The McKay correspondence

Yukari Ito (Kavli IPMU) *

Abstract

The original McKay correspondence was observed by John McKay for finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ and developed as a geometric correspondence with the quotient singularity $\mathbb{C}^2/G$. In Kinosaki, the author introduced the McKay correspondence in dimension three and showed recent progress and open problems with some examples.

1 Introduction

In this section, we introduce McKay correspondence which was originally observed by McKay in 1979 [20]. This is a bijective correspondence between the exceptional divisors of the minimal resolution of simple singularities and non-trivial representations of the group $G \subset \mathrm{SL}(2,\mathbb{C})$.

(i) Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $X := \mathbb{C}^2/G$. Then we call the singularities of X as simple singularities and there is a unique minimal resolution of X,

$$f : \widetilde{X} \longrightarrow X.$$

Then we can see the correspondence between the exceptional divisors and the non-trivial representations:

1) Take irreducible representations of the group G up to the isomorphism

$$\rho_1, \cdots, \rho_k$$

i.e., $\rho_i : G \mapsto \mathrm{GL}(n_i,\mathbb{C})$, and let ρ be a regular representation of G in $\mathrm{SL}(2,\mathbb{C})$.

*yukari.ito@ipmu.jp, The author is partially supported by the Grant-in-aid for scientific research(C) (No. 18K03209) of JSPS in Japan.

2) Take tensor product:

$$\rho_i \otimes \rho = \sum a_{ij}\rho_j,\ (1 \le i,j \le k).$$

Then we obtain the coefficients a_{ij}.

3) Construct the extended Dynkin diagram by the following manner:

(i) If $a_{ij} = 0$, then there are no edge between vertices i and j.

(ii) If $a_{ij} = 1$, then there are an edge between vertices i and j.

If we forget one vertex which corresponds to the trivial representation, then we get a dual graph $\widetilde{\Gamma}$ of the exceptional divisors of the minimal resolution.

Theorem 1 (McKay[20])**.** *$\widetilde{\Gamma}$ is isomorphic to the affine Dynkin diagram of an irreducible root system. This root system is of type A_n if G is cyclic of order $n+1$, of type D_n if G is binary dihedral of order $4(n-2)$, and of type E_6 (resp. E_7 resp. E_8) if G is the binary tetrahedral (resp. octahedral resp. icosahedral) group.*

(ii) Gonzalez-Springberg and Verdier construct a direct geometric correspondence between the set $\mathrm{Irr}(G)$ of irreducible representations of G and the set $\mathrm{Irr}(D)$ of irreducible components of the exceptional divisor D in the minimal resolution $f : \widetilde{X} \to X$ of the singularity of $X = \mathbb{C}^2/G$ [9]. Let $\rho : G \to GL(E)$ be a non-trivial irreducible representation of G. By $\mathcal{E} \to \mathbb{C}^2$, we denote the associated G-equivariant vector bundle on $\mathbb{C}^2$. So the associated locally free sheaf on $\mathbb{C}^2$ is equal to $\mathcal{O}_{\mathbb{C}^2} \otimes_{\mathbb{C}} E$ with canonical G-action. Since G acts freely on $\mathbb{C}^2 - \{0\}$ and $\mathcal{E}$ is a G-vector bundle. $\mathcal{E}$ defines a vector bundle $\mathcal{E}'$ on the quotient $X - \{0\} = (\mathbb{C}^2 - \{0\})/G$. Let $\widetilde{\mathcal{E}} := f^*(\mathcal{E}')$ be the pull-buck of this bundle on $\widetilde{X} - D \cong X - \{0\}$, and denote by $i : \widetilde{X} - D \to \widetilde{X}$ the inclusion map. If s is a global section of $\mathcal{E}$, then s induces a global section of $\mathcal{E}'$ and $\widetilde{\mathcal{E}}$, so a defines a section $\pi(s)$ of the sheaf $i_*(\widetilde{\mathcal{E}})$ on $\widetilde{X}$. Denote $\pi(\mathcal{E})$ or $\pi(\rho)$ the subsheaf of $i_*(\widetilde{\mathcal{E}})$ generated by the sections $\pi(s)$.

