BOUNDED COMPLEMENTS FOR ϵ-LC GENERALIZED FANO PAIRS

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Abstract. This is a brief report of the author's talk at the Kinosaki Algebraic Geometry Symposium 2020 on his joint work with Y. Nakamura and Y. Gongyo. Complements for generalized Fano pairs play an important role a series of papers by Birkar proving the BAB Theorem. Applying ACC for generalized lct and perturbing the coefficients the boundaries $\mathbb{Q}$-linearly, we generalize Birkar's construction of bounded complements for generalized lc Fano pairs.

1. Introduction

Throughout this paper, we work over an uncountable algebraically closed field of characteristic 0, for instance, the complex number field $\mathbb{C}$.

In [1], Birkar showed the following existence of bounded complements for generalized Fano pairs.

Theorem 1.1. [1, Theorem 1.10] *Let d and p be natural numbers and $I \in [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d, p and I satisfying the following. Assume $(X, B+M)$ is a generalized pair such that*

- *$(X, B+M)$ is generalized lc of dimension d,*
- *the coefficients of $B \in I$ and pM' is Cartier,*
- *X is of Fano type, and*
- *$-(K_X + B + M)$ is nef.*

Then there is a strong n-complement $K_X + B^+ + M$ of $K_X + B + M$.

Following [1], Birkar proved the following Theorem in [2].

Theorem 1.2. *Fix a positive integer d and a positive real number ϵ. The projective varieties X satisfying*

(1) $\dim X = d$,
(2) there exists a boundary B such that (X, B) is ϵ-lc, and
(3) $K_X + B \sim_{\mathbb{R}} 0$ and B is big,

form a bounded family.

Theorem 1.2 was known as the Borisov-Alexeev-Borisov (BAB) Conjecture for decades before Birkar proved it.

Using ACC for generalized lct (Theorem 2.5) and Theorem 1.2, Filipazzi and Moraga generalized Theorem 1.1 into the following theorem in [4].

Theorem 1.3. [4, Theorem 1.3]
Fix a natural number d, a closed DCC set of rational numbers $\Lambda \subseteq [0,1] \cap \mathbb{Q}$, a natural number p, and a non-negative real number ϵ. Then there exists a natural number n depending only on d, Λ, p and ϵ, such that if $(X, B+M)$ is a generalized ϵ-lc generalized pair of dimension d, X is of Fano type, the coefficients of $B \in \Lambda$, $-(K_X + B + M)$ is nef, pM' is Cartier, and either Λ is finite, or $M' = 0$ or $\epsilon = 0$, then there exists a generalized strong ϵ-lc n-complement $K_X + B^+ + M$ of $K_X + B + M$.

The original definition of boundedness of (generalized) complements is only for pairs or generalized pairs with rational coefficients of the boundaries. Han, Liu and Shokurov generalized this definition for usual pairs with boundaries of real coefficients, and showed the following theorem.

Theorem 1.4. [5, Theorem 1.3]
Fix a natural number d and a closed DCC set of non-negative real numbers $\Lambda \subseteq [0,1] \cap \mathbb{Q}$. Then there exists a natural number n depending only on d and Λ, such that if (X, B) is an lc pair of dimension d, X is of Fano type, the coefficients of $B \in \Lambda$ and $-(K_X + B)$ is nef, then there exists a strong lc n-complement $K_X + B^+$ of $K_X + B$.

We remark that in [5], generalized pairs are not considered.
Combining the theorems above, we have the following theorems, which are the main results of this article.

Theorem 1.5. *Fix a natural number d, a finite set of real numbers $I \subseteq [0,1]$, and a non-negative real number ϵ. Then there exists a natural number n and a finite set $\Gamma \subseteq [0,1]$ depending only on d, I and ϵ, such that if $(X, B+M)$ is a generalized ϵ-lc generalized pair of dimension d, X is of Fano type, the coefficients of B and $M' \in I$ and $-(K_X+B+M)$ is nef, then there exists a strong generalized (ϵ, n, Γ)-complement $K_X + B^+ + M$ of $K_X + B + M$. In particular, $(X, B + M)$ is $\mathbb{R}$-complementary.*

Theorem 1.6. *Fix a natural number d, a DCC set of real numbers $\Lambda \subseteq [0,1]$, and a non-nagetive real number ϵ. Suppose that either*

- $\epsilon = 0$,
- $M = 0$, *or*
- $d = 2$.

