}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & \xi_{ij} \end{pmatrix}, \qquad M_i^{(0)} = \begin{pmatrix} 0 & F_i \\ 1 & 0 \end{pmatrix}.$$

For 2×2 matrices $A_1, \ldots, A_m$, put $\prod_{k=1}^m A_k := A_1 A_2 \ldots A_m$. Then the following theorem holds.

Theorem 2.3 ([17, Theorem 2.1]). *With the above notation, put*

$$K_{ij}^{(k)+} := \frac{1}{a_{i1}^{(k)}}\Big(\Big(a_{i1}^{(k)} g_{ij,11}^{(k)} - a_{i0}^{(k)} g_{ij,21}^{(k)}\Big) E + g_{ij,21}^{(k)} M_i^{(0)}\Big), \tag{3}$$

$$K_{ij}^{(k)-} := \frac{\xi_{ij}}{a_{i1}^{(k)} \det(G_{ij})}\Big(\Big(a_{i1}^{(k)} g_{ij,11}^{(k)} - a_{i0}^{(k)} g_{ij,21}^{(k)}\Big) E - g_{ij,21}^{(k)} M_i^{(0)}\Big) \tag{4}$$

for each $k = 1, \ldots, m$. *Let* $n_1, \ldots, n_m$ *be* m *integers, and let* $[n]$ *be the list* $[n_1, \ldots, n_m]$. *Then* $\mathcal{L}^{[n]} := \mathcal{L}^{\otimes n_1}_{(\mathcal{M}^{(1)}, M^{(1)})} \otimes \cdots \otimes \mathcal{L}^{\otimes n_m}_{(\mathcal{M}^{(m)}, M^{(m)})}$ *is associated to the normal representation* $\big(\{G_{ij}^{[n]}\}, \{M_i^{[n]}\}\big)_{\mathfrak{U}}$ *with*

$$G_{ij}^{[n]} := \prod_{k=1}^{m} \Big(K_{ij}^{(k)}(n_k)\Big)^{|n_k|} G_{ij}^{(0)}, \qquad M_i^{[n]} := M_i^{(0)},$$

where $K_{ij}^{(k)}(n_k) := K_{ij}^{(k)+}$ *if* $n_k \geq 0$, *and* $K_{ij}^{(k)}(n_k) := K_{ij}^{(k)-}$ *otherwise.*

In the proof of Theorem 2.3, we have the following proposition.

Proposition 2.4 ([17, Proposition 2.5]). *Let* $(\mathcal{M}, M)$ *be an admissible pair for a non-singular double cover* $\phi : X \to Y$, *and let* $(\{G_{ij}\}, \{M_i\})_{\mathfrak{U}}$ *be a good representation of* $(\mathcal{M}, M)$. *Let* $\iota : X \to X$ *be the covering transformation of* ϕ. *Then the followings hold:*

$$\iota^* \mathcal{L}_{(\mathcal{M}, M)} \cong \mathcal{L}_{(\mathcal{M}, -M)}, \tag{5}$$

$$\mathcal{L}_{(\mathcal{M}, M)} \otimes \iota^* \mathcal{L}_{(\mathcal{M}, M)} \cong \phi^*\Big((\det \mathcal{M}) \otimes \mathcal{O}_Y(L)\Big), \tag{6}$$

$$\mathcal{L}^{-1}_{(\mathcal{M}, M)} \cong \phi^*\Big((\det \mathcal{M})^{-1} \otimes \mathcal{O}_Y(-L)\Big) \otimes \mathcal{L}_{(\mathcal{M}, -M)}. \tag{7}$$

Moreover, $\mathcal{L}^{-1}_{(\mathcal{M}, M)}$ *is associated to*

$$\left(\left\{\frac{\xi_{ij}}{\det(G_{ij})} G_{ij}\right\}, \{-M_i\}\right)_{\mathfrak{U}}.$$

3 A subgroup of $\mathrm{Pic}(X)$

We have seen the correspondence between admissible pairs for a non-singular double cover $\phi : X \to Y$ and line bundles on X. Hence it is effective for understanding $\mathrm{Pic}(X)$ to study $\mathrm{AD}_\phi(Y)$. However, it seems difficult to find a morphism $M : \mathcal{M}(-L) \to \mathcal{M}$ satisfying $M^2 = F \cdot \mathrm{id}_{\mathcal{M}}$ for a general 2-bundle $\mathcal{M}$ on Y. In the case where $\mathcal{M} \cong \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2)$ for some divisors D_1, D_2 on Y, such a morphism M can be represented as

$$M = \begin{pmatrix} a_0 & a_2 \\ a_1 & -a_0 \end{pmatrix} : \mathcal{O}_Y(D_1 - L) \oplus \mathcal{O}_Y(D_2 - L) \to \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2), \tag{8}$$

where a_0, a_1 and a_2 are global sections of $\mathcal{O}_Y(L)$, $\mathcal{O}_Y(L - D_1 + D_2)$ and $\mathcal{O}_Y(L + D_1 - D_2)$, respectively, satisfying $a_0^2 + a_1 a_2 = F$.

