RELATIONS BETWEEN TWO LOG MINIMAL MODELS OF LOG CANONICAL PAIRS

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1. Introduction

Throughout this paper we will work over the complex number field $\mathbb{C}$.

In the birational geometry, for a given smooth projective variety, it is expected that there exists a "good" birational model. The birational model is called *minimal model*, and an inductive procedure called *minimal model program* is expected to construct a minimal model. The existence of a minimal model for smooth projective varieties is one of the most important conjectures in the birational geometry. Currently, the conjecture is studied in the framework of pairs of a normal projective variety and an effective divisor such that the pairs have mild singularity, called log canonical pairs (in this framework minimal models are called *log minimal models*). Various special cases of the conjecture are known in all dimensions (see, for example, [BCHM], [HH]).

Suppose that a given log canonical pair has at least one log minimal model. In the case of log canonical surface, the log minimal model is uniquely determined. On the other hand, when the underlying variety of the log canonical pair has dimension at least 3, the log minimal model is not uniquely determined. For any given two log minimal model for the log canonical pair, it is interesting to study common properties that the two log minimal model have. In the case of $\mathbb{Q}$-factorial Kawamata log terminal pairs, which is a special class of log canonical pairs, it is known that the two log minimal models of a $\mathbb{Q}$-factorial Kawamata log terminal pair is connected by a sequence of flops ([BCHM], [K]).

In this article, we explain a generalization of the result to not necessarily $\mathbb{Q}$-factorial log canonical pairs. We also introduce an example which illustrates gaps between the class of $\mathbb{Q}$-factorial Kawamata log terminal pairs and the class of (not necessarily $\mathbb{Q}$-factorial) log canonical pairs.

2. Background and known results

We start with the definition of singularities of pairs.

A *pair* (X, Δ) consist of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor Δ on X whose coefficients belong to $[0, 1]$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The divisor Δ in (X, Δ) is called a *boundary divisor*, and $K_X + \Delta$ is called the *log canonical divisor*.

Definition 2.1 (Log canonical pairs and Kawamata log terminal pairs)**.** Let (X, Δ) be a pair. For any projective birational morphism $f : Y \to X$, we can write

$$K_Y = f^*(K_X + \Delta) + \sum_j a(E_j, X, \Delta)E_j,$$

where $a(E_j, X, \Delta)$ are rational numbers and E_j are distinct prime divisors. Then $a(E_j, X, \Delta)$ is called the *discrepancy* of E_j with respect to (X, Δ). The pair (X, Δ) is a *log canonical pair* (resp. a *Kawamata log terminal pair*) if $a(E_j, X, \Delta) \geq -1$ (resp. > -1) for all f and all E_j. When the underlying variety X is $\mathbb{Q}$-factorial, Kawamata log terminal pairs (X, Δ) are called *$\mathbb{Q}$-factorial Kawamata log terminal pair*.

We denote by $(X, \Delta)/Z$ a log canonical pair (X, Δ) equipped with a projective morphism $X \to Z$ of normal quasi-projective varieties. If there is no risk of confusion, $(X, \Delta)/Z$ is also called a log canonical pair.

In the birational geometry, the following conjecture on the existence of log minimal model for log canonical pairs is one of the most important open problems.

Conjecture 2.2. *Let $(X, \Delta)/Z$ be a log canonical pair equipped with a projective morphism $X \to Z$ such that $K_X + \Delta$ is pseudo-effective over Z. Then, there is a sequence of steps of a $(K_X + \Delta)$-minimal model program over Z*

$$(X, \Delta) \dashrightarrow (X', \Delta')$$

such that $K_{X'} + \Delta'$ is nef over Z.

By construction of $(K_X + \Delta)$-minimal model program, it follows that (X', Δ') is a log canonical pair. We call $(X', \Delta')/Z$ a *log minimal model* of (X, Δ) over Z.

Conjecture 2.2 for log canonical pairs $(X, \Delta)/Z$ with $\dim X \leq 3$ is known by Kawamata, Kollár, Mori, Reid, Shokurov, and others. In higher-dimensional case, we can run a minimal model program for all log canonical pairs ([F], [B2], [HX]) and Conjecture 2.2 is known in the following cases.