Then there exists a natural number n and a finite set $\Gamma \subseteq [0,1]$ depending only on d, Λ and ϵ, such that if $(X, B+M)$ is a generalized ϵ-lc generalized pair of dimension d, X is of Fano type, the coefficients of B and $M \in \Lambda$ and $-(K_X + B + M)$ is nef, then there exists a strong (ϵ, n, Γ)-complement $K_X + B^+M^+$ of $K_X + B + M$.

2. Preliminaries

We first define generalized pairs and their singularities.

2.1. **Generalized Pairs.**

Definition 2.1. A *generalized pair* $(X, B + M)$ consists of

- a normal variety X with a projective morphism $X \to Z$,
- an effective $\mathbb{R}$-divisor B on X, and
- a b-$\mathbb{R}$-Cartier b-divisor over X represented by some projective birational morphism $\varphi : X' \to X$ and $\mathbb{R}$-Cartier divisor M' on X'

such that M' is nef over Z and that $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \varphi_* M'$. We call the sum $B + M$ the *generalized boundary of* $(X, B + M)$.

Since M' is viewed as a b-divisor over X, we may always replace it by its pullback on any birational model over X' and replace X' accordingly without changing the generalized pair $(X, B + M)$. In this article, the base variety Z will always be assumed to be a point and omitted.

2.2. **Generalized Log Discrepancies.**

Definition 2.2. For a prime divisor E over X, we define the *generalized log discrepancy* $a_E(X, B + M)$ as follows. Possibly replacing X' by a higher model, we may assume that φ is a log resolution of (X, B) and that E is a divisor on X'. We define an $\mathbb{R}$-divisor B' on X' by

$$K_{X'} + B' + M' = \varphi^*(K_X + B + M).$$

Then we define $a_E(X, B + M) := 1 - \operatorname{coeff}_E B'$. The image $\varphi(E)$ is called the *center* of E on X and we denote it by $c_X(E)$. We define the *generalized minimal log discrepancy* $\operatorname{mld}(X, B + M)$ as

$$\operatorname{mld}(X, B + M) := \inf_E a_E(X, B + M),$$

where the infimum is taken over all prime divisors E over X. We say that $(X, B+M)$ is *generalized log canonical* (*generalized lc* for short) if $\operatorname{mld}(X, B + M) \geq 0$ holds. For $\epsilon \in \mathbb{R}_{\geq 0}$ we say that $(X, B + M)$ is *generalized ϵ-lc* if $\operatorname{mld}(X, B + M) \geq \epsilon$ holds.

We remark that a (usual) pairs can be view as a generalized pair with $M = 0$. In this way, Definition 2.2 extends the definition of the singularities for pairs.

2.3. **Fano pairs.** A projective pair (X, B) is a *Fano* (resp. *weak Fano*) pair if it is lc and $-(K_X + B)$ is ample (resp. $-(K_X + B)$ is nef and big). A projective variety X is called Fano, if $(X, 0)$ is Fano. It is called $\mathbb{Q}$-*Fano* if it is klt and Fano. It is called *of Fano type* if (X, B) is klt weak Fano for some boundary B.

2.4. **Bounded pairs.** A collection of varieties $\mathcal{D}$ is said to be *bounded* (resp. *birationally bounded*) if there exists $h\colon \mathcal{Z} \to S$ a projective morphism of schemes of finite type such that each $X \in \mathcal{D}$ is isomorphic (resp. birational) to $\mathcal{Z}_s$ for some closed point $s \in S$.