Now the correspondence between the graph Γ and the resolution graph of the singularity X is given by the first Chern classes of the sheaves $\pi(\rho)$. Denote by $\mathrm{Irr}^0(G) \subset \mathrm{Irr}(G)$ the set of non-trivial irreducible representation of G. Then

Theorem 2 (Gonzalez-Springberg and Verdier[9])**.** *For each $\rho \in Irr^0(G)$ the sheaf $\pi(\rho)$ on $\widetilde{X}$ is locally free of rank $deg(\rho)$. There is a bijection*

$\phi : Irr^0(G) \to Irr(G)$ *such that for all* $d \in Irr(D)$

$$c_1\left(\pi(\rho)\right) \cdot d = \begin{cases} 0 & d \neq \phi(\rho) \\ 1 & d = \phi(\rho) \end{cases}$$

Furthermore

$$\phi(\rho_i)\phi(\rho_j) = a_{ij},$$

for all $\rho_i, \rho_j \in Irr^0(G),\ \rho_i \neq \rho_j$.

Artin and Verdier [1] proved this more generally with reflexive modules and this theory was developed by Esnault and Knörrer ([7], [8]) for more general quotient surface singularities. After Riemenchneider's definition of speciality [23], Wunram [28] constructed a nice generalized McKay correspondence for any quotient surface singularities in 1988.

2 Three dimensional McKay correspondence

2.1 Existence of a crepant resolution

As an analogue of two dimensional McKay correspondence, let us consider finite subgroup G of $\mathrm{SL}(3, \mathbb{C})$. The quotient singularity $\mathbb{C}^3/G$ is canonical but not terminal. Around 1985, Calabi-Yau manifold appeared in Super String theory and they considered Orbifold Euler characteristic $\chi(M, G)$. When $M = \mathbb{C}^3$ and G is a finite subgroup of $\mathrm{SL}(3, \mathbb{C})$, $\chi(\mathbb{C}^3, G) = \#\{$conjugacy class of $G\}$ and Hirzebruch and Höfer conjectured existence of a crepant resolution of $\mathbb{C}^3/G$ ([10]) and it was proved positively:

Theorem 3 (Markushevich[19, 2, 18], Roan[24, 25, 26], Ito[13, 14])**.** *Let* G *be a finite subgroup of* $\mathrm{SL}(3, \mathbb{C})$*. Then there exists a resolution of singularities* $\pi : \widetilde{X} \longrightarrow \mathbb{C}^3/G$ *with* $\omega_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}$ *and* $\chi(\widetilde{X}) = \sharp\{$*conjugacy classes of* $G\}$.

When a group G is abelian, the quotient $\mathbb{C}^3/G$ s a toric variety, whose corresponding cone σ is

$$\sigma = \{x_1e_1 + x_2e_2 + x_3e_3 | x_i \geq 0\} \text{ in } N_{\mathbb{R}} = \mathbb{R}^3,$$

where $N = \mathbb{Z}^3 + \frac{1}{r}(a, b, c)\mathbb{Z}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Toric crepant resolution can be given by a triangulation of the triangle $\triangle$ with vertices $\{e_1, e_2, e_3\}$ and $\frac{1}{r}(a, b, c)$ is corresponding to $g = \mathrm{diag}(\epsilon^a, \epsilon^b, \epsilon^c)$ of G ($\epsilon = r$-th root of 1).

Example 4. *When G is a cyclic group of order 6 generated by a matrix diag$(\epsilon, \epsilon^2, \epsilon^3)$ where ϵ is 6-th root of 1. It is called singularity of type $\frac{1}{6}(1,2,3)$. The triangle $\triangle$ can be drawn as follows and you can obtain crepant resolution by subdivision of the triangle $\triangle$ with vertices e_1, e_2, e_3, $\frac{1}{6}(1,2,3)$, $\frac{1}{6}(2,4,0)$, $\frac{1}{6}(3,0,3)$ and $\frac{1}{6}(4,2,0)$. And there are five crepant resolutions of this quotient singularity.*