- $\dim X = 4$ ([B1]),
- (X, Δ) is a Kawamata log terminal pair and $K_X + \Delta$ or Δ is big over Z ([BCHM]),
- $\Delta = B + A$ such that $B \geq 0$, $A \geq 0$, and A is ample over Z ([HH]), and
- a $\mathbb{Q}$-factorial dlt model of (X, Δ) has a log minimal model over Z ([HH]).

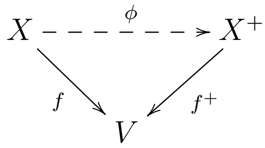
But, Conjecture 2.2 is currently widely open for higher-dimensional log canonical pairs.

From now on, we pay attention to log canonical pairs $(X, \Delta)/Z$ for which Conjecture 2.2 holds. For simplicity, pick a log canonical pair $(X, \Delta)/Z$ with $Z = \mathrm{Spec}\mathbb{C}$ for which Conjecture 2.2 holds. Let $\psi_1 \colon (X, \Delta) \dashrightarrow (X'_1, \Delta'_1)$ and $\psi_2 \colon (X, \Delta) \dashrightarrow (X'_2, \Delta'_2)$ be two sequences of steps of $(K_X + \Delta)$-minimal model program to log minimal models (X'_1, Δ'_1) and (X'_2, Δ'_2), respectively. If $\dim X = 2$, then the negativity lemma and construction of ψ_i $(i = 1, 2)$ implies that the induced birational map $\phi \colon X'_1 \dashrightarrow X'_2$ is an isomorphism and $\phi_* \Delta'_1 = \Delta'_2$. But, when $\dim X \geq 3$, we only know that the induced birational map $\phi \colon X'_1 \dashrightarrow X'_2$ is isomorphic in codimension one and $\phi_* \Delta'_1 = \Delta'_2$. Therefore, it is very natural to consider the following question.

Question 2.3. *Let $(X, \Delta)/Z$ be a log canonical pair equipped with a projective morphism $X \to Z$. Suppose that there are two sequences of steps of $(K_X + \Delta)$-minimal model program $\psi_1 \colon (X, \Delta) \dashrightarrow (X'_1, \Delta'_1)$ and $\psi_2 \colon (X, \Delta) \dashrightarrow (X'_2, \Delta'_2)$ to log minimal models (X'_1, Δ'_1) and (X'_2, Δ'_2). Under the situation, what common properties do (X'_1, Δ'_1) and (X'_2, Δ'_2) have?*

To introduce known result concerned with Question 2.3, we define flops.

Definition 2.4 (Flop)**.** Let $(X, \Delta)/Z$ be a log canonical pair equipped with a projective morphism $X \to Z$. A *flop* for $K_X + \Delta$ over Z is the following diagram over Z



such that

- the diagram is a D-flip for some $\mathbb{Q}$-Cartier divisor D, more precisely,
  - f is a small birational morphism, $-D$ is f-ample, and $\rho(X/V) = 1$, and
  - f^+ is a small birational morphism and $\phi_* D$ is an f^+-ample $\mathbb{Q}$-Cartier divisor,

  and
- $K_X + \Delta$ is numerically trivial over V.

In the case of $\mathbb{Q}$-factorial Kawamata log terminal pairs, as shown below, it is known that the birational map between two log minimal models are sequences of flops.

Theorem 2.5 (cf. [BCHM])**.** *Let $(X_1, \Delta_1)/Z$ and $(X_2, \Delta_2)/Z$ be two $\mathbb{Q}$-factorial Kawamata log terminal pairs such that $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are nef over Z. Let $\phi\colon X_1 \dashrightarrow X_2$ be a small birational map over Z such that $\phi_*\Delta_1 = \Delta_2$. Then, ϕ is a composition of flops for $K_X + \Delta$ over Z.*

Remark 2.6. Kawamata [K] proved a similar result for $\mathbb{Q}$-factorial terminal pairs (a special class of Kawamata log terminal pairs) and a birational map with a weaker assumption. More precisely, given $\mathbb{Q}$-factorial terminal pairs $(X_1, \Delta_1)/Z$ and $(X_2, \Delta_2)/Z$ such that $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are nef over Z and a (not necessarily small) birational map $\phi\colon X_1 \dashrightarrow X_2$ such that $\phi_*\Delta_1 = \Delta_2$, then ϕ is a composition of flops for $K_X + \Delta$ over Z.

