STRUCTURE OF FANO FIBRATIONS OF VARIETIES ADMITTING AN INT-AMPLIFIED ENDOMORPHISM

SHOU YOSHIKAWA

1. Introduction

Let X be a normal $\mathbb{Q}$-factorial klt projective variety over an algebraically closed field k of characteristic zero. We say that a surjective endomorphism $f\colon X \to X$ over k is *int-amplified* if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample. For example, non-invertible polarized endomorphisms are int-amplified. Admitting an int-amplified endomorphism imposes strong conditions on the structure of X. Indeed, Nakayama [Nak02] proved that if X is a smooth rational surface admitting a non-invertible surjective endomorphism, then X is toric. Recently, Meng [Men17] proved the following theorem.

Theorem 1.1 ([Men17], cf. [MZ18a]). *We assume that X has an int-amplified endomorphism. There exists a quasi-étale finite cover $\mu\colon \widetilde{X} \to X$, that is, μ is an étale in codimension one finite morphism such that the albanese morphism $\mathrm{alb}_{\widetilde{X}}$ is a fiber space whose general fiber is rationally connected.*

Following the above results, we discuss the next question.

Question 1.2 (cf. [MZ18b, Question 6.6]). *We assume that X has an int-amplified endomorphism. After replacing with a quasi-étale finite cover, is a general fiber of the albanese morphism of X toric? In particular, if X is smooth and rationally connected, then is X toric?*

First, we recall the notion of Fano type. Given a projective morphism $Z \to B$ of normal varieties, we say that Z is of *Fano type* over B if there exists an effective $\mathbb{Q}$-Weil divisor D on Z such that (Z, D) is klt and $-(K_Z + D)$ is ample over B (see § 2 for the details). When B is a point, we simply say that Z is of Fano type. We note that if Z is of Fano type over B, then a general fiber is of Fano type. For example, toric varieties are of Fano type and projective bundles over a variety B are of Fano type over B. Zhang [Zha06] and Hacon-Mckernan [HM07] proved that varieties of Fano type are rationally connected. On the other hand, smooth and rationally connected varieties are not necessarily of Fano type in general. Hence the following theorem strengthens Theorem 1.1 and gives a partial answer to Question 1.2.

Theorem 1.3. *We assume that X has an int-amplified endomorphism. There exists a quasi-étale finite cover $\mu\colon \widetilde{X} \to X$ such that the albanese morphism $\mathrm{alb}_{\widetilde{X}}\colon \widetilde{X} \to A$ is a fiber space and $\widetilde{X}$ is of Fano type over A.*

Furthermore, if X is smooth and rationally connected, then $\widetilde{X}$ has to coincide with X and A has to coincide with a point. Hence, as a corollary of Theorem 1.3, we obtain the following result, which gives an affirmative answer to [BG17, Conjecture 1.2] in the smooth and rationally connected case.

Corollary 1.4. *We assume that X has an int-amplified endomorphism. If X is smooth and rationally connected, then it is of Fano type.*

2. Key example

Before explaining the proof of Theorem 1.3, we observe the following example, which appears in [MY19, Section 7].

Example 2.1. *Let E be an elliptic curve and $[m]$ a multiplication by m for all integers m. Since $[m]$ is $[-1]$-equivariant, we obtain the following commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\mu} & \mathbb{P}^1 \\ {\scriptstyle [m]}\downarrow & & \downarrow{\scriptstyle h} \\ E & \xrightarrow{\mu} & \mathbb{P}^1, \end{array}$$

where μ is the quotient map by $[-1]$ and h is the endomorphism induced by $[m]$. Let $Q_1, \ldots, Q_4$ be the 2-torsion points on E and $P_1 = \mu(Q_1), \ldots, P_4 = \mu(Q_4)$. Let

$$\Delta = \frac{1}{2}(P_1 + P_2 + P_3 + P_4),$$

then $(\mathbb{P}^1, \Delta)$ satisfies the condition

$$R_{h,\Delta} := R_h + \Delta - h^*\Delta \geqslant 0,$$

indeed, $R_{h,\Delta} = 0$. We note that

$$R_{h,\Delta} \sim (K_{\mathbb{P}^1} + \Delta) - h^*(K_{\mathbb{P}^1} + \Delta),$$

thus we regard $R_{h,\Delta}$ as the ramification divisor of the pair (X, Δ) with respect to h. Pairs with effective ramification divisor play important role of the proof of Theorem 1.3.