2.5. **Descending Chain Condition (DCC) and Ascending Chain Condition (ACC).**

Definition 2.3. A set of real numbers $\mathscr{S}$ is said to *satisfy descending chain condition (DCC for short)* if for every non-empty subset S of $\mathscr{S}$, there is a minimum element in S. A set of real numbers $\mathscr{S}$ is said to *satisfy ascending chain condition (ACC for short)* if $-\mathscr{S}$ satisfies DCC. $\mathscr{S}$ is called a DCC (resp. ACC) set if it satisfies DCC (resp. ACC).

We then define the generalized log canonical thresholds.

Definition 2.4. Suppose that $(X, B + M)$ is generalized lc. Let D be an effective $\mathbb{R}$-divisor on X and N' be an $\mathbb{R}$-Cartier divisor on X'. We assume that N' is nef over Z, and $D + N$ is $\mathbb{R}$-Cartier where $N := \varphi_* N'$. Then for each $t \in \mathbb{R}_{\geq 0}$, the pair $\big(X, (B + tD) + (M + tN)\big)$ becomes a generalized pair. We define the *generalized lc threshold* of $D + N$ with respect to $(X, B + M)$ as

$$\sup\big\{t \in \mathbb{R}_{\geq 0} \bigm| \big(X, (B + tD) + (M + tN)\big) \text{ is generalized lc}\big\}.$$

To construct bounded strong complements, we will need the following theorem.

Theorem 2.5 ([3])**.** *Let d be a positive integer and let $I \subset [0, +\infty)$ be a DCC subset. Then there exists an ACC set J with the following conditions: if X, B, M, D and N satisfy*

- *$(X, B + M)$ is a generalized pair with $\dim X = d$,*
- *$\big(X, (B + D) + (M + N)\big)$ is also a generalized pair, and*
- *$B, D \in I$ and $M, N \in I$,*

then the the generalized lc thresholds of $D + N$ with respect to $(X, B + M)$ belongs to J.

Theorem 2.5 is a generalization of its original version in [6] dealing with usual pairs.

The following theorem follows from Theorem 2.5 (c.f.[4] and [5]).

Theorem 2.6. *Fix $d \in \mathbb{N}$ and a closed DCC set of real numbers $\Lambda \subseteq [0, \infty)$*

Then there exists a discrete subset $\Lambda_0 \subseteq \Lambda$ and a projection function (i.e. $g \circ g = g$) $g : \Lambda \to \Lambda_0$ which preserves the odder "$\leq$" on Λ depending only on d, Λ and ϵ, such that if $(X, B + M)$ is a generalized lc generalized pair of dimension d satisfying the following conditions

- *we can write $B = \sum_j b_j B_j$ and $M' = \sum_j m_j M'_j$ for some $\mathbb{Q}$-Cartier Weil divisors B_j and some Cartier divisors M'_j with their push-forwards M_j on X being $\mathbb{Q}$-Cartier, and*
- *$b_j \in \Lambda$ and $m_j \in \Lambda$ for all j,*

then $(X, g(B + M)) := (X, \sum_i g(b_i)B_i + \sum_i g(m_i)M_i)$ is generalized lc.

2.6. **Complements.** We now define complements for generalized pairs

Definition 2.7. [1, 2.18] Let $(X, B + M)$ be generalized pair with coefficients of $B \in [0, 1]$. Let n be a natural number. An *(generalized) n-complement (resp. $\mathbb{R}$-complement) of $K_X + B + M$ is of the form $K_X + B^+ + M$ (resp. $(K_X + B^+ + M^+)$, such that*

- *$(X, B^+ + M)$ is generalized lc (resp. $(X, B^+ + M^+)$ is generalized lc),*
- *$n(K_X + B^+ + M) \sim 0$ and nM' is Cartier (resp. $K_X + B^+ + M^+ \sim_{\mathbb{R}} 0$), and*
- *$n(B^+) \geq n\{B\} + \lfloor (n + 1)B \rfloor$ (resp. $B^+ \geq B$ and $M'^+ - M'$ is nef).*

An n-complement is strong if moreover $B^+ \geq B$. Let ϵ be a non-negative real number. A complement $K_X + B^+ + M^+$ is called (generalized) ϵ-lc or klt if the corresponding generalized pair $(X, B^+ + M^+)$ is. We say $(X, B + M)$ is $\mathbb{R}$-complementary if there exits an $\mathbb{R}$-complement of $K_X + B + M$.