Definition 3.1. Let $\phi : X \to Y$ be a non-singular double cover.

(i) We say that a line bundle $\mathcal{L}$ on X *splits* with respect to ϕ if $\phi_*\mathcal{L}$ is the direct sum of two line bundles on Y.

(ii) Let $\mathrm{sPic}_\phi(X)$ denote the subgroup of $\mathrm{Pic}(X)$ generated by line bundles which split with respect to ϕ ("s" of sPic means "sub" or "split");

$$\mathrm{sPic}_\phi(X) := \Big\langle [\mathcal{L}] \in \mathrm{Pic}(X) \;\Big|\; \mathcal{L} \text{ splits with respect to } \phi \Big\rangle.$$

Remark 3.2. If $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D_1) \oplus \mathcal{O}_Y(D_2)$, then

$$\phi_*\big(\mathcal{L} \otimes \phi^* \mathcal{O}_Y(-D_2)\big) \cong \mathcal{O}_Y(D_1 - D_2) \oplus \mathcal{O}_Y$$

by projection formula. Since $\phi^* \mathcal{O}_Y(D_2) \in \mathrm{sPic}_\phi(X)$, the subgroup $\mathrm{sPic}_\phi(X)$ is generated by $\phi^*(\mathrm{Pic}(Y))$ and line bundles $\mathcal{L}$ satisfying $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ on X for some divisor D' on Y.

Lemma 3.3 ([17, Lemma 3.3])**.** *If Y is an open subset of a smooth projective variety $\overline{Y}$ with $\mathrm{codim}_{\overline{Y}}(\overline{Y} \setminus Y) \geq 2$, then $\mathrm{H}^0(Y, \mathcal{O}_Y) = \mathbb{C}$, and $\mathrm{sPic}_\phi(X)$ is generated by $\phi^*(\mathrm{Pic}(Y))$ and line bundles $\mathcal{L}$ with $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ such that either $\mathcal{O}_Y(D') \cong \mathcal{O}_Y$ or $\mathrm{H}^0(Y, \mathcal{O}_Y(D')) = 0$.*

Proof. For an irreducible divisor C on Y, let $\overline{C}$ denote the closure of C on $\overline{Y}$. Let D be a divisor on Y, and put $\overline{D} := \sum_{i=1}^k \overline{C}_i$, where $D = \sum_{i=1}^k C_i$ is the irreducible decomposition of D. Then $\mathcal{O}_Y(D) = \mathcal{O}_{\overline{Y}}(\overline{D})|_Y$. By [9, Proposition 1.6], we have $\mathrm{H}^0\left(Y, \mathcal{O}_Y(D)\right) = \mathrm{H}^0\left(\overline{Y}, \mathcal{O}_{\overline{Y}}(\overline{D})\right)$. Thus

$$\mathrm{H}^0(Y, \mathcal{O}_Y) = \mathrm{H}^0\left(\overline{Y}, \mathcal{O}_{\overline{Y}}\right) = \mathbb{C}.$$

For a line bundle $\mathcal{L}$ on X with $\phi_*\mathcal{L} \cong \mathcal{O}_Y(D) \oplus \mathcal{O}_Y$, if $\mathcal{O}_Y(D) \not\cong \mathcal{O}_Y$ and $\mathrm{H}^0(Y, \mathcal{O}_Y(D)) \neq 0$, then $\mathrm{H}^0(Y, \mathcal{O}_Y(-D)) = 0$ and

$$\phi_*\Big(\mathcal{L} \otimes \phi^*\big(\mathcal{O}_Y(-D)\big)\Big) \cong \mathcal{O}_Y(-D) \oplus \mathcal{O}_Y .$$

Therefore the assertion holds true by Remark 3.2. □

Theorem 3.4 is a criterion for splitting of the push-forward for a line bundle on X (see Remark 3.2 and Lemma 3.3).