We note the underlying varieties of Kawamata log terminal pairs have only rational singularity. By this fact and Theorem 2.5, we obtain a partial answer to Question 2.3.

Theorem 2.7. *Let $(X_1, \Delta_1)/Z$, $(X_2, \Delta_2)/Z$ and $\phi\colon X_1 \dashrightarrow X_2$ be as in Theorem 2.5. For $i = 1, 2$, we denote the morphism $X_i \to Z$ by π_i. Then the followings hold.*

- *$R^p\pi_{1*}\mathcal{O}_{X_1} \simeq R^p\pi_{2*}\mathcal{O}_{X_2}$ for every $p > 0$, and*
- *the Cartier index of $K_{X_1} + \Delta_1$ and the Cartier index of $K_{X_2} + \Delta_2$ coincide.*

We will discuss the generalization of Theorem 2.7 to log canonical pairs.

3. Log canonical case

Before introducing the main results, we define extremal contractions and log canonical centers.

Definition 3.1 (Extremal contraction)**.** A contraction $f\colon X \to Y$ of normal quasi-projective varieties is an *extremal contraction* if for any two Cartier divisors D_1 and D_2, there are $a_1, a_2 \in \mathbb{Z}$ which are not both zero and a Cartier divisor D_Y on Y such that $a_1 D_1 - a_2 D_2 \sim f^* D_Y$.

Note that isomorphisms are extremal contractions in this article. By definition, any non-isomorphic extremal contraction $X \to Y$ satisfies $\rho(X/Y) = 1$, but the converse is not true in general. For example, let X be an elliptic curve and Y a point. Then $\rho(X/Y) = 1$ but there exists a numerically trivial Cartier divisor on X which is not a torsion.

Definition 3.2 (Log canonical centers)**.** Let (X, Δ) be a log canonical pair. Then a *log canonical center* of (X, Δ) is the image of a prime divisor P appearing in a birational model $Y \to X$ whose discrepancy $a(P, X, \Delta)$ (see Definition 2.1) is -1.

When (X, Δ) is a Kawamata log terminal pair, then there is no log canonical center by Definition 2.1.

We are ready to state the main results.

Theorem 3.3 (Main result I, [H, Theorem 1.1])**.** *Let $(X, \Delta)/Z$ and $(X', \Delta')/Z$ be log canonical pairs such that $K_X + \Delta$ and $K_{X'} + \Delta'$ are nef over Z. Suppose that there is a small birational map $\phi\colon X \dashrightarrow X'$ over Z such that*

- *$\phi_* \Delta = \Delta'$, and*
- *there is an open subset $U \subset X$ such that ϕ is an isomorphism on U and all log canonical centers of (X, Δ) intersect U.*

Then, there are projective small birational morphisms $f\colon \overline{X} \to X$ and $f'\colon \overline{X}' \to X'$ from normal quasi-projective varieties such that f and f' are compositions of extremal contractions and the induced birational map $f'^{-1} \circ \phi \circ f\colon \overline{X} \dashrightarrow \overline{X}'$ is a composition of flops for $K_{\overline{X}} + f_*^{-1}\Delta$ over Z

$$\begin{array}{ccccccccccccc}
\overline{X} & = & \overline{X}_0 & \overset{\varphi_0}{\dashrightarrow} & \overline{X}_1 & \dashrightarrow \cdots \dashrightarrow & \overline{X}_i & \overset{\varphi_i}{\dashrightarrow} & \overline{X}_{i+1} & \dashrightarrow \cdots \dashrightarrow & \overline{X}_l & = & \overline{X}' \\
 & & & \searrow \ V_0 \ \swarrow & & & & \searrow \ V_i \ \swarrow & & & & &
\end{array}$$

satisfying the following property:

(∗) *$\overline{X}_i$ is $\mathbb{Q}$-factorial if and only if $\overline{X}_{i+1}$ is $\mathbb{Q}$-factorial for any $0 \le i < l$, and each φ_i induces an isomorphic linear map $\varphi_{i*}\colon N^1(\overline{X}_i/Z)_{\mathbb{R}} \to N^1(\overline{X}_{i+1}/Z)_{\mathbb{R}}$.*

Theorem 3.4 (Main result II, see [H, Theorem 1.2])**.** *Let $(X, \Delta)/Z$, $(X', \Delta')/Z$ and $\phi\colon X \dashrightarrow X'$ be as in Theorem 3.3. We denote $X \to Z$ and $X' \to Z$ by π and π', respectively. Then the followings hold.*

- *$R^p\pi_*\mathcal{O}_X \xrightarrow{\sim} R^p\pi'_*\mathcal{O}_{X'}$ for every $p > 0$, and*
- *for any Cartier divisor D on X such that $D \equiv_Z r(K_X + \Delta)$ for some $r \in \mathbb{R}$, the birational transform $\phi_* D$ is Cartier and $\phi_* D \equiv_Z r(K_{X'} + \Delta')$. In particular, for every integer l, the divisor $l(K_X + \Delta)$ is Cartier if and only if $l(K_{X'} + \Delta')$ is Cartier.*

Remark 3.5. The second assertion of Theorem 3.4 implies the following fact: With notation as in Theorem 3.4, when Z is a point, the natural morphism $\mathrm{Pic}(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ is an isomorphism if and only if $\mathrm{Pic}(X')_{\mathbb{R}} \to N^1(X')_{\mathbb{R}}$ is so. Moreover, [H, Theorem 1.2] shows that X has a small $\mathbb{Q}$-factorialization (i.e., a small projective birational morphism from a $\mathbb{Q}$-factorial variety) if and only if X' has a small $\mathbb{Q}$-factorialization.

To check that Theorem 3.3 is a generalization of Theorem 2.5, we consider the $\mathbb{Q}$-factorial Kawamata log terminal case of Theorem 3.3. Let $(X, \Delta)/Z$ and $(X', \Delta')/Z$ be $\mathbb{Q}$-factorial Kawamata log terminal pairs such that $K_X + \Delta$ and $K_{X'} + \Delta'$ are nef over Z. Since there is no log canonical center of (X, Δ) and (X', Δ'), the existence of $\phi\colon X \dashrightarrow X'$ implies that X and X' are isomorphic in codimension one and Δ' is the birational transform of Δ on X'. Then there exist small birational morphisms $f\colon \overline{X} \to X$ and $f'\colon \overline{X}' \to X'$ as in Theorem 3.3 such that the induced birational map $f'^{-1} \circ \phi \circ f\colon \overline{X} \dashrightarrow \overline{X}'$ is a composition of flops. Because X (resp. X') is $\mathbb{Q}$-factorial and f (resp. f') is small, it follows that f (resp. f') is an isomorphism. This shows that ϕ is a composition of flops. Therefore, we see that Theorem 3.3 is a generalization of Theorem 2.5. By a similar way, we see that Theorem 3.4 is a generalization of Theorem 2.7.

We compare Theorem 3.3 and Theorem 2.5. In Theorem 3.3, the small birational map ϕ between the given two log canonical pairs need to have a good property related to log canonical centers. Moreover, we have to take small birational modifications to connect the log canonical pairs by a sequence of flops. On the other hand, in Theorem 2.5, we do not need any special assumption of the small birational map between the given two $\mathbb{Q}$-factorial Kawamata log terminal pairs, and the birational map can be decomposed to flops. These differences come from gaps between (not necessarily $\mathbb{Q}$-factorial) log canonical pairs and $\mathbb{Q}$-factorial Kawamata log terminal pairs. The following example shows that $\mathbb{Q}$-factoriality is necessary to decompose the birational map $\phi\colon X \dashrightarrow X'$ as in Theorem 3.3 into flops.