We remark that in the origin version of Definition 2.7, n-complements are defined only when M is of rational coefficients. We extent this definition in order to deal with generalized pairs with possible irrational coefficients. In the following, we will consider only strong complements. We remark that if $n(K_X + B^+ + M) \sim 0$, nM is integral and $B^+ \geq B$, then $nB^+ \geq n\{B\} + \lfloor (n + 1)B \rfloor$ ([1, 6.1]).

The following definition of strong (generalized) (ϵ, n, Γ)-complements for generalized pairs is modified from [5, Definition 1.9] for generalized pairs.

Definition 2.8. Let $(X, B + M)$ be a generalized pair, ϵ a non-negative real number and $\Gamma = \{a_1, a_2, ..., a_k\} \subseteq [0, 1]$ a finite set such that $\sum_{i=1}^k a_i = 1$. *A strong (generalized) (ϵ, n, Γ)-complement of $K_X + B + M$ is of the form $K_X + B^+ + M^+$, such that*

- *$(X, B^+ + M^+)$ is an $\mathbb{R}$-complement of $K_X + B + M$,*
- *$\sum_{i=1}^k a_i(B_i^+ + M_i^+) = B^+ + M^+$ for some generalized boundaries $B_i^+ + M_i^+$, and*
- *for each $i = 1, 2, ..., k$, $K_X + B_i^+ + M_i^+$ is an ϵ-lc n-complement of itself.*

An (n, Γ)-complement means a $(0, n, \Gamma)$-complement.

3. Complements for Generalized Pairs with Real Coefficients

In this section we introduce our main tools and give a sketch of proof of our main results. First, we have the following theorem of uniform perturbation preserving lc property.

3.1. **Uniform Perturbations Perserving Generalized Log Canonicity.**

Theorem 3.1. *Fix $d \in \mathbb{Z}_{>0}$. Let $r_1, \ldots, r_\ell$ be positive real numbers and let $r_0 = 1$. Assume that $r_0, r_1, \ldots, r_\ell$ are $\mathbb{Q}$-linearly independent. Let $s_1^B, \ldots, s_{c^B}^B : \mathbb{R}^{\ell+1} \to \mathbb{R}$ (resp. $s_1^M, \ldots, s_{c^M}^M : \mathbb{R}^{\ell+1} \to \mathbb{R}$) be $\mathbb{Q}$-linear functions (that is, the extensions of*

$\mathbb{Q}$-linear functions from $\mathbb{Q}^{\ell+1}$ to $\mathbb{Q}$ by taking the tensor product $\otimes_{\mathbb{Q}}\mathbb{R}$). Assume that $s_i^B(r_0,\ldots,r_\ell) \geq 0$ and $s_j^M(r_0,\ldots,r_\ell) \geq 0$ for each i and j. Then there exists a positive real number $\epsilon > 0$ such that the following holds: For any $\mathbb{Q}$-Gorenstein normal variety X of dimension d, a birational contraction $f : X' \to X$, $\mathbb{Q}$-Cartier effective Weil divisors $B_1,\ldots,B_{c^B}$ on X, and nef Cartier divisors $M_1',\ldots,M_{c^M}'$ on X', if the generalized pair*

$$\Big(X, \sum_{1\leq i\leq c^B} s_i^B(r_0,\ldots,r_\ell)B_i + \sum_{1\leq i\leq c^M} s_i^M(r_0,\ldots,r_\ell)M_i\Big)$$

is generalized lc, then the pair

$$\Big(X, \sum_{1\leq i\leq c^B} s_i^B(r_0,\ldots,r_{\ell-1},t)B_i + \sum_{1\leq i\leq c^M} s_i^M(r_0,\ldots,r_{\ell-1},t)M_i\Big)$$

is also generalized lc for any $t \in [r_\ell-\epsilon, r_\ell+\epsilon]$.