Furthermore, we consider the following commutative diagram

$$\begin{array}{ccc} g \circlearrowright \mathbb{P}^1 \times E & \xrightarrow{\widetilde{\pi}} & E \circlearrowleft [m] \\ {\scriptstyle \widetilde{\mu}}\downarrow & & \downarrow{\scriptstyle \mu} \\ f \circlearrowright X & \xrightarrow[\pi]{} & \mathbb{P}^1 \circlearrowleft h, \end{array}$$

where $\widetilde{\mu}$ is the quotient by the involution

$$([x : y], a) \mapsto ([y : x], -a),$$

g is the int-amplified endomorphism with

$$g([x : y], a) = ([x^m : y^m], ma),$$

and π is the induced morphism. Then $\widetilde{\mu}$ is quasi-étale and π is a Mori fiber space. We note that X is not of Fano type and $\mathbb{P}^1 \times E$ is of Fano type over E, thus this diagram is desire one in Theorem 1.3. In other words, if π appear in the steps of MMP, we have to take the cover μ and $\widetilde{\mu}$.

In order to take a cover we use the triviality of the ramification divisor of $(\mathbb{P}^1, \Delta)$*. Indeed, by* $R_{h,\Delta} = 0$*, we have*

$$K_{\mathbb{P}^1} + \Delta \sim h^*(K_{\mathbb{P}^1} + \Delta)$$

and $K_{\mathbb{P}^1} + \Delta \sim_{\mathbb{Q}} 0$*. Since* $\mu^*(K_{\mathbb{P}^1} + \Delta) \sim 0$*,* μ *coincides with the index one cover of the pair* $(\mathbb{P}^1, \Delta)$*.*

Remark 2.2. In the above example, the ramification divisor R_f of f does not contains the pullback $\pi^* R_h$ of the ramification divisor of h. Indeed, since R_g has only horizontal components and $\widetilde{\mu}$ is quasi-étale, R_f has only horizontal component. However, h has ramification points, thus the pullback of such a point is not contained in R_f. This observation implies that it is difficult to understand the relation between the ramification divisors of an equivariant Mori fiber space and a base variety. On the other hand, we see the relation between the ramification divisors of suitable pairs, for example, we have $R_f \geqslant \pi^* R_{h,\Delta} = 0$ in Example 2.1. It is one of the motivations to consider the notion of ramification divisors of pair.

3. Sketch of proof of Theorem 1.3

Next, we briefly explain how to prove Theorem 1.3. First suppose that K_X is not pseudo-effective. Running a minimal model program (MMP, for short) for X, we obtain a birational map $\sigma_0 \colon X \dashrightarrow X'$ and a Mori fiber space $\pi_0 \colon X' \to X_1$. Then we construct an effective $\mathbb{Q}$-Weil divisor Δ_1 on X_1 as follows,

$$\mathrm{ord}_E(\Delta_1) = \frac{m_E - 1}{m_E}$$

for any prime divisor E on X_1, where m_E is a positive integer satisfying $\pi_0^* E = m_E F$ for some prime divisor F on X'. We note that this divisor coincides with Δ in Example 2.1 if π_0 is π in the example. Then the birational map $X \dashrightarrow X_1$ is equivariant under f up to replacing f into some power of f. The induced endomorphism is denoted by f_1. Then the ramification divisor R_{f_1,Δ_1} of the pair (X_1, Δ_1) with respect to f_1 is effective (see Example 2.1). Next, we further assume that $K_{X_1} + \Delta_1$ is not pseudo-effective. Running an MMP for (X_1, Δ_1), we obtain a birational map $\sigma_1 \colon X_1 \dashrightarrow X_1'$ and a $(K_{X_1} + \Delta_1)$-Mori fiber space $\pi_1 \colon X_1' \to X_2$. Then we construct an effective $\mathbb{Q}$-Weil divisor Δ_2 on X_2 as follows,

$$\mathrm{ord}_E(\Delta_2) = \frac{m_E - 1 + \mathrm{ord}_F(\Delta_1')}{m_E}$$

for any prime divisor E on X_2, where F is a prime divisor on X_1' satisfying $\pi_1^* E = m_E F$ with positive integer m_E. Then this pair also has effective ramification divisor. Repeating such a process, we obtain the following sequence of rational maps and morphisms