Example 3.6 ([H, Example 4.1])**.** Let (X, Δ) be a Kawamata log terminal pair such that X is projective and not $\mathbb{Q}$-factorial, $K_X + \Delta$ is nef and $\rho(X) = 1$. For example, take X as a normal projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$ and (X, Δ) as a Kawamata log terminal pair such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Let X' be a small $\mathbb{Q}$-factorialization of X, and let Δ' be the birational transform of Δ on X'. Then (X, Δ) and (X', Δ') cannot be connected by flops because $\rho(X) = 1$ which implies that there is no non-trivial contraction from X.

An important gap between log canonical pairs and Kawamata log terminal pairs is the existence of log canonical centers. The gap appears in the second condition of the birational map $\phi\colon X \dashrightarrow X'$ in the main results. The second condition of $\phi\colon X \dashrightarrow X'$ in the main results looks technical, so it is natural to expect that the same conclusions as in the main results hold true without the second condition of the birational map. Unfortunately, the expectation cannot be realized in general.

Theorem 3.7 (cf. [H, Section 4])**.** *There exist log canonical pairs (X_1, Δ_1) and (X_2, Δ_2) such that X_1 and X_2 are projective, $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are both nef, and there is a small birational map $\phi\colon X_1 \dashrightarrow X_2$ satisfying the following properties:*

- $\phi_*\Delta_1 = \Delta_2$,
- *X_1 is $\mathbb{Q}$-factorial but X_2 has no small $\mathbb{Q}$-factorialization,*
- $\dim H^1(X_1, \mathcal{O}_{X_1}) \neq \dim H^1(X_2, \mathcal{O}_{X_2})$, *and*
- *ϕ_*D is not $\mathbb{Q}$-Cartier for a numerically trivial Cartier divisor D on X_1.*

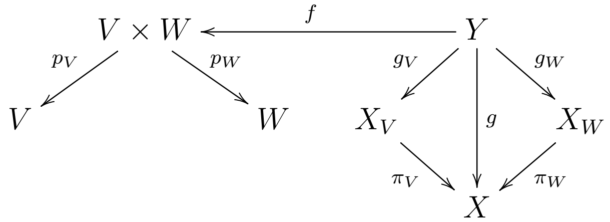
Let (X_1, Δ_1) and (X_2, Δ_2) be as in Theorem 3.7. The third and fourth properties of Theorem 3.7 implies that the same conclusion of Theorem 3.4 does not hold for (X_1, Δ_1) and (X_2, Δ_2). Furthermore, the second condition of Theorem 3.7 shows that $\widetilde{X}_1$ is $\mathbb{Q}$-factorial and $\widetilde{X}_2$ is not $\mathbb{Q}$-factorial for all small birational morphisms $f_1 \colon \widetilde{X}_1 \to X_1$ and $f_2 \colon \widetilde{X}_2 \to X_2$. Therefore, the condition $(*)$ of Theorem 3.3 does not hold for all induced small birational morphisms $f_2^{-1} \circ \phi \circ f_1 \colon \widetilde{X}_1 \dashrightarrow \widetilde{X}_2$. From them, the same conclusion of Theorem 3.3 does not hold for (X_1, Δ_1) and (X_2, Δ_2) in Theorem 3.7.

We explain the idea of construction of (X_1, Δ_1) and (X_2, Δ_2) as in Theorem 3.7.