Applying Theorem 3.1 inductively we have the following theorem.

Theorem 3.2. *Let $d \in \mathbb{N}$ and let $I \subset [0,+\infty)$ be a finite set. Then there exists $\delta > 0$ such that every $\mathbb{Q}$-linear function $f : \operatorname{Span}_{\mathbb{Q}} I \to \mathbb{Q}$ with $\max\{|f(a)-a|\}_{a\in I} \leq \delta$ satisfies the following. If $(X, B+M)$ is a generalized lc pair such that*

- $\dim X = d$,
- $B = \sum_i b_iB_i$ *for some $b_i \in I$ and effective Cartier divisor B_i, and*
- $M' = \sum_i m_iM_i'$ *for some $m_i \in I$ and some nef Cartier divisor M_i',*

then $(X, f(B+M)) := (X, \sum_i f(b_i)B_i + \sum_i f(m_i)M_i)$ is also generalized lc.

3.2. **Discreteness of Generalized Log Discrepancies.**

Theorem 3.3. *Let $d, r \in \mathbb{Z}_{>0}$ and let $I \subset [0,+\infty)$ be a finite set. Let $P(d,r,I)$ be the set of all generalized lc pairs $(X, B+M)$ with the following conditions:*

- $\dim X = d$,
- rK_X *is Cartier,*
- $B = \sum_i b_iB_i$ *for some $b_i \in I$ and effective Cartier divisor B_i.*
- $M = \sum_i m_iM_i$ *for some $m_i \in I$ and divisors M_i such that $M_i = f_*M_i'$ for some birational contraction $f : X' \to X$ and a nef Cartier divisor M_i'.*

Then the following set

$$B_{\text{gen}}(d,r,I) := \left\{ a_E(X,B+M) \;\middle|\; \begin{array}{l} (X,B+M) \in P(d,r,I), \\ E \text{ is a divisor over } X. \end{array} \right\}$$

is a discrete subset of $[0,+\infty)$.

In particular, the set

$$A'_{\text{gen}}(d,r,I) := \big\{\operatorname{mld}(X,B+M) \;\big|\; (X,B+M) \in P(d,r,I).\big\}$$

is a discrete subset of $[0,+\infty)$.

Theorem 3.1 and Theorem 3.3 was first showed by Nakamura in [7] for usual pairs.

3.3. **Uniform Perturbations Perturbing MLD Consistently.** Using Theorem 3.2 and Theorem 3.3, by the linearity of log discrepancies with respect to the coefficients of the generalized boundaries, we have the following lemma immediately.

Lemma 3.4. *Let $d, r \in \mathbb{Z}_{>0}$ and let $I \subseteq [0,1]$ be a finite subset. Let $P(d,r,I)$ be the set of generalized lc pairs defined in Theorem 3.3. Let $f : \operatorname{Span}_{\mathbb{Q}} I \to \mathbb{Q}$ be any $\mathbb{Q}$-linear function. Then for any $(X, B+M) \in P(d,r,I)$, and any divisor E over X, $a_E(X,B+M) \in \operatorname{Span}_{\mathbb{Q}} I$ and $f(a_E(X,B+M)) = a_E(X, f(B)+f(M))$. For any fixed positive real number $\epsilon \in \operatorname{Span}_{\mathbb{Q}} I$, there is a positive real number δ, depending only on d, r, ϵ and I such that if $|f(a)-a| \leq \delta$ for every $a \in I$ and*

$\mathrm{mld}(X, B+M) = \epsilon$, *then* $f(\mathrm{mld}_x(X, B+M)) = \mathrm{mld}(X, f(B)+f(M))$ *for any* $x \in X$. *Moreover, for any divisor* F *on* X, F *computes* $\mathrm{mld}_x(X, f(B)+f(M))$ *if and only if it computes* $\mathrm{mld}(X, B+M)$.