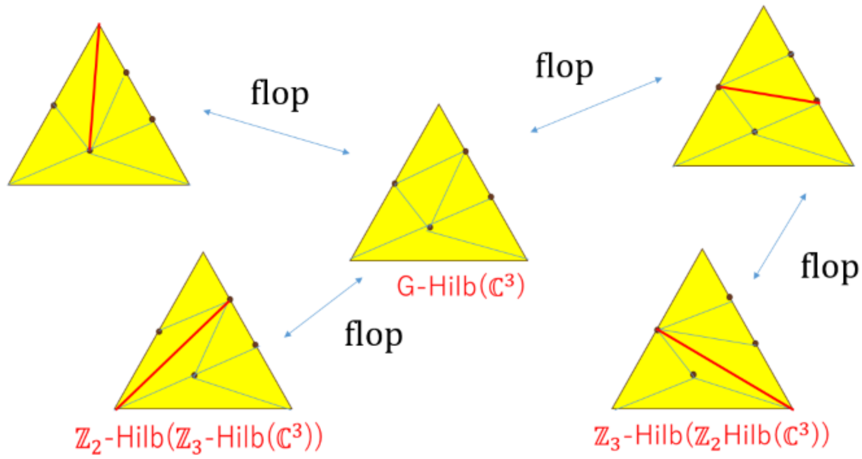

Figure 1: crepant resolutions of $\frac{1}{6}(1,2,3)$

2.2 G-Hilbert Scheme

For one quotient $\mathbb{C}^3/G$, we can obtain several crepant resolutions in general. In particular, we have one good projective crepant resolution, so-called G-Hilbert scheme.

Definition 5. *G-Hilbert scheme G-Hilb$(\mathbb{C}^n)$ is a set of G-invariant ideals I in $\mathbb{C}[x_1, \cdots, x_n]$ such that $\mathbb{C}[x_1, \cdots, x_n]/I \cong \mathbb{C}[G]$.*

When $n = 2$, the G-Hilbert scheme G-Hilb$(\mathbb{C}^2)$ is the minimal resolution of $\mathbb{C}^2/G$ for $G \subset \mathrm{GL}(2,\mathbb{C})$. It was proved by Ito and Nakamura for $G \subset \mathrm{SL}(2,\mathbb{C})$[15], Kidoh for cyclic $G \subset \mathrm{GL}(2,\mathbb{C})$ [17] and Ishii for any small groups in $\mathrm{GL}(2,\mathbb{C})$ [11].

When $n = 3$, G-Hilb$(\mathbb{C}^3)$ is a projective crepant resolution of $\mathbb{C}^3/G$ for $G \subset \mathrm{SL}(3,\mathbb{C})$. It was proved by Nakamura for abelian groups [21] and by

Bridgeland, King and Reid for general and they also showed "Mukai implies McKay", that is, the McKay correspondence is a derived equivalence in [3]

The G-Hilbert scheme G-Hilb($\mathbb{C}^n$) is also a moduli space of G-clusters Z, where Z is a G-invariant subscheme of $\mathbb{C}^n$ such that $H^0(\mathcal{O}_Z) = R_G$ regular representation as $\mathbb{C}[G]$ module. Moreover it is isomorphic to $M_\theta(Q, R)$ which is a moduli space of McKay quiver with relations where θ is 0-generated.

Then Craw and Ishii stated the following conjecture and proved it for abelian groups in [4].

Conjecture 6 (Craw and Ishii [4])**.** *For any finite subgroup $G \subset \mathrm{SL}(3, \mathbb{C})$, all projective crepant resolutions is isomorphic to M_θ, where θ is some GIT stability parameter.*

Theorem 7 (Craw and Ishii [4])**.** *The above conjecture is true for abelian subgroups of* $\mathrm{SL}(3, \mathbb{C})$.

2.3 Reid's recipe

When G is an abelian finite subgroup of $\mathrm{SL}(3, \mathbb{C})$, there is a recipe for geometric correspondence, so-called Reid's recipe ([22], [5]). This is a correspondence between a set of non-trivial irreducible representations and a set of exceptional divisors and curves.

Example 8. *In case of $\frac{1}{6}(1, 2, 3)$, the corresponding representations ρ_i are appear in the figure 2. ρ_5 corresponds to the exceptional divisor and each ρ_i $(i = 1, 2, 3, 4)$ corresponds to an exceptional curves.*

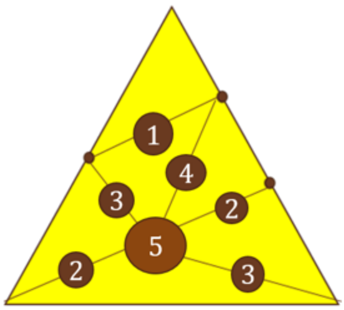

Figure 2: Reid's recipe for $\frac{1}{6}(1, 2, 3)$

3 Next steps

As we saw, many examples can be seen by toric geometry and we can see the geometric structure concretely. However, we would like to know more about non-abelian cases, higher dimensional crepant resolutions and the McKay correspondence.