Idea of proof of Theorem 3.7, see [H, Section 4]. We put $V = \mathbb{P}^n$, and let W be an elliptic curve. We define $p_V \colon V \times W \to V$ and $p_W \colon V \times W \to W$ by natural projections. Fix an ample $\mathbb{Q}$-divisor $H_V \sim_{\mathbb{Q}} -K_V$ and fix an ample $\mathbb{Q}$-divisor H_W on W, then put $H_{V\times W} = p_V^* H_V + p_W^* H_W$. We pick an integer $m > 0$ such that $mH_{V\times W}$ is a very ample Cartier divisor. We consider a $\mathbb{P}^1$-bundle

$$f \colon Y = \mathbb{P}_{V\times W}(\mathcal{O}_{V\times W} \oplus \mathcal{O}_{V\times W}(-mH_{V\times W})) \to V \times W.$$

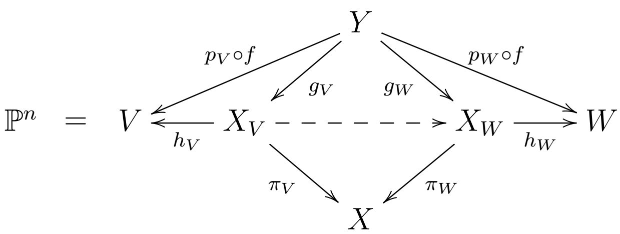
Let T be the unique section corresponding to $\mathcal{O}_Y(1)$, and put $A_Y = T + mf^* H_{V\times W}$. Since $-K_{V\times W} \sim_{\mathbb{Q}} p_V^* H_V$, we obtain $K_Y + T + A_Y + f^* p_V^* H_V \sim_{\mathbb{Q}} 0$. By construction, we also see that A_Y, $A_Y + f^* p_V^* H_V$ and $A_Y + f^* p_W^* H_W$ are all semi-ample, Y is smooth, and the pair (Y, T) is a log canonical pair. Let $g \colon Y \to X$, $g_V \colon Y \to X_V$ and $g_W \colon Y \to X_W$ be contractions induced by A_Y, $A_Y + f^* p_V^* H_V$ and $A_Y + f^* p_W^* H_W$, respectively. These morphisms are isomorphisms outside T. Furthermore, the induced birational maps $\pi_V \colon X_V \dashrightarrow X$ and $\pi_W \colon X_W \dashrightarrow X$ are morphism. We have constructed the following diagram.



By construction, the morphisms π_V and π_W are small birational morphisms. Then the induced birational map $X_V \dashrightarrow X_W$ is small.

Put $H_{X_V} = g_{V*} f^* p_V^* H_V$ and $H_{X_W} = g_{W*} f^* p_W^* H_W$. Then it follows that $f^* p_V^* H_V = g_V^* H_{X_V}$ and $f^* p_W^* H_W = g_W^* H_{X_W}$. Since H_V and H_W are ample, both H_{X_V} and H_{X_W} are semi-ample, and contractions induced by H_{X_V} and H_{X_W} are morphisms $h_V \colon X_V \to V$ and $h_W \colon X_W \to W$ satisfying $p_V \circ f = h_V \circ g_V$ and $p_W \circ f = h_W \circ g_W$, respectively. Then $H_{X_V} = h_V^* H_V$ and $H_{X_W} = h_W^* H_W$.

We have constructed the following diagram



and $\mathbb{Q}$-Cartier divisors

$$\bullet\, H_V \sim_{\mathbb{Q}} -K_V, \quad \bullet H_{X_V} = g_{V*}(p_V \circ f)^* H_V,$$
$$\bullet\, H_W, \qquad\qquad \bullet H_{X_W} = g_{W*}(p_W \circ f)^* H_W, \quad \text{and}$$
$$\bullet\, A_Y = T + m(p_V \circ f)^* H_V + m(p_W \circ f)^* H_W$$

satisfying

- Y is smooth and (Y, T) is a log canonical pair,
- $K_Y + T + A_Y + (p_V \circ f)^* H_V \sim_{\mathbb{Q}} 0$, and
- $(p_V \circ f)^* H_V = g_V^* H_{X_V}$ and $(p_W \circ f)^* H_W = g_W^* H_{X_W}$.

By [H, Proposition 4.2], the followings hold true.

(i) The pair $(X_W, 0)$ is $\mathbb{Q}$-factorial Kawamata log terminal pair,
(ii) the equality $\dim H^p(X_W, \mathcal{O}_{X_W}) = \dim H^p(W, \mathcal{O}_W)$ holds for every $p > 0$, and
(iii) the variety X_V is log canonical Fano.