Theorem 3.4 ([17, Theorem 2])**.** *Let D^+ be an effective divisor on X, and let D be the effective divisor on Y defined by $f = 0$ for $f \in \mathrm{H}^0\left(Y, \mathcal{O}_Y(D)\right)$ such that $\phi^*D = D^+ + \iota^*D^+$. Assume that $\mathrm{H}^0(Y, \mathcal{O}_Y) = \mathbb{C}$. If $\phi_* \mathcal{O}_X(D^+) \cong \mathcal{O}_Y(D') \oplus \mathcal{O}_Y$ for a divisor D' on Y satisfying either $\mathcal{O}_Y(D') \cong \mathcal{O}_Y$ or $\mathrm{H}^0\left(Y, \mathcal{O}_Y(D')\right) = 0$, then D and D' satisfy the following two conditions;*

(i) *D' is linearly equivalent to $D - L$, i.e., $\mathcal{O}_Y(D') \cong \mathcal{O}_Y(D - L)$; and*

(ii) *there are global sections a_0 and a_1 of $\mathcal{O}_Y(L)$ and $\mathcal{O}_Y(2L - D)$, respectively, such that $F = a_0^2 + f a_1$.*

Moreover, in the case where D^+ is irreducible, the converse holds true.

Remark 3.5. In the case of $Y = \mathbb{P}^1$, Jacobi [10] have studied the correspondence between line bundles on the hyperelliptic curve X and equations of the form $F = a_0^2 + a_1 a_2$ via the Jacobian variety. The study of [10] relates to Hitchin theory (cf. [18]).

Theorem 3.4 implies that generators of $\mathrm{sPic}_\phi(X)$ correspond to equations of the form $F = a_0^2 + a_1 a_2$. Hence we can expect that $\mathrm{sPic}_\phi(X)$ reflects the arrangement of B_ϕ in Y enough to describe the structure of $\mathrm{Pic}(X)$. In several examples below, the equation $\mathrm{sPic}_\phi(X) = \mathrm{Pic}(X)$ holds.

Example 3.6. Let X be a hyperelliptic curve, and let $\phi : X \to \mathbb{P}^1$ be a non-singular double cover. Since any rank 2-bundle on $\mathbb{P}^1$ splits, we have $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Example 3.7. Let $\phi : X \to \mathbb{P}^2$ be a double cover branched along a smooth conic B_ϕ. Note that $\deg(L) = 1$. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let D^+ be a ruling of $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. The image $D = \phi(D^+)$ is a tangent line of B_ϕ. Let $f \in \mathrm{H}^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\right)$ be a section defining D. Then B_ϕ is given by $a_0^2 + f a_1 = 0$ for some $a_0, a_1 \in \mathrm{H}^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\right)$. By Proposition 2.4 and Theorem 3.4, we obtain $\phi_* \mathcal{O}_X(D^+) \cong \phi_* \mathcal{O}_X(\iota^* D^+) \cong \mathcal{O}_Y^{\oplus 2}$. Since $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(D^+)$ and $\mathcal{O}_X(\iota^* D^+)$, we have $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Example 3.8. Let $\phi : X \to \mathbb{P}^2$ be a double cover branched along a smooth quartic B_ϕ. Note that $\deg(L) = 2$. Then X is isomorphic to the blowing-up of $\mathbb{P}^2$ at 7 points in general position. Moreover $\mathrm{Pic}(X)$ is generated by 8 (-1)-curves $E_0, \ldots, E_7$, where E_0 is the strict transform of a line passing through two bowing-up centers, and $E_1, \ldots, E_7$ are the exceptional divisors. The images $\phi(E_0), \ldots, \phi(E_7)$ are 8 of 28 bitangent lines of B_ϕ. Let $f_j \in \mathrm{H}^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\right)$ be a section defining $\phi(E_j)$ for each $j = 0, \ldots, 7$. Then there exist global sections $a_{j,k}$ $(k = 0, 1)$ of $\mathcal{O}_{\mathbb{P}^2}(k+2)$ such that B_ϕ is defined by $a_{j,0}^2 + f_j a_{j,1} = 0$. Hence $\phi_* \mathcal{O}_X(E_j) \cong \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}$ for each $j = 0, \ldots, 7$ by Theorem 3.4. Therefore we obtain $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

The following conjecture arises.

Conjecture 3.9. *If* $\mathrm{H}^0(Y, \mathcal{O}_Y) = \mathbb{C}$, *then* $\mathrm{Pic}(X) = \mathrm{sPic}_\phi(X)$.

Remark 3.10. In the cases of Examples 3.7 and 3.8, it is known that $\phi_* \mathcal{L}$ is indecomposable for general line bundles $\mathcal{L}$ on X in [13].