3.1 Non-abelian cases

Even if G is non-abelian finite subgroup of $\mathrm{SL}(3,\mathbb{C})$, Ishii, Ito and Nolla de Celis showed that an iterated G-Hilbert schemes "Hilb of Hilb" is also a moduli space which satisfies Conjecture 6 ([4]).

Theorem 9 (Ishii, Ito and Nolla de Celis [12])**.** *Let G be a finite subgroup of $\mathrm{SL}(3,\mathbb{C})$, N be the abelian normal subgroup of G. Then G/N-Hilb(N-$\mathrm{Hilb}(\mathbb{C}^3)$ is a projective crepant resolution of $\mathbb{C}^3/G$ and isomorphic to a moduli space M_θ, where θ is some GIT stablity parameter.*

By this theorem, we can check the conjecture partially, and there are more crepant resolutions that are not "Hilb of Hilb". Moreover, when G is a simple group, we cannot use this construction.

Example 10. *In case $\frac{1}{6}(1,2,3)$, there are five crepant resolutions.One of them is the G-$\mathrm{Hilb}(\mathbb{C}^3)$ and two of them are iterated G-$\mathrm{Hilb}(\mathbb{C}^3)$, they are $\mathbb{Z}_2$-$\mathrm{Hilb}(\mathbb{Z}_3$-$\mathrm{Hilb}(\mathbb{C}^3))$ and $\mathbb{Z}_3$-$\mathrm{Hilb}(\mathbb{Z}_2$-$\mathrm{Hilb}(\mathbb{C}^3))$. The remaining two crepant resolutions cannot be obtained as a G-Hilbert scheme or an iterated G-Hilbert scheme. (cf. Figure 1)*

When G in non-abelian, we would also like to know more about the geometric correspondence like Reid's recipe.

3.2 Higher dimensional cases

Existence of a crepant resolution is not known in general in higher dimension even if G is abelian. Moreover, G-Hilbert schemes are not crepant resolution in general.

Dais, Henk and Zieglar found some conditions to admit a crepant resolutions for four dimensional Gorenstein abelian quotient singularities [6].

Recently, Kohei Saito and Yusuke Sato constructed higher dimensional crepant resolutions using with Fujiki-Oka resolution [27].

3.3 Reid's recipe for non-abelian quotients

This part is a joint work with Ben Wormleighton and work in progress, We show one example which gives a good geometric correspondence for a non-ablelian quotient.

Example 11. *Let G be a trihedral group generated by*

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix} \ (\epsilon^7 = 1) \ \textit{and} \ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We can take the normal abelian subgroup N of type $\frac{1}{7}(1, 2, 4)$ and the crepant resolution is unique. Then Reid's recipe gives the geometric correspondence between non-trivial irreducible representations and exceptional divisors and curves as follows.

ρ_1, ρ_2 and ρ_4 correspond to the exceptional curves and ρ_3, ρ_5 and ρ_6 corresponds to three exceptional divisors in the crepant resolution of $\mathbb{C}^3/N$.

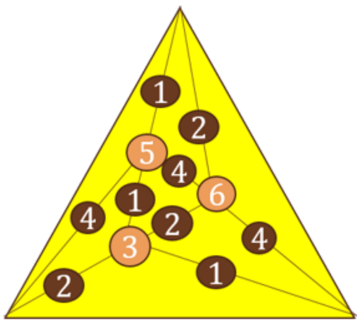

Figure 3: Reid's recipe for $\frac{1}{7}(1, 2, 4)$

To consider a crepant resolution of $\mathbb{C}^3/G$, we can identify three curves and three divisors and we have two more exceptional curves. On the other hand, there are 2 three-dimensional irreducible representations for G which are induced from N. Then the geometric correspondence for a crepant resolution of $\mathbb{C}^3/G$ become as follows.

One three dimensional representation $\rho_1 \oplus \rho_2 \oplus \rho_4$ corresponds to exceptional curve, the other three dimensional representation $\rho_3 \oplus \rho_5 \oplus \rho_6$ corresponds to the exceptional divisor and there are two more one dimensional irreducible representations which correspond to two other exceptional curves.

References

[1] M. Artin and J.L. Verdier, *Reflexive modules over rational double points*, Math. Ann. **270** (1985), 79–82.