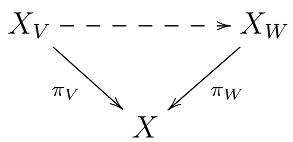
Since X_V is log canonical Fano, there is a $\mathbb{Q}$-divisor $\Delta_{X_V} \geq 0$ on X_V such that $K_{X_V} + \Delta_{X_V} \sim_{\mathbb{Q}} 0$ and the pair (X_V, Δ_{X_V}) is a log canonical pair. Let Δ_{X_W} be the birational transform of Δ_{X_V} on X_W. Then $K_{X_W} + \Delta_{X_W} \sim_{\mathbb{Q}} 0$ and (X_W, Δ_{X_W}) is a log canonical pair.

We set $(X_1, \Delta_1) = (X_W, \Delta_{X_W})$ and $(X_2, \Delta_2) = (X_V, \Delta_{X_V})$. Then X_1 and X_2 are projective, $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are $\mathbb{Q}$-linearly trivial, and there is a small birational map $\phi\colon X_1 \dashrightarrow X_2$. We show that (X_1, Δ_1) and (X_2, Δ_2) satisfy the conditions of Theorem 3.7. Firstly, it is clear by construction that $\phi_* \Delta_1 = \Delta_2$ which is the first condition of Theorem 3.7. Secondly, by (ii), we have $\dim H^1(X_1, \mathcal{O}_{X_1}) = \dim H^1(W, \mathcal{O}_W) = 1$. On the other hand, by (iii) and [F, Theorem 8.1] (or [F, Theorem 6.3 (ii)]), we have $\dim H^1(X_2, \mathcal{O}_{X_2}) = 0$. Thus

$$\dim H^1(X_1, \mathcal{O}_{X_1}) \neq \dim H^1(X_2, \mathcal{O}_{X_2})$$

which is the third condition of Theorem 3.7. Thirdly, we pick a non-torsion $M \equiv 0$ on W and put $D = h_W^* M$. If $\phi_* D$ is $\mathbb{Q}$-Cartier, then $\phi_* D \equiv 0$, so $\phi_* D \sim_{\mathbb{Q}} 0$ by [F, Theorem 13.1]. Then $D \sim_{\mathbb{Q}} 0$, hence $M \sim_{\mathbb{Q}} 0$ which contradicts our choice of M. Therefore, D satisfies the fourth condition of Theorem 3.7. This also shows that X_2 is not $\mathbb{Q}$-factorial. Finally, X_W is $\mathbb{Q}$–factorial by (i), and if X_2 has a small $\mathbb{Q}$-factorialization $\tilde{X} \to X_2$ then the induced birational map $X_1 \dashrightarrow \tilde{X}$ is small, so $\rho(X_1) = \rho(\tilde{X})$. Since $\tilde{X} \to X_2$ is not isomorphism, we have $\rho(\tilde{X}) > \rho(X_2)$. Furthermore, by explicit computations of the Picard numbers, we have $\rho(Y) = 3$, $\rho(X_1) = \rho(X_2) = 2$ and $\rho(X) = 1$. Therefore, $\rho(X_1) = \rho(\tilde{X}) > \rho(X_2) = \rho(X_1)$, a contradiction. So X_1 is $\mathbb{Q}$-factorial, but X_2 does not have any small $\mathbb{Q}$-factorialization. $\square$

The above construction of the diagram



is an interesting example of log canonical flip (see [H, Remark 4.5]). Indeed, the following properties hold.

- $\dim H^1(X_V, \mathcal{O}_{X_V}) = 0 \neq 1 = \dim H^1(X_W, \mathcal{O}_{X_W})$, and
- X_V is log canonical Fano but X_W is not even of log canonical Fano type, i.e., there is no effective $\mathbb{R}$-divisor B_W on X_W such that (X_W, B_W) is a log canonical pair and $-(K_{X_W} + B_W)$ is ample.

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