4 An idea to generate 2-bundles

In this section, we give an idea to generate 2-bundles through double covers. Schwarzenberger proved the following theorem.

Theorem 4.1 (Schwarzenberger [13, Theorem 3])**.** *Let Y be a non-singular surface. For any 2-bundle $\mathcal{E}$ on Y, there exist a non-singular double cover $\phi : X \to Y$ and a line bundle $\mathcal{L}$ on X such that $\phi_* \mathcal{L} \cong \mathcal{E}$.*

We give a generalization of this theorem. We call $\phi : X \to Y$ a *normal double cover* if ϕ is a finite surjective morphism of degree two from a normal variety X to a smooth variety Y over $\mathbb{C}$. Let $\mathrm{Cl}(X)$ be the divisor class group of X. Note that there is a canonical one-to-one correspondence between $\mathrm{Cl}(X)$ and the set of divisorial sheaves on X (cf. [14]). See [9] for general results on reflexive sheaves. By modifying the proof of [13, Theorem 3], we can prove the following theorem which is a generalization of [13, Theorem 3].

Theorem 4.2 ([17, Theorem 3])**.** *Let $\mathcal{E}$ be a 2-bundle on a smooth projective variety Y of dimension n over $\mathbb{C}$. There exist a normal double cover $\phi : X \to Y$ and a divisorial sheaf $\mathcal{L}$ on X such that $\mathcal{E} \cong \phi_*\mathcal{L}$.*

Proof. Let $\mathbb{P}_{\mathcal{E}}$ be the $\mathbb{P}^1$-bundle $\mathbf{Proj}(S(\mathcal{E}))$, and let $p : \mathbb{P}_{\mathcal{E}} \to Y$ be the projection, where $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$. Let $\mathcal{H}$ be a very ample line bundle on Y. The line bundle $\mathcal{H}$ gives the embedding $\Phi_{\mathcal{H}} : Y \hookrightarrow \mathbb{P}^s$ with $s+1 = \dim \mathrm{H}^0(Y, \mathcal{H})$. For k large enough, we have the following exact sequence:

$$\mathcal{H}^{\oplus r+1} \to \mathcal{E} \otimes \mathcal{H}^k \to 0$$

for some $r > 0$. This induces an embedding $i : \mathbb{P}_{\mathcal{E}} \hookrightarrow \mathbb{P}^N$ for $N = rs + r + s$ via the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \hookrightarrow \mathbb{P}^N$. Put $\widetilde{\mathcal{L}} := i^* \mathcal{O}_{\mathbb{P}^N}(1)$. Note that $i(p^{-1}(P))$ is a line in $\mathbb{P}^N$ for each $P \in Y$, and $p_*\widetilde{\mathcal{L}} \cong \mathcal{E} \otimes \mathcal{H}^k$. Hence $i : \mathbb{P}_{\mathcal{E}} \hookrightarrow \mathbb{P}^N$ induces an embedding $i' : Y \hookrightarrow \mathrm{Gr}_1(N)$ by $P \mapsto i(p^{-1}(P))$, where $\mathrm{Gr}_1(N)$ is the Grassmannian consisting of lines in $\mathbb{P}^N$.

For a quadratic hypersurface $Q \subset \mathbb{P}^N$, let $V(Q)$ be the subscheme of $\mathrm{Gr}_1(N)$ consisting of lines on Q. Note that $\mathrm{PGL}(N, \mathbb{C}) := \mathrm{Aut}(\mathbb{P}^N)$ acts transitively on both of $\mathbb{P}^N$ and $\mathrm{Gr}_1(N)$ such that $V(g(Q)) = g(V(Q))$ for any $g \in \mathrm{PGL}(N, \mathbb{C})$ and $Q \subset \mathbb{P}^N$. Since $\dim \mathrm{Gr}_1(N) = 2N - 2$, $\dim V(Q) = 2N - 5$ (cf. [6]) and $\dim Y = n$, we obtain $\dim Y \cap V(Q) = n - 3$ for a general hypersurface $Q \subset \mathbb{P}^N$ of degree 2 by [11, Theorem 2]. Put $X' := \mathbb{P}_{\mathcal{E}} \cap Q$ for a general quadratic hypersurface $Q \subset \mathbb{P}^N$ such that $\dim i'(Y) \cap V(Q) = n - 3$ and X' is smooth. Let $\mathcal{L}'$ be the restriction of $\widetilde{\mathcal{L}}$ to X'. For an affine open set U of Y, the Künneth formula for sheaves implies that

$$R^q p_* \widetilde{\mathcal{L}}^{-1}(U) = \mathrm{H}^q(U \times \mathbb{P}^1, \mathcal{O}_U \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$$

for all $q \geq 0$. Since $\widetilde{\mathcal{L}} \otimes \mathcal{J}_{X'} \cong \widetilde{\mathcal{L}}^{-1}$ for the ideal sheaf $\mathcal{J}_{X'} \cong \widetilde{\mathcal{L}}^{-2}$ of X', we have $p'_*\mathcal{L}' \cong \mathcal{E} \otimes \mathcal{H}^k$ by [13, Proposition 5].