Sketch of Proof of Theorem 1.5. We deal with the cases when $\epsilon > -$ and when $\epsilon = 0$ respectively.

Assume $\epsilon > 0$ first. By Theorem 1.2, we may assume that X is in a fixed bounded family. So there is a fixed natural number r such that rD is Cartier for all prime divisor D on X. Applying Lemma 3.4 on Theorem 1.3, we can show the theorem in this case.

Now we assume that $\epsilon = 0$. Applying Theorem 3.2 on Theorem 1.1, we can show the theorem in this case. $\square$

Conjecture 3.5. *Fix a natural number* d, *a closed DCC set of real numbers* $\Lambda \subseteq [0, \infty)$, *and a non-negative real number* ϵ.

Then there exists a discrete subset $\Lambda_0 \subseteq \Lambda$ *and a projection function (i.e.* $g \circ g = g$*)* $g : \Lambda \to \Lambda_0$ *which preserves the odder "*$\leq$*" on* Γ *depending only on* d, Λ *and* ϵ, *such that if* $(X, B+M)$ *is a generalized* ϵ*-lc generalized pair of dimension* d *satisfying the following conditions*

- X *is of Fano type,*
- *we can write* $B = \sum_j b_j B_j$ *and* $M' = \sum_j m_j M'_j$ *for some* $\mathbb{Q}$*-Cartier Weil divisors* B_j *and some Cartier divisors* M'_j *with their push-forwards* M_j *on* X *being* $\mathbb{Q}$*-Cartier,*
- $b_j \in \Lambda$ *and* $m_j \in \Lambda$ *for all* j, *and*
- $-(K_X + B + M)$ *is nef*

then $(X, g(B) + g(M))$ *is* ϵ*-lc and* $-(K_X + g(B) + g(M))$ *is nef. Moreover, there exists a sequence of projection functions* $g_k : \Lambda \to \Lambda$ *with discrete images, such that* $g_k(\lambda) - \lambda \in [0, \frac{1}{k}]$ *for any* $\lambda \in \Lambda$, *and we can take* g *above to be* g_k *for any* k *large enough.*

Remark 3.6. We list the known cases of Conjecture 1.3.

The case when $\epsilon > 0$ and $M' = 0$ is shown in the proof of [4, Theorem 1.3]. The case when $\epsilon = 0$ is shown in [4, Lemma 3.2]. Note that in [4], $\Lambda \subseteq \mathbb{Q}$ is assumed. We observe from the proof that if we only want to show Conjecture 3.5, the rationality assumption is not used and can be removed.

The case when $M' = 0$ and $\epsilon = 0$ is shown separately in [5, Theorem 5.20, (6)].

We can show the case when $d = 2$ by applying Theorem 1.2 and the following lemma.

Lemma 3.7. *Fix integers* $n > 0$ *and* $c \geq 0$. *Then there exists a positive number* N *depending on* n *and* c, *such that the following holds.*

Let (X, D) *be a log smooth surface. Let* $\varphi : X' \to X$ *be a smooth birational model over* X. *We can write* $\varphi^*(D) = \sum_i e_i E_i + \overline{D}$, *where* $\overline{D}$ *is the strict transform of* D *and* E_i *are the exceptional divisors of* φ. *Suppose* $A = \sum_i a_i E_i + \overline{D}$ *is an effective nef Cartier divisor on* X', *and* $(D+H)^2 \leq n$ *for some ample Cartier divisor* H *on* X. *Suppose* $a_i \geq c$ *for every* i *and* $a_j = c$ *for some* a_j. *Then there exists* $a_i = c$ *such that* $e_i < N$.

Proof. The proof is straightforward and is omitted. $\square$

Proof of Theorem 1.6. Combining the know cases of Conjecture 3.5 and Theorem 1.5, the theorem follows. $\square$

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