[2] J. Bertin and D. Markushevich, *Singularités quotients non abéliennes de dimension 3 et variétés de Calabi-Yau*, MathȦnn; **299**, (1994), 105–116.

[3] T. Bridgeland, A. King and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. AMS, **14**, (2001), 535–554.

[4] A. Craw and A. Ishii, *Flops of G-Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Math. J., **124** (2), (2004), 259–307.

[5] A. Craw, *An explicit construction of the McKay correspondence for A-Hilb* $\mathbb{C}^3$, Journal of Algebra 285 (2005), no. 2, 682-705.

[6] D. I. Dais, M. Henk, and G. M. Ziegler, *All Abelian quotient c.i.-singularities admit projective crepant resolutions in all dimensions*, Adv. in Math. **139** (1998) 194–239.

[7] E. Esnault, *Reflexive modules on quotient surface singularities*, J. Reine Angew Math, **362** (1985), 63–71.

[8] E. Esnault and H. Knörrer, *Reflexive modules over rational double points*, Math. Ann. **272** (1985), 545–548.

[9] G. Gonzalez-Sprinberg and J.-L. Verdier, *Points doubles rationnels et représentations de groupes*, C. R. Acad. Sci. Paris, Sér. I, Math. , **283**, (1981), 111–113.

[10] F. Hirzebruch and T. Höfer, *On the Euler number of an orbifold*, MathȦnn; **286**, (1990), 255–260.

[11] A. Ishii, *On the McKay correspondence for a finite small subgroup of* $GL(2,\mathbb{C})$, J.Reine Angew. Math. **549** (2002), 221-233.

[12] A. Ishii, Y. Ito and A. Nolla de Celis, *On G/N-Hilb of N-Hilb*, Kyoto J. Math. **53**, no. 1 (2013), 91-130.

[13] Y. Ito, *Crepant resolution of trihedral singularities and the orbifold Euler characteristic*, InternJourof Math; **6**, (1995), 33–43.

[14] Y. Ito, *Gorenstein quotient singularities of monomial type in dimension three*, J. Math. Sci. Univ. Tokyo **2**, (1995), 419–440.

[15] Y. Ito and I. Nakamura, *Hilbert schemes and simple singularities*, New Trends in Algebraic Geometry, Proc. of EuroConference on Algebraic Geometry, Warwick 1996, ed. by K. Hulek et al., CUP, (1999), 151–233.

[16] Y. Ito and B. Wormleighton, in preparation.

[17] R. Kidoh, *Hilbert schemes and cyclic quotient singularities*, Hokkaido Mathematical Journal, **30** (2001), 91-103.

[18] D. Markushevich, *Resolution of* $\mathbb{C}^3/H_{168}$, Math. Ann. **308**, (1997), 279–289.

[19] D. G. Markushevich, M. A. Olshanetsky and A. M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, CommMathPhys; **111**, (1987), 247–274.

[20] J. McKay, *Graphs, singularities, and finite group*, in Santa Cruz conference on finite groups (Santa Cruz, 1979), Proc. Symp. Pure Math. **37**, AMS, (1980), 183–186.

[21] I. Nakamura, *Hilbert schemes of abelian group orbits*, J. Algebraic Geom., **10**, (2001), 757-779.

[22] M. Reid, *McKay correspondence*, Proc. of algebraic geometry symposium (Kinosaki, Nov 1996), T. Katsura (ed.), (1997), 14-41.

[23] O. Riemenchneider, *Characterization and application of special reflexive modules on rational surface singularities*, Institut Mittag-Leffler Report No.3 (1987).

[24] S.-S. Roan, *On the generalization of Kummer surfaces*, J. Diff. Geometry **30**, (1989), 523–537.

[25] S.-S. Roan, *On* $c_1 = 0$ *resolution of quotient singularity*, Intern. Jour. of Math., **5**, (1994), 523–536.

[26] S.-S. Roan, *Minimal Resolution of Gorenstein Orbifolds in Dimension Three*, Topology, **35**, (1996), 273-559.

[27] K. Sato and Y. Sato, *Crepant Property of Fujiki-Oka Resolutions for Gorenstein Abelian Quotient Singularities*, preprint, arXiv:2004.03522.

[28] J. Wunram, *Reflexive modules on quotient surface singularities*, Math ann. **279** (1988), 583–598.