Then the restriction $p' := p|_{X'} : X' \to Y$ is a generically finite morphism of degree 2. Let $U' := \{P \in Y \mid (p')^{-1}(P) \text{ is finite}\}$. By Stein factorization of p', we obtain a birational morphism $f' : X' \to X''$ and a finite morphism $g' : X'' \to Y$ such that $g' \circ f' = p'$. Take the normalization $\kappa : X \to X''$, and put $\phi := g' \circ \kappa : X \to Y$, which is a normal double cover. Let $\mathcal{L}$ be the double dual $(\kappa^* f_* \mathcal{L}')^{\vee\vee}$ of $\kappa^* f_* \mathcal{L}'$. Then $\mathcal{L}$ is a divisorial sheaf on X, and $\phi_*\mathcal{L}|_{U'} \cong p'_*\mathcal{L}'|_{U'}$ since f' and κ are isomorphic over U'. Since $\phi_*\mathcal{L}$ is reflexive by [9, Corollary 1.7] and $\mathrm{codim}_Y(Y \setminus U') = 3$, $\phi_*\mathcal{L} \cong p'_*\mathcal{L}' \cong \mathcal{E} \otimes \mathcal{H}^k$. Therefore, $\phi_*(\mathcal{L} \otimes \phi^*\mathcal{H}^{-k}) \cong \mathcal{E}$. □

As the idea of [7], we can apply our method to divisorial sheaves on normal double covers as follows. Let $\phi : X \to Y$ be a normal double cover. Let X° be the smooth locus $X \setminus \mathrm{Sing}(X)$ of X, and put $Y^\circ := \phi(X^\circ)$. Then the restriction $\phi^\circ : X^\circ \to Y^\circ$ of ϕ is a non-singular double cover. For a divisorial sheaf $\mathcal{L}$ on X, the restriction $\mathcal{L}^\circ$ of $\mathcal{L}$ to X° is a line bundle on X°, and $i_*\mathcal{L}^\circ = \mathcal{L}$ and $j_*\phi^\circ_*\mathcal{L}^\circ = \phi_*\mathcal{L}$ hold, where $i : X^\circ \to X$ and $j : Y^\circ \to Y$ are the inclusion maps. Hence computation of push-forwards of line bundles on X° can be applied to that of divisorial sheaves on X via j_*.

If Conjecture 3.9 is true, then Theorem 4.2 implies that any 2-bundle on $\mathbb{P}^n$ can be generated by the following method:

(i) Take a reduced divisor $B : F = 0$ of even degree on $\mathbb{P}^n$ with several representation of the form $F = a_0^2 + a_1 a_2$;

(ii) let $\phi : X \to \mathbb{P}^n$ be the normal double cover branched at B, and let $\phi^\circ : X^\circ \to Y^\circ$ be the non-singular double cover as above;

(iii) take several line bundles $\mathcal{L}_1, \dots, \mathcal{L}_m$ on X° such that $\phi^\circ_*\mathcal{L}_i$ is split, and compute 2-bundles $\phi^\circ_*(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_m^{n_m})$ on Y° by Theorem 2.3;

(iv) then $j_*\phi^\circ_*(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_m^{n_m})$ are reflexive sheaves of rank two.

Remark 4.3. Let $\phi : X \to \mathbb{P}^2$ be a non-singular double cover branched along smooth curve of degree $2r$. In [13], $\phi_*\mathcal{L}$ was studied for several line bundles $\mathcal{L}$ on X in the case of $r = 1, 2$. Ottaviani [12] and Vallès [19] studied the direct images of line bundles on X.

If $\phi : X \to \mathbb{P}^n$ is non-singular, then the reflexive sheaves in (iv) are 2-bundles. A problem of this method is when a reflexive sheaf of (iv) is a 2-bundle.

Problem 4.4. Give a condition for a reflexive sheaf $j_*\phi^\circ_*(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_m^{n_m})$ in (iv) to be a 2-bundle.

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