$$
\begin{array}{ccccccc}
X \overset{\sigma_0}{\dashrightarrow} X' & & & & & & \\
\downarrow{\scriptstyle \pi_0} & & & & & & \\
(X_1, \Delta_1) & \overset{\sigma_1}{\dashrightarrow} & (X_1', \Delta_1') & & & & \\
 & & \downarrow{\scriptstyle \pi_1} & & & & \\
 & & (X_2, \Delta_2) & \overset{\sigma_1}{\dashrightarrow} & & & \\
 & & & & \ddots & & \\
 & & & & & \overset{\sigma_r}{\dashrightarrow} & (X_r', \Delta_r') \\
 & & & & & & \downarrow{\scriptstyle \pi_r} \\
 & & & & & & (W, \Delta_W),
\end{array}
$$

where $K_W + \Delta_W$ is pseudo-effective. If K_X or $K_{X_1} + \Delta_1$ is pseudo-effective, we define (W, Δ_W) as $(X, 0)$ or (X_1, Δ_1), respectively. After iterating f, we prove that there exist an f-equivarient birational map $X \dashrightarrow Y$ and a sequence of Mori fiber spaces from Y to W such that the following diagram commutes

$$
\begin{array}{ccccccc}
X \overset{\sigma_0}{\dashrightarrow} X' & & & \dashrightarrow & & & Y \\
\downarrow{\scriptstyle \pi_0} & & & & & & \downarrow \\
(X_1, \Delta_1) & \overset{\sigma_1}{\dashrightarrow} & (X_1', \Delta_1') & & & & Y_1 \\
 & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow \\
 & & (X_2, \Delta_2) & \overset{\sigma_1}{\dashrightarrow} & & & \vdots \\
 & & & & \ddots & & \downarrow \\
 & & & & & \overset{\sigma_r}{\dashrightarrow} & (X_r', \Delta_r') \\
 & & & & & & \downarrow{\scriptstyle \pi_r} \\
 & & & & & & (W, \Delta_W).
\end{array}
$$

Since the above rational maps and morphisms are f-equivariant, W has an int-amplified endomorphism h and $R_{\Delta_W} := R_h + \Delta_W - h^*\Delta_W$ is an effective divisor, where R_h is the ramification divisor of h. The effectivity of R_{Δ_W} implies that $-(K_W + \Delta_W)$ is pseudo-effective (see [Men17]), hence $K_W + \Delta_W$ is $\mathbb{Q}$-linearly trivial. Then we prove that W has a finite cover by an abelian variety A. Moreover we can lift this cover to X as follows,

$$
\begin{array}{ccc}
\widetilde{X} & \overset{\widetilde{\pi}}{\dashrightarrow} & A \\
{\scriptstyle \mu}\downarrow & & \downarrow \\
X & \underset{\pi}{\dashrightarrow} & W,
\end{array}
$$

where μ is a quasi-étale finite morphism, and in particular, $\widetilde{\pi}$ and π are morphisms.

Finally, we prove that $\widetilde{X}$ is of Fano type over A. Note that Y is of Fano type over W. Since being of Fano type over W is invariant under every equivariant birational map with respect to an int-amplified endomorphism, X is also of Fano type over W. Moreover, since μ is quasi-étale, $\widetilde{X}$ is also of Fano type over A. In conclusion, we obtain Theorem 1.3.

References

[BG17] Amaël Broustet and Yoshinori Gongyo, *Remarks on log Calabi-Yau structure of varieties admitting polarized endomorphisms*, Taiwanese J. Math. **21** (2017), no. 3, 569–582. MR 3661381

[HM07] Christopher D. Hacon and James Mckernan, *On Shokurov's rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136. MR 2309156

[Men17] Sheng Meng, *Building blocks of amplified endomorphisms of normal projective varieties*, Mathematische Zeitschrift (2017), 1–21.

[MY19] Yohsuke Matsuzawa and Shou Yoshikawa, *Int-amplified endomorphisms on normal projective surfaces*, arXiv preprint arXiv:1902.06071 (2019).

[MZ18a] Sheng Meng and De-Qi Zhang, *Building blocks of polarized endomorphisms of normal projective varieties*, Adv. Math. **325** (2018), 243–273. MR 3742591

[MZ18b] ———, *Normal projective varieties admitting polarized or int-amplified endomorphisms*, Acta Mathematica Vietnamica (2018), 1–16.

[Nak02] Noboru Nakayama, *Ruled surfaces with non-trivial surjective endomorphisms*, Kyushu J. Math. **56** (2002), no. 2, 433–446. MR 1934136

[Zha06] Qi Zhang, *Rational connectedness of log* **Q***-Fano varieties*, J. Reine Angew. Math. **590** (2006), 131–142. MR 2208131

Graduate school of Mathematical Sciences, the University of Tokyo, Komaba, Tokyo, 153-8914, Japan

Email address: yoshikaw@ms.u-tokyo.ac.jp